Nonautonomous Young Differential Equations Revisited

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Abstract In this paper we prove that under mild conditions a nonautonomous Young differential equation possesses a unique solution which depends continuously on initial conditions. The proofs use estimates in p -variation norms, the construction of greedy sequence of times, and Gronwall-type lemma with the help of Shauder theorem of fixed points.

Keywords Stochastic differential equations (SDE) · Fractional Brownian motion (fBm) · Young integral · p -variation

1 Introduction

This paper deals with the Young differential equation of the form

$$dx_t = f(t, x_t)dt + g(t, x_t)d\omega_t, \quad t \geq 0 \quad (1.1)$$

where $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are continuous functions, ω is a $\mathbb{R}^m$ -valued function of finite p -variation norm for some $1 < p < 2$. Such type of system is generated from stochastic differential equations driven by fractional Brownian noises, as seen e.g. in [20]. Equation (1.1) is understood in the integral form

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\omega_s, \quad t \geq 0, \quad (1.2)$$

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where the first integral is of Riemannian type, meanwhile the second integral can be defined in the Young sense [23]. The existence and uniqueness of the solution of (1.2) are studied by several authors. When f, g are time-independent, system (1.2) is proved in [23] and [22] and [18] to have a unique solution in a certain space of continuous functions with bounded p -variation.

The result is then generalized for the case $2 \leq p < 3$ in Lyons' seminal paper [19] in which rough path theory is introduced to define the second integral in (1.2) and also the integration equation (see [8], [16] and [17]). An alternative theory of controlled paths in Gubinelli's work [10] simplifies and generalizes the concept of integration and differential equations, leading to the concept of rough differential equations (see recent works in [3] and [2] for $2 \leq p < 3$, or [14] for controlled differential equations as Young integrals). According to their settings, f, g are time independent and g is often assumed to be differentiable upto a certain order and bounded in its derivatives.

A different approach following Zähle [24] by using fractional derivatives can be seen in [21] which derives very weak conditions for time varying f and g in (1.1), in particular g need to be only C^1 with bounded and Hölder continuous first derivative, to ensure the existence and uniqueness of the solution in the space of Hölder continuous functions. Later, one finds that there is a connection between the rough path approach and the techniques in fractional calculus, see e.g. [11] and [12].

Our aim in this paper is to close the gap between the two methods for nonautonomous Young equations by proving that, under similar assumptions to those of Nualart and Rascanu [21], the existence and uniqueness theorem for system (1.1) still holds in the space of continuous functions with bounded p -variation norm. For that to work, we construct the so-called *greedy* sequence of times (see [4, Definition 4.7]) such that the solution can be proved to exists uniquely in each interval of the consecutive times of the greedy sequence, and is then concatenated to form a global solution. It is important to note that since we are using estimates for p -variation norms, we do not apply the classical arguments of contraction mappings, but use Shauder-Tychonoff fixed point theorem as seen in [18] and a Gronwall-type lemma.

Another issue is the generation of flow which was asserted in [17] for the autonomous systems. The idea is to construct the shift dynamical system in the extended space of finite p -variation norm for the whole real line time, and to prove that the system generates a nonautonomous dynamical system satisfying the cocycle property (see [3]). When applying to stochastic differential equations driven by fractional Brownian motions, by considering an appropriate probability space, one can prove that the system generates a random dynamical system (see [3, 5, 9]). However in the nonautonomous situation, one only expects a generation of a two-parameter flow on the phase space.

The paper is organized as follows. In Sect. 2, the Young integral is introduced and a version of greedy sequence of times is presented. In Sect. 3, we prove the existence and uniqueness of the global solution of system (1.2) in Theorem 3.6, for this we need to formulate a Gronwall-type lemma. Proposition 3.7 gives an estimate of q -var norm of solution via p -var norm of the driver ω . We also prove the existence and uniqueness of the solution of the backward equation (3.23) in Theorem 3.8. In Sect. 4, the fact in Theorem 4.1 that two trajectories do not intersect helps to conclude that the Cauchy operator or the Ito map of (1.2) generates a continuous two parameter flow. In the autonomous case it generates a continuous nonautonomous dynamical system which helps to form a topological skew product flow.

2 Preliminaries

2.1 Young Integral

In this section we recall some facts about Young integral, more details can be seen in [8]. Let $C([a, b], \mathbb{R}^d)$ denote the space of all continuous paths $x : [a, b] \rightarrow \mathbb{R}^d$ equipped with sup norm $\|\cdot\|_{\infty, [a, b]}$ given by $\|x\|_{\infty, [a, b]} = \sup_{t \in [a, b]} |x_t|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. For $p \geq 1$ and $[a, b] \subset \mathbb{R}$, a continuous path $x : [a, b] \rightarrow \mathbb{R}^d$ is of finite p -variation if

$$\|x\|_{p\text{-var}, [a, b]} := \left(\sup_{\Pi(a, b)} \sum_{i=1}^n |x_{t_{i+1}} - x_{t_i}|^p \right)^{1/p} < \infty, \quad (2.1)$$

where the supremum is taken over the whole class of finite partition of $[a, b]$. The subspace $\widehat{C}^p([a, b], \mathbb{R}^d) \subset C([a, b], \mathbb{R}^d)$ of all paths x with finite p -variation and equipped with the p -var norm

$$\|x\|_{p\text{-var}, [a, b]} := |x_a| + \|x\|_{p\text{-var}, [a, b]},$$

is a nonseparable Banach space [8, Theorem 5.25, p. 92]. Notice that if $x \in \widehat{C}^p([a, b], \mathbb{R}^d)$ then the mapping $(s, t) \rightarrow \|x\|_{p\text{-var}, [s, t]}$ is continuous on the simplex $\Delta[a, b] := \{(s, t) | a \leq s \leq t \leq b\}$, see [8, Proposition 5.8, p. 80].

Furthermore, the closure of $C^\infty([a, b], \mathbb{R}^d)$ in $\widehat{C}^p([a, b], \mathbb{R}^d)$ is a separable Banach space denoted by $\widehat{C}^{0,p}([a, b], \mathbb{R}^d)$ which can be defined as the space of all continuous paths x such that

$$\lim_{\delta \rightarrow 0} \sup_{\Pi(a, b), |\Pi| \leq \delta} \sum_i |x_{t_{i+1}} - x_{t_i}|^p = 0.$$

It is easy to prove (see [8, Corollary 5.33, p. 98]) that for $1 \leq p < p'$ we have

$$\widehat{C}^p([a, b], \mathbb{R}^d) \subset \widehat{C}^{0,p'}([a, b], \mathbb{R}^d).$$

Also, for $0 < \alpha \leq 1$ denote by $C^{\alpha\text{-Hol}}([a, b], \mathbb{R}^d)$ the Banach space of all Hölder continuous paths $x : [a, b] \rightarrow \mathbb{R}^d$ with exponent α , equipped with the norm

$$\begin{aligned} \|x\|_{\alpha\text{-Hol}, [a, b]} &:= |x_a| + \|x\|_{\alpha\text{-Hol}, [a, b]} \\ &= |x_a| + \sup_{(s, t) \in \Delta[a, b]} \frac{|x_t - x_s|}{(t - s)^\alpha} < \infty. \end{aligned} \quad (2.2)$$

Clearly, if $x \in C^{\alpha\text{-Hol}}([a, b], \mathbb{R}^d)$ then for all $s, t \in [a, b]$ we have

$$|x_t - x_s| \leq \|x\|_{\alpha\text{-Hol}, [a, b]} |t - s|^\alpha.$$

Hence, for all p such that $p\alpha \geq 1$ we have

$$\|x\|_{p\text{-var}, [a, b]} \leq \|x\|_{\alpha\text{-Hol}, [a, b]} (b - a)^\alpha < \infty. \quad (2.3)$$

Therefore, $C^{1/p\text{-Hol}}([a, b], \mathbb{R}^d) \subset \widehat{C}^p([a, b], \mathbb{R}^d)$.

As introduced in [21], the Besov space $W_b^{1/p, \infty}([a, b], \mathbb{R}^d)$ of measurable functions $g : [a, b] \rightarrow \mathbb{R}^d$ such that

$$\sup_{a < s < t < b} \left(\frac{|g_t - g_s|}{(t - s)^{1/p}} + \int_s^t \frac{|g_y - g_s|}{(y - s)^{1+1/p}} dy \right) < \infty$$

is a subspace of $C^{1/p\text{-Hol}}([a, b], \mathbb{R}^d)$. Hence $W_b^{1/p, \infty}([a, b], \mathbb{R}^d) \subset \widehat{C}^p([a, b], \mathbb{R}^d)$.

Lemma 2.1 Let $x \in \widehat{C}^p([a, b], \mathbb{R}^d)$, $p \geq 1$. If $a = a_1 < a_2 < \dots < a_k = b$, then

$$\sum_{i=1}^{k-1} \|x\|_{p\text{-var}, [a_i, a_{i+1}]}^p \leq \|x\|_{p\text{-var}, [a_1, a_k]}^p \leq (k-1)^{p-1} \sum_{i=1}^{k-1} \|x\|_{p\text{-var}, [a_i, a_{i+1}]}^p.$$

Proof The proof is similar to the one in [8, p. 84], by using triangle inequality and power means inequality

$$\frac{1}{n} \sum_{i=1}^n z_i \leq \left(\frac{1}{n} \sum_{i=1}^n z_i^r \right)^{1/r}, \quad \forall z_i \geq 0, r \geq 1.$$

□

Definition 2.2 A continuous map $\bar{\omega} : \Delta[a, b] \rightarrow \mathbb{R}^+$ is called a control if it is zero on the diagonal and superadditive, i.e

- (i), For all $t \in [a, b]$, $\bar{\omega}_{t,t} = 0$,
- (ii), For all $s \leq t \leq u$ in $[a, b]$, $\bar{\omega}_{s,t} + \bar{\omega}_{t,u} \leq \bar{\omega}_{s,u}$.

The functions $(s, t) \rightarrow (t-s)^\theta$ with $\theta \geq 1$, and $(s, t) \rightarrow \|x\|_{p\text{-var}, [s,t]}^q$, where x is of bounded p -variation norm on $[a, b]$ and $q \geq p$ are some examples of control function. The following lemma gives a useful property of controls in relation with variations of a path (see [8] for more properties of control functions).

Lemma 2.3 Let $\bar{\omega}^j$ be a finite sequence of control functions on $[0, T]$, $C_j > 0$, $j = \overline{1, k}$, $p \geq 1$ and $x : [0, T] \rightarrow \mathbb{R}^d$ be a continuous path satisfying $|x_t - x_s| \leq \sum_{i=j}^k C_j \bar{\omega}^j(s, t)^{1/p}$, $\forall s < t \in [0, T]$, then

$$\|x\|_{p\text{-var}, [s,t]} \leq \sum_{j=1}^k C_j \bar{\omega}^j(s, t)^{1/p}, \quad \forall s < t \in [0, T]. \quad (2.4)$$

Proof Consider an arbitrary finite partition $\Pi = (s_i), i = 0 \dots, n+1$, of $[s, t]$. By assumption and Minskowski inequality we have

$$\begin{aligned} \left(\sum_{i=0}^n |x_{s_{i+1}} - x_{s_i}|^p \right)^{1/p} &\leq \left[\sum_{i=0}^n \left(\sum_{j=1}^k C_j \bar{\omega}^j(s_i, s_{i+1})^{1/p} \right)^p \right]^{1/p} \\ &\leq \sum_{j=1}^k \left(\sum_{i=0}^n C_j^p \bar{\omega}^j(s_i, s_{i+1}) \right)^{1/p} \leq \sum_{j=1}^k C_j \bar{\omega}^j(s, t)^{1/p}. \end{aligned}$$

This implies the conclusion of the lemma. □

Now, consider $x \in \widehat{C}^q([a, b], \mathbb{R}^{d \times m})$ and $\omega \in \widehat{C}^p([a, b], \mathbb{R}^m)$, $p, q \geq 1$, if Riemann-Stieltjes sums for finite partition $\Pi = \{a = t_0 < t_1 < \dots < t_n = b\}$ of $[a, b]$ and any $\xi_i \in [t_i, t_{i+1}]$

$$S_\Pi := \sum_{i=0}^n x_{\xi_i} (\omega_{t_{i+1}} - \omega_{t_i}), \quad (2.5)$$

converges as the mesh $|\Pi| := \max_{0 \leq i \leq n-1} |t_{i+1} - t_i|$ tends to zero, we call the limit is the Young integral of x w.r.t ω on $[a, b]$ denoted by $\int_a^b x_t d\omega_t$. It is well known that if $p, q \geq 1$ and

$\frac{1}{p} + \frac{1}{q} > 1$, the Young integral $\int_a^b x_t d\omega_t$ exists (see [23, pp. 264–265]). Moreover, if x^n and ω^n are of bounded variation, uniformly bounded in $\widehat{C}^q([a, b], \mathbb{R}^{d \times m})$, $\widehat{C}^p([a, b], \mathbb{R}^m)$ and converges uniformly to x, ω respectively, then the sequence of the Riemann-Stieltjes integral $\int_a^b x_t^n d\omega_t^n$ approach $\int_a^b x_t d\omega_t$ as $n \rightarrow \infty$ (see [8]). This integral satisfies additive property by the construction, and the so-called Young-Loeve estimate [8, Theorem 6.8, p. 116]

$$\left| \int_s^t x_u d\omega_u - x_s[\omega_t - \omega_s] \right| \leq K \|x\|_{q\text{-var}, [s, t]} \|\omega\|_{p\text{-var}, [s, t]}, \quad (2.6)$$

where

$$K := (1 - 2^{1-\theta})^{-1}, \quad \theta := \frac{1}{p} + \frac{1}{q} > 1. \quad (2.7)$$

Lemma 2.4 *For $1 \leq p, 1 \leq q$ such that $\theta = \frac{1}{p} + \frac{1}{q} > 1$ and $x \in \widehat{C}^q([a, b], \mathbb{R}^{d \times m})$, $\omega \in \widehat{C}^p([a, b], \mathbb{R}^m)$, the following estimate holds*

$$\left\| \int_a^\cdot x_u d\omega_u \right\|_{p\text{-var}, [a, b]} \leq \|\omega\|_{p\text{-var}, [a, b]} (|x_a| + (K + 1) \|x\|_{q\text{-var}, [a, b]}), \quad (2.8)$$

where K is determined by (2.7).

Proof The conclusion is a direct sequence of (2.6) and [8, Proposition 5.10(i), p. 83]. $\square$

Due to Lemma 2.4, the integral $t \mapsto \int_a^t x_s d\omega_s$ is a continuous bounded p -variation path. Note that the definition of Young integral does depend on the direction of integration in a simple way like the Riemann-Stieltjes integral. Namely, it is easy to see that

$$\begin{aligned} \int_b^a x_u d\omega_u &= \lim_{\Pi(a, b), |\Pi| \rightarrow 0} \sum_{i=1}^n x_{\xi_i} (\omega_{t_i} - \omega_{t_{i+1}}) \\ &= - \lim_{\Pi(a, b), |\Pi| \rightarrow 0} \sum_{i=1}^n x_{\xi_i} (\omega_{t_{i+1}} - \omega_{t_i}) = - \int_a^b x_u d\omega_u. \end{aligned} \quad (2.9)$$

2.2 The Greedy Sequence of Times

The original idea of a greedy sequence was introduced in [4, Definition 4.7]. Given $\alpha > 0$, a compact interval $I \in \mathbb{R}$ and a control $\bar{\omega} : \Delta(I) \rightarrow \mathbb{R}^+$, the construction of such a sequence aims to have a “greedy” approximation to the supremum in the definition of the so-called *accumulated α -local $\bar{\omega}$ -variation* (see [4, Definition 4.1])

$$M_{\alpha, I}(\bar{\omega}) = \sup_{\Pi(I), \bar{\omega}_{t_i, t_{i+1}} \leq \alpha} \sum_{t_i \in \Pi(I)} \bar{\omega}_{t_i, t_{i+1}}.$$

In particular, $\bar{\omega}_{s, t}$ is chosen to be $\|\cdot\|_{p\text{-var}, [s, t]}^p$ in [4].

A similar version for stopping times was developed before in [9] and then has been studied further recently by [7] for stability of the system. Here we propose another version of greedy sequence of times which matches with the nonautonomous setting.

Denote by $\widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ the space of all continuous functions $\omega : \mathbb{R} \rightarrow \mathbb{R}^m$ such that for any $T > 0$ the restrictions of ω to $[-T, T]$ is of $\widehat{C}^p([-T, T], \mathbb{R}^m)$. Equip $\widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ with the metric

$$d(\omega^1, \omega^2) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|\omega^1 - \omega^2\|_{p\text{-var}, [-n, n]}}{1 + \|\omega^1 - \omega^2\|_{p\text{-var}, [-n, n]}}.$$

Let $n \in \mathbb{N}$, observe that metric d satisfies

$$\begin{aligned} d(\omega^1, \omega^2) &\leq \|\omega^1 - \omega^2\|_{p\text{-var}, [-n, n]} + 2^{-n}, \\ \|\omega^1 - \omega^2\|_{p\text{-var}, [-n, n]} &\leq \frac{2^n d(\omega^1, \omega^2)}{1 - 2^n d(\omega^1, \omega^2)}, \end{aligned} \quad (2.10)$$

where the second inequality holds for any fixed n and ω^1, ω^2 close enough such that $2^n d(\omega^1, \omega^2) < 1$. Hence every Cauchy sequence $(\omega^k)_k$ w.r.t. metric d is also a Cauchy sequence when restricted to $\widehat{C}^p([-n, n], \mathbb{R}^m)$, thus converges to a limit $\omega^* \in \widehat{C}^p([-n, n], \mathbb{R}^m)$ which is uniquely defined pointwise, so $\omega^* \in \widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$. Therefore, $(\widetilde{C}^p(\mathbb{R}, \mathbb{R}^m), d)$ is a complete metric space.

Remark 2.5 (i) *Truncation:* Another consequence of (2.10) is that the truncated version of $\omega \in \widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ in any $\widehat{C}^p([-n, n], \mathbb{R}^m)$ differs very little w.r.t. metric d from the original ω if we choose n large enough. Moreover, if a function is continuous w.r.t. ω on any restriction in $\widehat{C}^p([-n, n], \mathbb{R}^m)$ for any $n > 0$ then it is also continuous w.r.t. ω in $\widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ with respect to metric d .

(ii) *Concatenation:* Let $a < b < c$. Suppose that $\omega^1 \in \widehat{C}^p([a, b], \mathbb{R}^m)$, $\omega^2 \in \widehat{C}^p([b, c], \mathbb{R}^m)$ and $\omega_b^1 = \omega_b^2$. Then $\omega^1 \cdot 1_{[a, b]} + \omega^2 \cdot 1_{[b, c]}$ belongs to $\widehat{C}^p([a, c], \mathbb{R}^m)$.

For any given $\lambda, \mu > 0$ we construct a strict increasing sequence of times $\{\tau_n\}$,

$$\tau_n : \widetilde{C}^p(\mathbb{R}, \mathbb{R}^m) \longrightarrow \mathbb{R}^+,$$

such that $\tau_0 \equiv 0$ and

$$|\tau_{i+1}(\omega) - \tau_i(\omega)|^\lambda + \|\omega\|_{p\text{-var}, [\tau_i(\omega), \tau_{i+1}(\omega)]} = \mu. \quad (2.11)$$

To do so, first define $\tau : \widetilde{C}^p(\mathbb{R}, \mathbb{R}^m) \longrightarrow \mathbb{R}^+$ such that

$$\tau(\omega) := \sup\{t \geq 0 : t^\lambda + \|\omega\|_{p\text{-var}, [0, t]} \leq \mu\}.$$

Observe that the function $\kappa(t) := t^\lambda + \|\omega\|_{p\text{-var}, [0, t]}$ is continuous and strictly increasing w.r.t. t with $\kappa(0) = 0$ and $\kappa(\infty) = \infty$, therefore due to the continuity there exists a unique $\tau = \tau(\omega) > 0$ such that

$$\tau^\lambda + \|\omega\|_{p\text{-var}, [0, \tau]} = \mu. \quad (2.12)$$

Thus τ is well defined. Next, we construct the time sequence inductively as follows. Set $\tau_0 := 0$, $\tau_1(\omega) := \tau(\omega)$. Suppose that we have defined $\tau_n(\omega)$ for $n \geq 1$, looking at the following equality as an equation of $\delta_n(\omega) \in \mathbb{R}^+$, like above we find a unique $\delta_n(\omega)$ such that

$$\mu = \delta_n^\lambda(\omega) + \|\omega(\cdot + \tau_n(\omega))\|_{p\text{-var}, [0, \delta_n(\omega)]},$$

hence we can set

$$\tau_{n+1}(\omega) := \tau_{n-1}(\omega) + \delta_n(\omega), \quad (2.13)$$

where $\delta_n(\omega)$ is determined above. Thus we have defined a time sequence $\{\tau_n\}$ for all $n = 0, 1, 2, \dots$. Such a sequence then satisfies (2.11).

Now, we fix $\omega \in \widetilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ and consider the number of times of the greedy sequence inside an arbitrary finite interval of $\mathbb{R}^+$. We write τ_n for $\tau_n(\omega)$ to simplify the notation. For given $T > 0$, we introduce the notation

$$N(T, \omega) := \sup\{n : \tau_n \leq T\} < \infty. \quad (2.14)$$

or more generally, for any $0 \leq a < b < \infty$,

$$N(a, b, \omega) := \sup\{n : \tau_n \leq b\} - \inf\{n : \tau_n \geq a\}. \quad (2.15)$$

Lemma 2.6 *Let $p' \geq \max\{p, \frac{1}{\lambda}\}$ be arbitrary, the following estimate holds*

$$N(T, \omega) \leq \frac{2^{p'-1}}{\mu^{p'}} \left(T^{p'\lambda} + \|\omega\|_{p\text{-var}, [0, T]}^{p'} \right). \quad (2.16)$$

More generally,

$$N(a, b, \omega) \leq \frac{2^{p'-1}}{\mu^{p'}} \left[(b-a)^{p'\lambda} + \|\omega\|_{p\text{-var}, [a, b]}^{p'} \right]. \quad (2.17)$$

Proof We have for all $n \in \mathbb{N}^*$

$$\begin{aligned} n\mu^{p'} &= \sum_{i=0}^{n-1} \mu^{p'} = \sum_{i=0}^{n-1} \left[|\tau_{i+1} - \tau_i|^\lambda + \|\omega\|_{p\text{-var}, [\tau_i, \tau_{i+1}]}^{p'} \right]^{p'} \\ &\leq 2^{p'-1} \left[\sum_{i=0}^{n-1} |\tau_{i+1} - \tau_i|^{p'\lambda} + \left(\sum_{i=0}^{n-1} \|\omega\|_{p\text{-var}, [\tau_i, \tau_{i+1}]}^p \right)^{p'/p} \right] \\ &\leq 2^{p'-1} \left[(\tau_n - \tau_0)^{p'\lambda} + \left(\sum_{i=0}^{n-1} \|\omega\|_{p\text{-var}, [\tau_i, \tau_{i+1}]}^p \right)^{p'/p} \right] \\ &\leq 2^{p'-1} \left[\tau_n^{p'\lambda} + \|\omega\|_{p\text{-var}, [0, \tau_n]}^{p'} \right]. \end{aligned} \quad (2.18)$$

Consequently, we obtain

$$N(T, \omega) \leq \frac{2^{p'-1}}{\mu^{p'}} \left[T^{p'\lambda} + \|\omega\|_{p\text{-var}, [0, T]}^{p'} \right].$$

Similarly, (2.17) holds. $\square$

Remark 2.7 (i) Since the left-hand side of (2.18) tends to infinity as n goes to ∞ its right hand side can not be bounded. This implies that $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$.
(ii) We can construct a time sequence starts at $\tau_0 = t_0$, an arbitrary point in $\mathbb{R}$, and on $(-\infty, t_0]$ in a similar manner.

3 Existence and Uniqueness Theorem

In this section, we are working with the restriction of any trajectory ω in a given time interval $[0, T]$ by considering it as an element in $\widehat{C}^p([0, T], \mathbb{R}^m)$, for a certain $p \in (1, 2)$ (see Remark 2.5 for the relation between $\omega \in \widehat{C}^p(\mathbb{R}, \mathbb{R}^m)$ and its restrictions). Consider the Young differential equation in the integral form as:

$$x_t = x_0 + \int_0^t f(s, x_s) ds + \int_0^t g(s, x_s) d\omega_s, \quad t \in [0, T]. \quad (3.1)$$

We recall here a result in [21] on existence and uniqueness of solution of (3.1), which was proved using contraction mapping arguments with ω in a Besov-type space. In this paper we however would like to derive a proof in $\widehat{C}^p$ applying Schauder fixed point theorem and greedy sequence of times tool. First we need to formulate some assumptions on the coefficient functions f and g of (3.1).

(**H**₁) $g(t, x)$ is differentiable in x and there exist some constants $0 < \beta, \delta \leq 1$, a control function $h(s, t)$ defined on $\Delta[0, T]$ and for every $N \geq 0$ there exists $M_N > 0$ such that the following properties hold:

$$(H_g) : \begin{cases} (i) \text{ Lipschitz continuity} \\ |g(t, x) - g(t, y)| \leq L_g |x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall t \in [0, T], \\ (ii) \text{ Local Hölder continuity} \\ |\partial_{x_i} g(t, x) - \partial_{y_i} g(t, y)| \leq M_N |x - y|^\delta, \\ \quad \forall x, y \in \mathbb{R}^d, \quad |x|, |y| \leq N, \quad \forall t \in [0, T], \\ (iii) \text{ Generalized Hölder continuity in time} \\ |g(t, x) - g(s, x)| + |\partial_{x_i} g(t, x) - \partial_{x_i} g(s, x)| \leq h(s, t)^\beta \\ \quad \forall x \in \mathbb{R}^d, \quad \forall s, t \in [0, T], \quad s < t. \end{cases}$$

(**H**₂) There exists $a > 0$ and $b \in L^{\frac{1}{1-\alpha}}([0, T], \mathbb{R}^d)$, where $\frac{1}{2} \leq \alpha < 1$, and for every $N \geq 0$ there exists $L_N > 0$ such that the following properties hold:

$$(H_f) : \begin{cases} (i) \text{ Local Lipschitz continuity} \\ |f(t, x) - f(t, y)| \leq L_N |x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad |x|, |y| \leq N, \quad \forall t \in [0, T], \\ (ii) \text{ Boundedness} \\ |f(t, x)| \leq a|x| + b(t), \quad \forall x \in \mathbb{R}^d, \quad \forall t \in [0, T]. \end{cases}$$

(**H**₃) The parameters in **H**₁ and **H**₂ satisfy the inequalities $\delta > p - 1$, $\beta > 1 - \frac{1}{p}$, $\delta\alpha > 1 - \frac{1}{p}$.

We would like to study the existence and uniqueness of the solution of (3.1) under the given conditions that $x \in \widehat{C}^q([0, T], \mathbb{R}^d)$ with appropriate constant $q > 0$.

By the assumption $p \in (1, 2)$ and the condition **H**₃, $1 - \frac{1}{p} < \min \left\{ \beta, \delta\alpha, \frac{\delta}{p}, \frac{1}{2} \right\}$, we can choose consecutively constants q_0, q such that

$$1 - \frac{1}{p} < \frac{1}{q_0} < \min \left\{ \beta, \delta\alpha, \frac{\delta}{p}, \frac{1}{2} \right\}, \quad (3.2)$$

$$\frac{1}{q_0\delta} \leq \frac{1}{q} < \min \left\{ \alpha, \frac{1}{p} \right\}. \quad (3.3)$$

Then, we have

$$\frac{1}{p} + \frac{1}{q_0} > 1, \quad q_0\beta > 1, \quad q_0 \geq q_0\delta \geq q > p, \quad q\alpha > 1. \quad (3.4)$$

We now consider $x \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$ with some $[t_0, t_1] \subset [0, T]$. Define the mapping given by

$$\begin{aligned} F(x)_t &= x_{t_0} + I(x)_t + J(x)_t \\ &:= x_{t_0} + \int_{t_0}^t f(s, x_s) ds + \int_{t_0}^t g(s, x_s) d\omega_s, \quad \forall t \in [t_0, t_1]. \end{aligned} \quad (3.5)$$

Note that a fixed point of F is a solution of (3.1) on $[t_0, t_1]$ with the boundary condition $x(t_0) = x_{t_0}$ (the initial condition x_{t_0} of (3.1) is then not given).

Introduce the notations

$$M := \max \left\{ L_g, aT^{1-\alpha}, |g(0, 0)| + h(0, T)^\beta, \|b\|_{L^{\frac{1}{1-\alpha}}} \right\}, \quad (3.6)$$

$$M'_N := \max\{L_N, M_N, M\}, \quad \forall N > 0. \quad (3.7)$$

It can be seen from the above assumptions that $|g(t, x)| \leq |g(t, 0)| + L_g|x|$ and $|g(t, 0)| \leq |g(0, 0)| + h(0, T)^\beta$, hence

$$|g(t, x)| \leq |g(0, 0)| + h(0, T)^\beta + L_g|x| \leq M(1 + |x|). \quad (3.8)$$

For the next proposition we need the following auxiliary lemma.

Lemma 3.1 Assume that $H_1 - H_3$ are satisfied.

(i) If $x \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$ then $g(\cdot, x) \in \widehat{C}^{q_0}([t_0, t_1], \mathbb{R}^{d \times m})$ and

$$\|g(\cdot, x)\|_{q_0\text{-var}, [t_0, t_1]} \leq M(1 + \|x\|_{q\text{-var}, [t_0, t_1]}). \quad (3.9)$$

(ii) For all $s < t$ and for all $x_i \in \mathbb{R}^d$ such that $|x_i| \leq N$, $i = 1, 2, 3, 4$, then

$$\begin{aligned} |g(s, x_1) - g(s, x_3) - g(t, x_2) + g(t, x_4)| &\leq L_g|x_1 - x_2 - x_3 + x_4| + |x_2 - x_4|h(s, t)^\beta \\ &\quad + M_N|x_2 - x_4|(|x_1 - x_2|^\delta + |x_3 - x_4|^\delta). \end{aligned}$$

(iii) For any $x, y \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$ such that $x_{t_0} = y_{t_0}$ and $\|x\|_{\infty, [t_0, t_1]} \leq N$, $\|y\|_{\infty, [t_0, t_1]} \leq N$ we have

$$\begin{aligned} \|g(\cdot, x) - g(\cdot, y)\|_{q_0\text{-var}, [t_0, t_1]} &\leq M'_N \|x - y\|_{q\text{-var}, [t_0, t_1]} \\ &\quad \left(2 + \|x\|_{q\text{-var}, [t_0, t_1]}^\delta + \|y\|_{q\text{-var}, [t_0, t_1]}^\delta \right). \end{aligned} \quad (3.10)$$

Proof (i) For $s < t$ in $[t_0, t_1]$, we have

$$\begin{aligned} |g(t, x_t) - g(s, x_s)| &\leq |g(t, x_t) - g(t, x_s)| + |g(t, x_s) - g(s, x_s)| \\ &\leq L_g|x_t - x_s| + h(s, t)^\beta. \end{aligned}$$

Let $\Pi = (s_i)_{i=1}^{n+1}$ be an arbitrary finite partition of $[t_0, t_1]$, $s_1 = t_0$, $s_{n+1} = t_1$. Since $q_0 \geq q$ and $q_0\beta > 1$ we have

$$\begin{aligned} \left(\sum_{i=1}^n |g(s_{i+1}, x_{s_{i+1}}) - g(s_i, x_{s_i})|^{q_0} \right)^{1/q_0} &\leq L_g \left(\sum_{i=1}^n |x_{s_{i+1}} - x_{s_i}|^{q_0} \right)^{1/q_0} \\ &\quad + \left(\sum_{i=1}^n h(s_i, s_{i+1})^{q_0\beta} \right)^{1/q_0} \\ &\leq L_g \|x\|_{q_0\text{-var}, [t_0, t_1]} + h(t_0, t_1)^\beta \\ &\leq L_g \|x\|_{q\text{-var}, [t_0, t_1]} + h(0, T)^\beta \\ &\leq M(1 + \|x\|_{q\text{-var}, [t_0, t_1]}) < \infty. \end{aligned}$$

Take the supremum over the set of all finite partition Π we get $g(\cdot, x) \in \widehat{C}^{q_0}([t_0, t_1], \mathbb{R}^{d \times m})$ and

$$\|g(\cdot, x)\|_{q_0\text{-var}, [t_0, t_1]} \leq M(1 + \|x\|_{q\text{-var}, [t_0, t_1]}).$$

(ii) This part is similar to [21, Lemma 7.1] with our function $h(s, t)^\beta$ playing the role of $|t - s|^\beta$ in [21, Lemma 7.1].

(iii) Note that $q_0\beta > 1$ and $q_0\delta \geq q$ hence

$$\begin{aligned}
 & \|g(\cdot, x) - g(\cdot, y)\|_{q_0\text{-var}, [t_0, t_1]} \\
 &:= \left(\sup_{\Pi([t_0, t_1])} \sum_i |g(s_{i+1}, x_{s_{i+1}}) - g(s_{i+1}, y_{s_{i+1}}) - g(s_i, x_{s_i}) + g(s_i, y_{s_i})|^{q_0} \right)^{1/q_0} \\
 &\leq L_g \sup_{\Pi([t_0, t_1])} \left(\sum_i |x_{s_{i+1}} - y_{s_{i+1}} - x_{s_i} + y_{s_i}|^{q_0} \right)^{1/q_0} \\
 &\quad + \|x - y\|_{\infty, [t_0, t_1]} \sup_{\Pi([t_0, t_1])} \left(\sum_i h(s_i, s_{i+1})^{q_0\beta} \right)^{1/q_0} \\
 &\quad + M_N \|x - y\|_{\infty, [t_0, t_1]} \sup_{\Pi([t_0, t_1])} \left[\left(\sum_i |x_{s_{i+1}} - x_{s_i}|^{q_0\delta} \right)^{1/q_0} \right. \\
 &\quad \left. + \left(\sum_i |y_{s_{i+1}} - y_{s_i}|^{q_0\delta} \right)^{1/q_0} \right] \\
 &\leq L_g \|x - y\|_{q\text{-var}, [t_0, t_1]} + \|x - y\|_{\infty, [t_0, t_1]} \\
 &\quad \times \left[h(t_0, t_1)^\beta + M_N \left(\|x\|_{q\text{-var}, [t_0, t_1]}^\delta + \|y\|_{q\text{-var}, [t_0, t_1]}^\delta \right) \right] \\
 &\leq M'_N \|x - y\|_{q\text{-var}, [t_0, t_1]} \left(2 + \|x\|_{q\text{-var}, [t_0, t_1]}^\delta + \|y\|_{q\text{-var}, [t_0, t_1]}^\delta \right).
 \end{aligned}$$

The lemma is proved. $\square$

For a proof of our main theorem on existence and uniqueness of solutions of an Young differential equation, we need the following proposition.

Proposition 3.2 Assume that $H_1 - H_3$ are satisfied. Let $0 \leq t_0 < t_1 \leq T$ be arbitrary, q be chosen as above satisfying (3.3) and F be defined by (3.5). Then for any $x \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$ we have $F(x) \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$, thus

$$F : \widehat{C}^q([t_0, t_1], \mathbb{R}^d) \longrightarrow \widehat{C}^q([t_0, t_1], \mathbb{R}^d).$$

Moreover, the following statements hold

(i) The q -variation of $F(x)$ satisfies

$$\begin{aligned}
 \|F(x)\|_{q\text{-var}, [t_0, t_1]} &\leq M(K + 2) (1 + \|x\|_{q\text{-var}, [t_0, t_1]}) \\
 &\quad \times ((t_1 - t_0)^\alpha + \|\omega\|_{p\text{-var}, [t_0, t_1]}).
 \end{aligned} \tag{3.11}$$

(ii) Let $N \geq 0$ be arbitrary but fixed. Suppose that $x, y \in \widehat{C}^q([t_0, t_1], \mathbb{R}^d)$ be such that $\|x\|_{\infty, [t_0, t_1]} \leq N$, $\|y\|_{\infty, [t_0, t_1]} \leq N$ and $x_{t_0} = y_{t_0}$, then we have

$$\begin{aligned}
 & \|F(x) - F(y)\|_{q\text{-var}, [t_0, t_1]} \\
 &\leq \|x - y\|_{q\text{-var}, [t_0, t_1]} ((t_1 - t_0) + \|\omega\|_{p\text{-var}, [t_0, t_1]}) \\
 &\quad \times M'_N (K + 1) \left(2 + \|x\|_{q\text{-var}, [t_0, t_1]}^\delta + \|y\|_{q\text{-var}, [t_0, t_1]}^\delta \right).
 \end{aligned} \tag{3.12}$$

Proof (i) Since $\frac{1}{p} + \frac{1}{q_0} > 1$, by virtue of (3.9), the Young integral $\int_0^t g(s, x_s) d\omega_s$ exists for all $t \in [t_0, t_1]$. Using (2.8), (3.5) and (3.8) we get

$$\begin{aligned}\|J(x)\|_{q\text{-var}, [t_0, t_1]} &\leq \|\omega\|_{p\text{-var}, [t_0, t_1]} [|g(t_0, x_{t_0})| + (K+1) \|g(\cdot, x_\cdot)\|_{q_0\text{-var}, [t_0, t_1]}] \\ &\leq \|\omega\|_{p\text{-var}, [t_0, t_1]} [M(1 + |x_{t_0}|) + M(K+1)(1 + \|x\|_{q\text{-var}, [t_0, t_1]})] \\ &\leq \|\omega\|_{p\text{-var}, [t_0, t_1]} M [(K+2) + |x_{t_0}| + (K+1) \|x\|_{q\text{-var}, [t_0, t_1]}].\end{aligned}$$

Now, by Hölder inequality and the assumption $\mathbf{H}_2$ we have

$$\int_s^t |b(u)| du \leq \left(\int_s^t |b(u)|^{\frac{1}{1-\alpha}} du \right)^{1-\alpha} \left(\int_s^t 1 du \right)^\alpha \leq \|b\|_{L^{\frac{1}{1-\alpha}}} (t-s)^\alpha \leq M(t-s)^\alpha.$$

Therefore, for $s < t$ in $[t_0, t_1]$ using the assumption $\mathbf{H}_2$ we have

$$\begin{aligned}\left| \int_s^t f(u, x_u) du \right| &\leq a \|x\|_{\infty, [s, t]} (t-s) + \|b\|_{L^{\frac{1}{1-\alpha}}} (t-s)^\alpha \\ &\leq (t-s)^\alpha \left(aT^{1-\alpha} \|x\|_{\infty, [t_0, t_1]} + \|b\|_{L^{\frac{1}{1-\alpha}}} \right) \\ &\leq (t-s)^\alpha M (1 + |x_{t_0}| + \|x\|_{q\text{-var}, [t_0, t_1]}).\end{aligned}$$

This implies

$$\|I(x)\|_{q\text{-var}, [t_0, t_1]} = \left\| \int_{t_0}^\cdot f(u, x_u) du \right\|_{q\text{-var}, [t_0, t_1]} \leq M(t_1 - t_0)^\alpha (1 + |x_{t_0}| + \|x\|_{q\text{-var}, [t_0, t_1]})$$

by [8, Proposition 5.10(i), p. 83] and the fact that the function $(s, t) \rightarrow (t-s)^{q\alpha}$ defined on $\Delta[t_0, t_1]$ is a control function for $q\alpha > 1$. Since

$$\|F(x)\|_{q\text{-var}, [t_0, t_1]} \leq \|I(x)\|_{q\text{-var}, [t_0, t_1]} + \|J(x)\|_{q\text{-var}, [t_0, t_1]}$$

(3.11) holds.

(ii) By virtue of (2.8), (3.10) and the condition $x_{t_0} = y_{t_0}$, we have

$$\begin{aligned}\|J(x) - J(y)\|_{p\text{-var}, [t_0, t_1]} &\leq \|\omega\|_{p\text{-var}, [t_0, t_1]} [|g(t_0, x_{t_0}) - g(t_0, y_{t_0})| \\ &\quad + (K+1) \|g(\cdot, x_\cdot) - g(\cdot, y_\cdot)\|_{q_0\text{-var}, [t_0, t_1]}] \\ &\leq \|\omega\|_{p\text{-var}, [s, t]} (K+1) \|g(\cdot, x_\cdot) - g(\cdot, y_\cdot)\|_{q_0\text{-var}, [t_0, t_1]} \\ &\leq (K+1) M'_N \|\omega\|_{p\text{-var}, [t_0, t_1]} \|x - y\|_{q\text{-var}, [t_0, t_1]} \\ &\quad \times \left(2 + \|x\|_{q\text{-var}, [t_0, t_1]}^\delta + \|y\|_{q\text{-var}, [t_0, t_1]}^\delta \right).\end{aligned}$$

Similarly,

$$\begin{aligned}|[I(x)_t - I(y)_t] - [I(x)_s - I(y)_s]| &\leq \int_s^t |f(u, x_u) - f(u, y_u)| du \\ &\leq L_N \|x - y\|_{q\text{-var}, [t_0, t_1]} (t-s),\end{aligned}$$

hence

$$\|I(x) - I(y)\|_{q\text{-var}, [t_0, t_1]} \leq M'_N \|x - y\|_{q\text{-var}, [t_0, t_1]} (t_1 - t_0).$$

Inequality (3.12) is a direct consequence of these estimates for $I(x)$ and $J(x)$. $\square$

Before proving the existence and uniqueness theorem, we need the following lemma of Gronwall type.

Lemma 3.3 (Gronwall-type Lemma) *Let $1 \leq p \leq q$ be arbitrary and satisfy $\frac{1}{p} + \frac{1}{q} > 1$. Assume that $\omega \in \widehat{C}^p([0, T], \mathbb{R})$ and $y \in \widehat{C}^q([0, T], \mathbb{R}^d)$ satisfy*

$$|y_t - y_s| \leq A_{s,t}^{1/q} + a_1 \left| \int_s^t y_u du \right| + a_2 \left| \int_s^t y_u d\omega_u \right|, \quad \forall s, t \in [0, T], \quad s < t, \quad (3.13)$$

for some fixed control function A on $\Delta[0, T]$ and some constants $a_1, a_2 \geq 0$. Then there exists a constant C independent of T such that for every $s, t \in [0, T], s < t$,

$$\|y\|_{q\text{-var}, [s,t]} \leq (|y_s| + A_0) e^{C(|t-s|^p + \|\omega\|_{p\text{-var}, [s,t]}^p)}, \quad (3.14)$$

where $A_0 = A_{0,T}^{1/q}$.

Proof Put

$$c := \max\{a_1, a_2(K+1)\},$$

in which K is defined in (2.7). We have

$$\begin{aligned} |y_t - y_s| &\leq A_{s,t}^{1/q} + a_1 \|y\|_{\infty, [s,t]}(t-s) + a_2 \|\omega\|_{p\text{-var}, [s,t]} (\|y\|_{\infty, [s,t]} + K \|y\|_{q\text{-var}, [s,t]}) \\ &\leq A_{s,t}^{1/q} + \max\{a_1 \|y\|_{\infty, [s,t]}, a_2 (\|y\|_{\infty, [s,t]} + K \|y\|_{q\text{-var}, [s,t]})\} (t-s + \|\omega\|_{p\text{-var}, [s,t]}). \end{aligned}$$

Fix the interval $[s, t] \subset [0, T]$ and apply the above inequality for arbitrary subinterval $[u, v] \subset [s, t]$ we obtain

$$\begin{aligned} |y_v - y_u| &\leq A_{u,v}^{1/q} + \max\{a_1 \|y\|_{\infty, [s,t]}, a_2 (\|y\|_{\infty, [s,t]} + K \|y\|_{q\text{-var}, [s,t]})\} (v-u + \|\omega\|_{p\text{-var}, [u,v]}) \\ &\leq A_{u,v}^{1/q} + \max\{a_1, a_2(K+1)\} (|y_s| + \|y\|_{q\text{-var}, [s,t]}) (v-u + \|\omega\|_{p\text{-var}, [u,v]}) \\ &\leq A_{u,v}^{1/q} + c(|y_s| + \|y\|_{q\text{-var}, [s,t]}) (v-u + \|\omega\|_{p\text{-var}, [u,v]}). \end{aligned}$$

Therefore, by virtue of Lemma 2.3, we get

$$\begin{aligned} \|y\|_{q\text{-var}, [s,t]} &\leq A_{s,t}^{1/q} + c(|y_s| + \|y\|_{q\text{-var}, [s,t]}) (t-s + \|\omega\|_{p\text{-var}, [s,t]}) \\ &\leq A_0 + c(|y_s| + \|y\|_{q\text{-var}, [s,t]}) (t-s + \|\omega\|_{p\text{-var}, [s,t]}). \end{aligned} \quad (3.15)$$

Now we construct the time sequence t_i with parameter $\{1, \frac{1}{2c}\}$ according to Sect. 2.2, that is

$$(t_{i+1} - t_i + \|\omega\|_{p\text{-var}, [t_i, t_{i+1}]}) = \frac{1}{2c}.$$

Then, by (3.15) for all $s, t \in [t_i, t_{i+1}], s < t$, we have $\|y\|_{q\text{-var}, [s,t]} \leq A_0 + \frac{1}{2}(|y_s| + \|y\|_{q\text{-var}, [s,t]})$, which implies $\|y\|_{q\text{-var}, [u,v]} \leq 2A_0 + |y_u|$ for all $u, v \in [t_i, t_{i+1}], u < v$. Therefore, $|y_{t_{i+1}}| \leq \|y\|_{\infty, [s, t_{i+1}]} \leq 2(A_0 + |y_s|)$ for all $s \in [t_i, t_{i+1}]$. By induction we obtain for any $s \in [t_k, t_{k+1}], 0 \leq k \leq i, i \in \{0, \dots, N(T, \omega)\}$, where $N(T, \omega)$ is defined by (2.14), the sequence of inequalities

$$2A_0 + |y_{t_{i+1}}| \leq 2(2A_0 + |y_{t_i}|) \leq \dots \leq 2^{i-k}(2A_0 + |y_{t_{k+1}}|) \leq 2^{i-k+1}(2A_0 + |y_s|).$$

Hence,

$$\|y\|_{q\text{-var}, [t_i, t_{i+1}]} \leq 2A_0 + |y_{t_i}| \leq 2^{i-k}(2A_0 + |y_s|), \quad \forall s \in [t_k, t_{k+1}], \quad 0 \leq k \leq i. \quad (3.16)$$

Now, we estimate the q -var norm of y in an arbitrary but fixed interval $[s, t] \subset [0, T]$. Recall the time sequence defined in (2.11). If there exists i such that $s < t_i < t$, put

$$\overline{N} := \sup\{n : t_n \leq t\}, \quad \underline{N} := \inf\{n : t_n \geq s\}, \quad N := \overline{N} - \underline{N} = N(s, t, \omega),$$

where $N(s, t, \omega)$ is defined in (2.15). We have $s \leq t_{\underline{N}} < t_{\underline{N}+1} < \dots < t_{\overline{N}} \leq t$ and

$$\begin{aligned} \|y\|_{q\text{-var}, [s, t_{\underline{N}}]} &\leq 2A_0 + |y_s|, \\ \|y\|_{q\text{-var}, [t_{\underline{N}+i}, t_{\underline{N}+i+1}]} &\leq 2^{i+1}(2A_0 + |y_s|), \quad i = 0, \dots, N-1, \\ \|y\|_{q\text{-var}, [t_{\overline{N}}, t]} &\leq 2A_0 + |y_{t_{\overline{N}}}| \leq 2^N(2A_0 + |y_s|). \end{aligned}$$

By Lemma 2.1 we have

$$\begin{aligned} \|y\|_{q\text{-var}, [s, t]} &\leq (N+1)^{\frac{q-1}{q}} \left(\|y\|_{q\text{-var}, [s, t_{\underline{N}}]}^q + \sum_{i=1}^{N-1} \|y\|_{q\text{-var}, [t_i, t_{i+1}]}^q + \|y\|_{q\text{-var}, [t_{\overline{N}}, t]}^q \right)^{1/q} \\ &\leq (N+1)^{\frac{q-1}{q}} (2A_0 + |y_s|) \left(\sum_{j=0}^N 2^{jq} \right)^{1/q} \\ &\leq (N+1)(2A_0 + |y_s|)2^N. \end{aligned}$$

In the case $[s, t] \subset [t_i, t_{i+1}]$ with some $i \in \{0, 1, \dots, N(T, \omega)\}$, we already have

$$\|y\|_{q\text{-var}, [s, t]} \leq 2A_0 + |y_s|.$$

To sum up, for any $[s, t] \subset [0, T]$ we have the estimate

$$\|y\|_{q\text{-var}, [s, t]} \leq (2A_0 + |y_s|)2^{2N}.$$

Combining with (2.17), we conclude that

$$\begin{aligned} \|y\|_{q\text{-var}, [s, t]} &\leq (2A_0 + |y_s|)2^{4^p C^p (|t-s|^p + \|\omega\|_{p\text{-var}, [s, t]}^p)} \\ &\leq (2A_0 + |y_s|)e^{C(|t-s|^p + \|\omega\|_{p\text{-var}, [s, t]}^p)}, \end{aligned}$$

where $C = 4^p C^p \ln 2$. The proof is complete. $\square$

Remark 3.4 (i) Gronwall Lemma is an important tool in the theory of ordinary differential equations, and the theory of Young differential equations as well. Some versions of Gronwall-type lemma can be seen in [21] and [6].

(ii) The conclusion of Lemma 3.3 is still true if one replaces A_0 by $A_{s,t}^{1/q}$.

(iii) It can be seen from the proof that in the conditions of Lemma 3.3 we have

$$\|y\|_{\infty, [0, T]} \leq (2A_0 + |y_0|)2^{N(T, \omega)+1} \leq (2A_0 + |y_0|)2^{(4cT)^p + 1 + (4c\|\omega\|_{p\text{-var}, [0, T]})^p}.$$

Corollary 3.5 *If in Lemma 3.3 we replace the condition (3.13) by the condition*

$$\|y\|_{q\text{-var}, [s, t]} \leq A_{s,t}^{1/q} + a_1(|y_s| + \|y\|_{q\text{-var}, [s, t]})(t-s + \|\omega\|_{p\text{-var}, [s, t]}) \quad (3.17)$$

for all $s < t$ in $[0, T]$, a positive constant $a_1 > 0$ and $\omega \in \widehat{C}^p([0, T], \mathbb{R}^m)$. Then there exists a constant C independent of T such that for every $s < t$ in $[0, T]$

$$\|y\|_{q\text{-var}, [s, t]} \leq (|y_s| + A_{s,t}^{1/q})e^{C(|t-s|^p + \|\omega\|_{p\text{-var}, [s, t]}^p)}. \quad (3.18)$$

We are now at the position to state and prove the main theorem of this section.

Theorem 3.6 (Existence and uniqueness of global solution) *Consider the Young differential equation (3.1), starting from an arbitrary initial time $t_0 \in [0, T]$,*

$$x_t = x_{t_0} + \int_{t_0}^t f(s, x_s) ds + \int_{t_0}^t g(s, x_s) d\omega_s, \quad t \in [t_0, T], \quad x_{t_0} \in \mathbb{R}^d.$$

with T being an arbitrary fixed positive number and $x_0 \in \mathbb{R}^d$ being an arbitrary initial condition. Assume that the conditions H_1 – H_3 hold. Then, this equation has a unique solution x in the space $\widehat{C}^q([t_0, T], \mathbb{R}^d)$, where q is chosen as above satisfying (3.3). Moreover, the solution is in $\widehat{C}^{p'}([t_0, T], \mathbb{R}^d)$, where $p' = \max\{p, \frac{1}{\alpha}\}$.

Proof The proof proceeds in several steps.

Step 1: In this step we will show the *local existence and uniqueness* of solution. Set

$$\mu := \frac{1}{2M(K+2)}, \quad (3.19)$$

where M is defined in (3.6) and K is defined in (2.7). Let $s_0 \in [t_0, T]$ be arbitrary but fixed. We recall here the time sequence τ_n with the parameters α, μ , i.e

$$\tau_0 = 0, \quad |\tau_{i+1} - \tau_i|^\alpha + \|\omega\|_{p\text{-var}, [\tau_i, \tau_{i+1}]} = \mu.$$

Put $r_0 = \min\{n : \tau_n > s_0\}$ and define $s_1 = \min\{\tau_{r_0}, T\}$. Then,

$$|s_1 - s_0|^\alpha + \|\omega\|_{p\text{-var}, [s_0, s_1]} \leq \mu. \quad (3.20)$$

We will show that the Eq. (3.1) restricted to $[s_0, s_1]$ has a unique solution.

Existence of local solutions.

Recall the mapping F defined by the formula (3.5) with t_0, t_1 replaced by s_0, s_1 , respectively. By Proposition 3.2 and (3.19)–(3.20), for s_0, s_1 determined above we have $F : \widehat{C}^q([s_0, s_1], \mathbb{R}^d) \rightarrow \widehat{C}^q([s_0, s_1], \mathbb{R}^d)$ and

$$\|F(x)\|_{q\text{-var}, [s_0, s_1]} = |F(x)_{s_0}| + \|F(x)\|_{q\text{-var}, [s_0, s_1]} \leq |x_{s_0}| + \frac{1}{2} (1 + \|x\|_{q\text{-var}, [s_0, s_1]}).$$

We show furthermore that if $x \in \widehat{C}^q([s_0, s_1], \mathbb{R}^d)$ then $F(x) \in C^{(q-\epsilon)\text{-var}}([s_0, s_1], \mathbb{R}^d)$ with small enough ϵ . Indeed, since $q > p$, $q\alpha > 1$, we can choose $\epsilon > 0$ such that $q - \epsilon \geq p$ and $(q - \epsilon)\alpha \geq 1$. For all $s < t$ in $[s_0, s_1]$, using (3.11) we have

$$\begin{aligned} |F(x)_t - F(x)_s| &\leq \|F(x)\|_{q\text{-var}, [s, t]} \\ &\leq M(K+2) (1 + \|x\|_{q\text{-var}, [s_0, s_1]}) ((t-s)^\alpha + \|\omega\|_{p\text{-var}, [s, t]}) \\ &\leq M(K+2) (1 + \|x\|_{q\text{-var}, [s_0, s_1]}) \\ &\quad \times \left[\left((t-s)^{(q-\epsilon)\alpha} \right)^{\frac{1}{q-\epsilon}} + \left(\|\omega\|_{p\text{-var}, [s, t]}^{(q-\epsilon)} \right)^{\frac{1}{q-\epsilon}} \right], \end{aligned}$$

hence

$$\|F(x)\|_{(q-\epsilon)\text{-var}, [s_0, s_1]} \leq M(K+2) (1 + \|x\|_{q\text{-var}, [s_0, s_1]}) ((s_1 - s_0)^\alpha + \|\omega\|_{p\text{-var}, [s_0, s_1]})$$

and the assertion follows by an application of Lemma 2.3. Consider the set

$$B_1 := \left\{ x \in \widehat{C}^q([s_0, s_1], \mathbb{R}^d) \mid x(s_0) = x_{s_0}, \|x\|_{q\text{-var}, [s_0, s_1]} \leq 2|x_{s_0}| + 1 \right\}.$$

Taking into account (3.12), $F : B_1 \rightarrow B_1$ is continuous. We show that B_1 is a closed convex set in the Banach space $\widehat{C}^q([s_0, s_1], \mathbb{R}^d)$, and F is a compact operator on B_1 . Indeed, for the former observation, note that if $z = \lambda x + (1 - \lambda)y$ for some $x, y \in B_1, \lambda \in [0, 1]$ then

$$z_{s_0} = \lambda x_{s_0} + (1 - \lambda)y_{s_0} = \lambda x_{s_0} + (1 - \lambda)x_{s_0} = x_{s_0}$$

and

$$\begin{aligned} \|z\|_{q\text{-var}, [s_0, s_1]} &= \|\lambda x + (1 - \lambda)y\|_{q\text{-var}, [s_0, s_1]} \\ &\leq \lambda \|x\|_{q\text{-var}, [s_0, s_1]} + (1 - \lambda)\|y\|_{q\text{-var}, [s_0, s_1]} \leq 2|x_{s_0}| + 1. \end{aligned}$$

Now, we prove that for any sequence $y^n \in F(B_1)$, there exists an subsequence converges in p -var norm to an element $y \in B_1$, i.e. $F(B_1)$ is relatively compact in B_1 . To do that, we will show that (y^n) are equicontinuous, bounded in $(q - \epsilon)$ -var norm. Namely, take the sequence $y^n = F(x^n) \in F(S), x^n \in B_1$. Then, by virtue of Lemma 2.3 we have

$$\sup_n \|y^n\|_{(q-\epsilon)\text{-var}, [s_0, s_1]} \leq |x_{s_0}| + 2M(K + 2)(1 + |x_{s_0}|)((s_1 - s_0)^\alpha + \|\omega\|_{p\text{-var}, [s_0, s_1]}).$$

It means that y^n are bounded in $C([s_0, s_1], \mathbb{R}^d)$ with sup norm, as well as bounded in $C^{(q-\epsilon)\text{-var}}([s_0, s_1], \mathbb{R}^d)$. Moreover, for all $n, s_0 \leq s \leq t \leq s_1$,

$$|y_t^n - y_s^n| \leq 2M(K + 2)(1 + |x_{s_0}|)((t - s)^\alpha + \|\omega\|_{p\text{-var}, [s, t]}),$$

which implies that (y^n) is equicontinuous. Applying Proposition 5.28 of [8], we conclude that y^n converges to some y along a subsequence in $\widehat{C}^q([s_0, s_1], \mathbb{R}^d)$. This proves the compactness of $\overline{F(B_1)}$. Hence, $F(B_1)$ is a relative compact set in $\widehat{C}^q([s_0, s_1], \mathbb{R}^d)$. We conclude that F is a compact operator from B_1 into itself. Therefore, by the Schauder-Tychonoff fixed point theorem (see e.g [25, Theorem 2.A, p. 56]), there exists a function $\hat{x} \in B_1$ such that $F(\hat{x}) = \hat{x}$, thus there exists a solution $\hat{x} \in B_1$ of (3.1) on the interval $[s_0, s_1]$.

Uniqueness of local solutions.

Now, we assume that x, y are two solutions in $\widehat{C}^q([s_0, s_1], \mathbb{R}^d)$ of the Eq. (3.1) such that $x_{s_0} = y_{s_0}$. It follows that $F(x) = x$ and $F(y) = y$. Put

$$N_0 = \max\{\|x\|_{q\text{-var}, [s_0, s_1]}, \|y\|_{q\text{-var}, [s_0, s_1]}\},$$

and $z = x - y$, we have $z_{s_0} = 0$ and

$$\|x\|_{\infty, [s_0, s_1]}, \|y\|_{\infty, [s_0, s_1]} \leq N_0.$$

By virtue of Proposition 3.2(ii), we obtain

$$\begin{aligned} \|z\|_{q\text{-var}, [s, t]} &= \|x - y\|_{q\text{-var}, [s, t]} = \|F(x) - F(y)\|_{q\text{-var}, [s, t]} \\ &\leq M'_{N_0}(K + 1)(1 + 2N_0^\delta)(|z_s| + \|z\|_{q\text{-var}, [s, t]})(t - s + \|\omega\|_{p\text{-var}, [s, t]}). \end{aligned} \quad (3.21)$$

Applying Corollary 3.5 to the function z , since $z_{s_0} = 0$ we conclude that $\|z\|_{q\text{-var}, [s_0, s_1]} = 0$. That implies $z \equiv 0$ on $[s_0, s_1]$. The uniqueness of the local solution is proved.

Step 2: Next, by virtue of the additivity of the Riemann and Young integrals, the solution can be concatenated. Namely, let $0 < t_1 < t_2 < t_3 \leq T$. Let x_t be a solution of the Eq. (3.1) on $[t_1, t_2]$ and y_t be a solution of the Eq. (3.1) on $[t_2, t_3]$ with $y(t_2) = x(t_2)$. Define a continuous function $z(\cdot) : [t_1, t_3] \rightarrow \mathbb{R}^d$ by setting $z(t) = x(t)$ on $[t_1, t_2]$ and $z(t) = y(t)$ on $[t_2, t_3]$. Then $z(\cdot)$ is the solution of the Young differential equation (3.1) on $[t_1, t_3]$. Conversely, if

z_t is a solution on $[t_1, t_3]$ then its restrictions on $[t_1, t_2]$ and on $[t_2, t_3]$ are solutions of the corresponding equation with the corresponding initial values.

Step 3: Finally, apply the estimates (2.17) to the case of μ being defined by (3.19), we can easily get the unique global solution to the Eq. (3.1) on $[t_0, T]$.

Put $n_0 = \min\{n : \tau_n > t_0\}$. The interval $[t_0, T]$ can be covered by $N(T, \omega) - n_0 + 1$ intervals $[t_i, t_{i+1}]$, $i = 0, N(T, \omega) - n_0 + 1$, determined by times $t_i = \tau_{n_0+i-1}$, $i = 1, \dots, N(T, \omega) - n_0$, with parameter μ being defined by (3.19) and $t_{N(T, \omega)+1} := T$. The arguments in **Step 1** are applicable to each of intervals $[t_i, t_{i+1}]$, $i = 0, N(T, \omega)$, implying the existence and uniqueness of solutions on those intervals. Then, starting at $x(t_0) = x_{t_0}$ the unique solution of (3.1) on $[t_0, t_1]$ is extended uniquely to $[t_1, t_2]$, then further by induction up to $[t_{N(T, \omega)-1}, t_{N(T, \omega)}]$ and lastly to $[t_{N(T, \omega)}, T]$. The solution x of (3.1) on $[t_0, T]$ then exists uniquely.

Furthermore, for all ϵ such that $q - \epsilon \geq p'$ the solution x belongs to $\widehat{C}^{q-\epsilon}([t_i, t_{i+1}], \mathbb{R}^d)$, for all $i = 0, N(T, \omega)$. Hence, $x \in \widehat{C}^{p'}([t_0, T], \mathbb{R}^d)$. $\square$

Proposition 3.7 Assume that the conditions $H_1 - H_3$ are satisfied. Let $0 \leq t_0 < T$. Denote by $x(\cdot) = x(t_0, \cdot, \omega, x_0)$ the solution of the Eq. (3.1) on $[t_0, T]$. Then there exist positive constants $C_1 = C_1(T)$, $C_2 = C_2(T)$ such that

$$\|x\|_{q\text{-var}, [t_0, T]} \leq C_1[1 + (T - t_0)^\alpha](1 + |x_0|)(1 + \|\omega\|_{p\text{-var}, [t_0, T]})e^{C_2\|\omega\|_{p\text{-var}, [t_0, T]}^{p'}}, \quad (3.22)$$

where $p' = \max\{p, \frac{1}{\alpha}\}$.

Proof Since x is a solution, $x = Fx$, hence by (3.11) we have

$$\begin{aligned} \|x\|_{q\text{-var}, [s, t]} &\leq M(K + 2) \left(1 + \|x\|_{q\text{-var}, [s, t]}\right) \left((t - s)^\alpha + \|\omega\|_{p\text{-var}, [s, t]}\right) \\ &\leq M(K + 2) \left((t - s)^\alpha + \|\omega\|_{p\text{-var}, [s, t]}\right) \\ &\quad + M(K + 2)(|x_s| + \|x\|_{q\text{-var}, [s, t]}) \left((t - s)^\alpha + \|\omega\|_{p\text{-var}, [s, t]}\right). \end{aligned}$$

Use the arguments similar to that of the proof of Lemma 3.3 we conclude that there exist $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that (3.22) is satisfied. $\square$

In order to study the flow generated by the solution of system (3.1) in the next section, we need also to consider the backward version of (3.1) in the following form

$$x_t = x_T + \int_t^T f(s, x_s)ds + \int_t^T g(s, x_s)d\omega_s, \quad t \in [0, T], \quad (3.23)$$

where $x_T \in \mathbb{R}^d$ is the initial value of the backward equation (3.23), the coefficient functions $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$ are continuous functions, and the driven force $\omega : [0, T] \rightarrow \mathbb{R}^m$ belongs to $\widehat{C}^p([0, T], \mathbb{R}^m)$.

Theorem 3.8 (Existence and uniqueness of solutions of backward equation) Consider the backward equation (3.23) on $[0, T]$. Assume that the conditions $H_1 - H_3$ hold. Then the backward equation (3.23) has a unique solution $x \in \widehat{C}^q([0, T], \mathbb{R}^d)$, where q is chosen as above satisfying (3.3).

Proof We make a change of variables

$$\begin{aligned} \hat{f}(u, x) &:= f(T - u, x), \quad \hat{g}(u, x) := g(T - u, x), \quad \hat{\omega}(u) := \omega(T - u), \quad y_u := x_{T-u}, \\ u &\in [0, T]. \end{aligned}$$

Then $x_T = y_0$, and by putting $v = T - t$ and $u = T - s$ we have

$$\int_t^T f(s, x_s) ds = \int_t^T f(T - u, x_{T-u}) ds = - \int_{T-t}^0 \hat{f}(u, y_u) du = \int_0^v \hat{f}(u, y_u) du.$$

Furthermore, by virtue of the property (2.9) of the Young integral we have

$$\int_t^T g(s, x_s) d\omega_s = \int_t^T g(T - u, x_{T-u}) d\omega_{T-u} = \int_0^v \hat{g}(u, y_u) d\hat{\omega}_u.$$

Therefore, the backward equation (3.23) is equivalent to the forward equation

$$y_v = y_0 + \int_0^v \hat{f}(u, y_u) du + \int_0^v \hat{g}(u, y_u) d\hat{\omega}_u, \quad v \in [0, T], \quad (3.24)$$

where $y_0 = x_T \in \mathbb{R}^d$. Now, we verify the conditions of Theorem 3.6 for the forward equation (3.24). First note that if $\omega \in \widehat{C}^p([0, T], \mathbb{R}^m)$ then $\hat{\omega} \in \widehat{C}^p([0, T], \mathbb{R}^m)$. Furthermore, the condition (H_1) obviously holds for $\hat{g}$ and the condition (i) of (H_2) holds for $\hat{f}$. For the condition (ii) of (H_2) we note that if it holds for f then

$$|\hat{f}(v, x)| = |f(T - v, x)| \leq a|x| + b(T - v) = a|x| + \hat{b}(v), \quad v \in [0, T],$$

where $\hat{b}(v) = b(T - v) \in L^{\frac{1}{1-\alpha}}(0, T; \mathbb{R}^d)$ because $(H_2)(ii)$ is satisfied for f . Thus, $(H_2)(ii)$ is satisfied for $\hat{f}$. Consequently, Theorem 3.6 is applicable to the forward equation (3.24) implying that (3.24) has unique solution $y \in \widehat{C}^q([0, T], \mathbb{R}^d)$. Since (3.24) is equivalent to the backward equation (3.23) we have the theorem proved. $\square$

Theorem 3.9 Suppose that the assumptions of Theorem 3.6 are satisfied. Denote by $X(t_0, \cdot, \omega, x_0)$ the solution of (3.1) starting from x_0 at time t_0 , i.e. $X(t_0, t_0, \omega, x_0) = x_0$. Then the solution mapping

$$X : [0, T] \times [0, T] \times \widehat{C}^p([0, T], \mathbb{R}^m) \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \\ (s, t, \omega, z) \mapsto X(s, t, \omega, z),$$

is continuous.

Proof First observe that, fixing $(\omega, x_0) \in \widehat{C}^p([0, T], \mathbb{R}^m) \times \mathbb{R}^d$ and looking at forward and backward equations (3.1) and (3.23), we can extend the solution $X(t_0, \cdot, \omega, x_0)$ of (3.1), with the initial value x_0 at t_0 to the whole $[0, T]$. The proof is divided into several steps.

Step 1 (Continuity w.r.t x_0):

By Proposition 3.7, we can choose N_0 (depending on x_0, ω) such that

$$\|X(t_0, \cdot, \omega', x'_0)\|_{q\text{-var}, [0, T]} \leq N_0$$

for all $t_0 \in [0, T]$, $|x_0 - x'_0| \leq 1$, $\|\omega - \omega'\|_{p\text{-var}, [0, T]} \leq 1$. We use here, for short, notation $y = X(t_0, \cdot, \omega', x_0)$, $y' = X(t_0, \cdot, \omega', x'_0)$. Using arguments similar to that of the proof of Proposition 3.2(ii), we have

$$\begin{aligned} & |(y - y')_t - (y - y')_s| \\ & \leq \int_s^t |f(u, y_u) - f(u, y'_u)| du + \left| \int_s^t g(u, y_u) - g(u, y'_u) d\omega_u \right| \\ & \leq M'_{N_0}(t - s) \|y - y'\|_{\infty, [s, t]} \\ & \quad + M'_{N_0}(K + 1) \|\omega'\|_{p\text{-var}, [s, t]} \left(|y_s - y'_s| + \|y - y'\|_{q\text{-var}, [s, t]} \right) (2 + 2N_0^\delta) \\ & \leq M'_{N_0}(K + 1)(2 + 2N_0^\delta) \left(|y_s - y'_s| + \|y - y'\|_{q\text{-var}, [s, t]} \right) \left(t - s + \|\omega'\|_{p\text{-var}, [s, t]} \right). \end{aligned}$$

Due to Corollary 3.5, there exist constants C_3, C_4 depending on parameters of the equation (3.1) and N_0 , such that

$$\|y - y'\|_{q\text{-var},[0,T]} \leq |y_0 - y'_0| C_3 e^{C_4 \|\omega'\|_{p\text{-var},[0,T]}^p} \leq |y_0 - y'_0| C_3 e^{C_4 (1 + \|\omega\|_{p\text{-var},[0,T]})^p}.$$

Therefore,

$$|x_t - y_t| \leq |x_{t_0} - y_{t_0}| + \|x - y\|_{q\text{-var},[t_0,t]} \leq |x_0 - x'_0| \left(C_3 e^{C_4 (1 + \|\omega\|_{p\text{-var},[0,T]})^p} + 1 \right).$$

Consequently, we find a positive constants $C_1(T, \omega, x_0)$ such that for all $t_0, t \in [0, T]$, all ω' such that $\|\omega' - \omega\|_{p\text{-var},[0,T]} < 1$, we have

$$|X(t_0, t, \omega', x'_0) - X(t_0, t, \omega', x_0)| \leq C_1(T, \omega, x_0) |x_0 - x'_0|. \quad (3.25)$$

Step 2 (Continuity w.r.t. ω):

Let $\omega' \in \widehat{C}^p([0, T], \mathbb{R}^m)$ be such that $\|\omega' - \omega\|_{p\text{-var},[0,T]} \leq 1$. We use here, for short, notation $x = X(t_0, \cdot, \omega, x_0)$, $x' = X(t_0, \cdot, \omega', x_0)$. For all $s < t$ in $[0, T]$, we have

$$\begin{aligned} & |(x' - x)_t - (x' - x)_s| \\ &= \left| \int_s^t [f(u, x'_u) - f(u, x_u)] du + \int_s^t [g(u, x'_u) - g(u, x_u)] d\omega_u + \int_s^t g(u, x'_u) d(\omega' - \omega)_u \right| \\ &\leq L_{N_0}(t - s) \|x' - x\|_{\infty, [s,t]} + M(K + 1)(1 + \|x'\|_{q\text{-var}, [s,t]}) \|\omega' - \omega\|_{p\text{-var}, [s,t]} \\ &\quad + \|\omega'\|_{p\text{-var}, [s,t]} M'_{N_0}(K + 1) \left(|(x' - x)_s| + \|x' - x\|_{q\text{-var}, [s,t]} \right) \\ &\quad \times \left(2 + \|x'\|_{q\text{-var}, [0,T]}^\delta + \|x\|_{q\text{-var}, [0,T]}^\delta \right) \\ &\leq C_5 \|\omega' - \omega\|_{p\text{-var}, [s,t]} + C_6 \left(t - s + \|\omega'\|_{p\text{-var}, [s,t]} \right) \left(|(x' - x)_s| + \|x' - x\|_{q\text{-var}, [s,t]} \right) \\ &\leq C_5 \|\omega' - \omega\|_{p\text{-var}, [s,t]} + C_6 \left(t - s + \|\omega'\|_{p\text{-var}, [s,t]} \right) \left(|(x' - x)_s| + \|x' - x\|_{q\text{-var}, [s,t]} \right), \end{aligned}$$

where C_5, C_6 depend on N_0 . Consequently, by virtue of Lemma 2.3 we get

$$\begin{aligned} \|x' - x\|_{q\text{-var}, [s,t]} &\leq C_3 \|\omega' - \omega\|_{p\text{-var}, [s,t]} + C_4 \left(t - s + \|\omega\|_{p\text{-var}, [s,t]} \right) \\ &\quad \times \left(|(x' - x)_s| + \|x' - x\|_{q\text{-var}, [s,t]} \right). \end{aligned}$$

Now, since $x'_{t_0} - x_{t_0} = 0$, using Corollary 3.5 on $[t_0, t]$ (or $[t, t_0]$ and use backward equation if $t < t_0$) we find positive constant $C_2(T, \omega, x_0)$ such that

$$\|x' - x\|_{q\text{-var}, [t_0,t]} \leq C_2(T, \omega, x_0) \|\omega' - \omega\|_{p\text{-var}, [t_0,t]} \leq C_2(T, \omega, x_0) \|\omega' - \omega\|_{p\text{-var}, [0,T]}.$$

Therefore, for all $t_0, t \in [0, T]$,

$$|X(t_0, t, \omega', x_0) - X(t_0, t, \omega, x_0)| \leq C_2(T, \omega, x_0) \|\omega' - \omega\|_{p\text{-var}, [0,T]}. \quad (3.26)$$

Step 3 (Continuity in all variables):

Now we fix (t_1, t_2, ω, x_0) and let $(t'_1, t'_2, \omega', x'_0)$ be in a neighborhood of (t_1, t_2, ω, x_0) such that

$$|t_1 - t'_1|, |t_2 - t'_2|, \|\omega - \omega'\|_{p\text{-var}, [0,T]}, |x_0 - x'_0| \leq 1.$$

By triangle inequality and (3.25), (3.26), we have

$$\begin{aligned} & |X(t'_1, t'_2, \omega', x'_0) - X(t_1, t_2, \omega, x_0)| \\ & \leq |X(t'_1, t'_2, \omega', x'_0) - X(t'_1, t'_2, \omega', x_0)| + |X(t'_1, t'_2, \omega', x_0) - X(t'_1, t'_2, \omega, x_0)| \\ & \quad + |X(t'_1, t'_2, \omega, x_0) - X(t_1, t'_2, \omega, x_0)| + |X(t_1, t'_2, \omega, x_0) - X(t_1, t_2, \omega, x_0)| \\ & \leq (C_1(T, \omega, x_0) + C_2(T, \omega, x_0))(|x'_0 - x_0| + \|\omega' - \omega\|_{p\text{-var}, [0, T]}) \\ & \quad + |X(t'_1, t'_2, \omega, x_0) - X(t'_1, t'_2, \omega, X(t_1, t'_1, \omega, x_0))| + \|X(t_1, \cdot, \omega, x_0)\|_{q\text{-var}, [t_2, t'_2]} \end{aligned}$$

It is obvious that when the triple $(|x'_0 - x_0|, \|\omega' - \omega\|_{p\text{-var}, [0, T]}, |t'_2 - t_2|)$ tends to 0 we have $(C_1(T, \omega, x_0) + C_2(T, \omega, x_0))(|x - x_0| + \|\omega' - \omega\|_{p\text{-var}, [0, T]}) \rightarrow 0$ and $\|X(t_1, \cdot, \omega, x_0)\|_{q\text{-var}, [t_2, t'_2]} \rightarrow 0$. As for the remaining term, let $|t'_1 - t_1|$ be small enough so that $|X(t_1, t'_1, \omega, x_0) - x_0| \leq 1$, using (3.25) again we obtain

$$\begin{aligned} & |X(t'_1, t'_2, \omega, X(t_1, t'_1, \omega, x_0)) - X(t'_1, t'_2, \omega, x_0)| \\ & \leq C_1(T, \omega, x_0)|X(t_1, t'_1, \omega, x_0) - x_0| \\ & \leq C_1(T, \omega, x_0) \|X(t_1, \cdot, \omega, x_0)\|_{q\text{-var}, [t_1, t'_1]}, \end{aligned}$$

hence $|X(t'_1, t'_2, \omega, X(t_1, t'_1, \omega, x_0)) - X(t'_1, t'_2, \omega, x_0)| \rightarrow 0$ as $|t'_1 - t_1| \rightarrow 0$. Summing up the above arguments, we conclude that X is continuous. $\square$

Remark 3.10 The time interval in Theorems 3.6 to 3.9 needs not be $[0, T]$. It can be $[t_0, t_0 + T]$ for any $t_0 \in \mathbb{R}$, $T > 0$.

4 Topological Flow Generated by Young Differential Equations

In this section we show that the solution of a nonautonomous Young differential equation generates a two-parameter flow on the phase space $\mathbb{R}^d$, thus we can study the long term behavior of the solution flow using the tools of the theory of dynamical systems. We also discuss the autonomous situation, in which we show that the solution then satisfies the cocycle property, thus generates a topological skew product flow. The reader is referred to the work [15] and [18], [17] for the smoothness and diffeomorphism property of the flow.

For simplicity of the presentation, we will assume from now on that for any given $T > 0$ all hypotheses $\mathbf{H}_1 - \mathbf{H}_3$ hold on $[0, T]$.

4.1 Topological Two-Parameter Flows for Nonautonomous Systems

Theorem 4.1 (Different trajectories do not intersect) *Assume that the conditions $\mathbf{H}_1 - \mathbf{H}_3$ hold. Let x_t and $\hat{x}_t$ be two solutions of the Young differential equation (3.1) on $[0, T]$. If $x_a = \hat{x}_a$ for some $a \in [0, T]$ then $x_t = \hat{x}_t$ for all $t \in [0, T]$. In other words, two solutions of the differential equation (3.1) either coincide or do not intersect.*

Proof Suppose that $x_a = \hat{x}_a$ for some $a \in [0, T]$. If $a = 0$ then by the uniqueness of the solution provided by Theorem 3.6, $x_t = \hat{x}_t$ for all $t \in [0, T]$. Let $a \in (0, T]$. Since the restrictions of the functions x_t and $\hat{x}_t$ on $[a, T]$ are solutions of the Eq. (3.1) starting from a , Theorem 3.6 implies that $x_t = \hat{x}_t$ for all $t \in [a, T]$. Now, consider the restrictions of the functions x_t and $\hat{x}_t$ on $[0, a]$. They are solutions of the equations

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\omega_s, \quad t \in [0, a],$$

with the initial values x_0 and $\hat{x}_0$ respectively. Since $x_a = \hat{x}_a$, the two functions x_t and $\hat{x}_t$ are solutions of the same backward equation

$$x_t = x_a - \int_t^a f(s, x_s)ds - \int_t^a g(s, x_s)d\omega_s, \quad t \in [0, a], \quad (4.1)$$

with the same initial value x_a . Theorem 3.8 asserts the uniqueness of solution of (4.1) on $[0, a]$, hence x_t must coincide with $\hat{x}_t$ on $[0, a]$ and the theorem is proved. $\square$

Remark 4.2 (Locality of Young differential equations) By virtue of Theorems 3.6, 3.8 and 4.1, under the assumptions of Theorem 3.6, the Eq. (3.1) has locality properties like ODE: we can solve it locally and extend the solution both forward and backward, and any two solutions meeting each other at some time should coincide in the common interval of definitions.

Now, in analog with the theory of ordinary differential equation we give a definition of the Cauchy operator of the Eq. (1.1), which is an operator in $\mathbb{R}^d$ acting along trajectories of (1.1).

Definition 4.3 (Cauchy operator) Suppose that on any compact interval of $\mathbb{R}$ the conditions $H_1 - H_3$ hold. For any $-\infty < t_1 \leq t_2 < +\infty$, any $\omega \in \tilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ the Cauchy operator $X(t_1, t_2, \omega, \cdot)$ of the Eq. (1.1) is defined as follows:

$$X(t_1, t_2, \omega, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$$

is the mapping along trajectories of (1.1) from time moment t_1 to time moment t_2 , i.e., for any vector $x_{t_1} \in \mathbb{R}^d$ we define $X(t_1, t_2, \omega, x_{t_1})$ to be the vector $x_{t_2} \in \mathbb{R}^d$ which is the value of the solution x of the equation

$$x_t = x_{t_1} + \int_{t_1}^t f(s, x_s)ds + \int_{t_1}^t g(s, x_s)d\omega_s, \quad t \in [t_1, t_2],$$

evaluated at time t_2 .

Theorem 4.4 Assume that the conditions $H_1 - H_3$ hold on any compact interval of $\mathbb{R}$. For any $-\infty < t_1 \leq t_2 < +\infty$ the Cauchy operator $X(t_1, t_2, \omega, \cdot)$ of (1.1) is a homeomorphism. Moreover, $X(t_1, t_1, \omega, \cdot) = id$.

Proof By Theorem 4.1 the Cauchy operator $X(t_1, t_2, \omega, \cdot)$ is an injection. Using arguments of the proof of Theorem 4.1 we get that the equation

$$x_t = x_{t_1} + \int_{t_1}^t f(s, x_s)ds + \int_{t_1}^t g(s, x_s)d\omega_s, \quad t \in [t_1, t_2], \quad (4.2)$$

with the terminal value $x_{t_2} \in \mathbb{R}^d$ and unknown initial value x_{t_1} , is equivalent to the following initial value problem for the backward equation on $[t_1, t_2]$

$$x_t = x_{t_2} - \int_t^{t_2} f(s, x_s)ds - \int_t^{t_2} g(s, x_s)d\omega_s, \quad t \in [t_1, t_2], \quad (4.3)$$

with initial value $x_{t_2} \in \mathbb{R}^d$, hence Theorem 3.8 is applicable and provides existence of solution for any terminal value x_{t_2} of the forward equation on $[t_1, t_2]$. Consequently, the Cauchy operator $X(t_1, t_2, \omega, \cdot)$ is a surjection, thus a bijection.

It is clear from the proof of Theorems 3.6 and 3.9 that the solutions of (1.1) depend continuously on the initial values. Therefore, the Cauchy operator $X(t_1, t_2, \omega, \cdot)$ acts continuously on $\mathbb{R}^d$. Similar conclusion holds for the inverse $X^{-1}(t_1, t_2, \omega, \cdot)$ by using backward equation. Hence $X(t_1, t_2, \omega, \cdot)$ is a homeomorphism and trivially $X(t_1, t_1, \omega, \cdot) = id$. $\square$

Following [13, p. 114], below we introduce the concept of two parameter flows.

Definition 4.5 (*Two-parameter flow*) A family of mappings $X_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ depending on two real variables $s, t \in [a, b] \subset \mathbb{R}$ is called a *two-parameter flow of homeomorphisms of $\mathbb{R}^d$ on $[a, b]$* if it satisfies the following conditions:

- (i) For any $s, t \in [a, b]$ the mapping $X_{s,t}$ is a homeomorphism of $\mathbb{R}^d$;
- (ii) $X_{s,s} = id$ for any $s \in [a, b]$;
- (iii) $X_{s,t}^{-1} = X_{t,s}$ for any $s, t \in [a, b]$;
- (iv) $X_{s,t} = X_{u,t} \circ X_{s,u}$ for any $s, t, u \in [a, b]$.

Theorem 4.6 (Two-parameter flow generated by Young differential equations) *Assume that the conditions $H_1 - H_3$ hold on any compact interval of $\mathbb{R}$. The family of Cauchy operators of (1.1) generates a two parameter flow of homeomorphisms of $\mathbb{R}^d$. Namely, for $-\infty < t_1 \leq t_2 < +\infty$ and $\omega \in \tilde{C}^p(\mathbb{R}, \mathbb{R}^m)$ we define $X(t_1, t_2, \omega, \cdot)$ according to Definition 4.3 and setting $X(t_2, t_1, \omega, \cdot) := X^{-1}(t_1, t_2, \omega, \cdot)$, then the family $X(t_1, t_2, \omega, \cdot)$, $t_1, t_2 \in [0, T]$, is a two parameter flow of homeomorphisms of $\mathbb{R}^d$ on $[0, T]$. Furthermore, the flow is continuous.*

Proof Conditions (i)–(ii) of Definition 4.5 follow from Theorem 4.4.

Condition (iii) of Definition 4.5 follows from the definition $X(t_2, t_1, \omega, \cdot) := X^{-1}(t_1, t_2, \omega, \cdot)$ for $t_1 \leq t_2$. Actually, it is seen from the proof of Theorem 4.4 that the inverse $X(t_2, t_1, \omega, \cdot)$ satisfies the backward equation (4.3).

Condition (iv) of Definition 4.5 follows from the definition of the Cauchy operators and Theorem 4.1.

The continuity of the flow follows directly from Theorem 3.9. $\square$

4.2 Topological Skew Product Flows for Autonomous Systems

In this subsection we restrict the discussion to the autonomous system

$$dx_t = f(x_t)dt + g(x_t)d\omega_t \quad (4.4)$$

where f, g are time independent and satisfy the assumptions $H_1 - H_3$. We consider ω in the space $\tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m) := \{\omega \in \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m), \omega(0) = 0\}$ and introduce the shift operator $\theta : \mathbb{R} \times \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m) \rightarrow \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m)$ by

$$\theta_t \omega(\cdot) := \omega(t + \cdot) - \omega(t).$$

It is easy to check that $\theta_{t+s}\omega = \theta_t \circ \theta_s \omega$ for all $t, s \in \mathbb{R}$ and $\omega \in \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m)$. Moreover, it is followed from [3, Theorem 5] that θ is continuous w.r.t. (t, ω) , thus $(\tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m), d, \theta)$ is a continuous dynamical system. On the other hand, it follows from definition of Young integral that

$$\int_{s+\tau}^{t+\tau} y_u d\omega(u) = \int_s^t y_{u+\tau} d\theta_\tau \omega(u), \quad \forall s, t, \tau \in \mathbb{R}$$

(see [5] for a version using fractional derivatives). Hence from the existence and uniqueness theorem, the solution $X(t, s, \omega, x_0)$ of the Young equation (4.4) satisfies

$$X(t, s, \omega, x_0) = X(t - s, 0, \theta_s \omega, x_0), \quad \forall t, s \in \mathbb{R},$$

therefore the mapping $\varphi : \mathbb{R} \times \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by $\varphi(t, \omega)x_0 := X(t, 0, \omega, x_0)$ possesses a cocycle property

$$\varphi(t + s, \omega)x_0 = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)x_0, \quad \forall x_0 \in \mathbb{R}^d, \omega \in \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m), t, s \in \mathbb{R}.$$

The problem of generation of the random dynamical systems [1] from stochastic differential equations driven by fractional Brownian noise has been discussed in [3,5,9], to name a few, where they solve the stochastic equation in the path-wise sense as in (4.4) for each realization ω of the fractional Brownian motion. Here in our deterministic setting, due to the fact that the shift dynamical system θ and the solution Cauchy operator are continuous, it follows that the skew product flow defined by

$$\begin{aligned}\Theta : \mathbb{R} \times \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}^d &\rightarrow \tilde{C}_0^p(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}^d \\ \Theta_t(\omega, x_0) &:= (\theta_t \omega, X(t, \omega, x_0))\end{aligned}\quad (4.5)$$

is a continuous mapping which satisfies the group property, i.e. $\Theta_{t+s} = \Theta_t \circ \Theta_s$, for all $t, s \in \mathbb{R}$. Therefore it is a topological skew-product dynamical system.

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