Persistence in Forward Nonautonomous Competitive Systems of Parabolic Equations

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Abstract In the present paper forward nonautonomous competitive systems of two parabolic second order partial differential equations are studied. The concept of forward uniform persistence for such systems is introduced. Sufficient conditions, expressed in terms of the principal spectrum, are given for those systems to be forward uniformly persistent.

Keywords Forward nonautonomous competitive system of parabolic equation · System of parabolic equations of Kolmogorov type · Principal spectrum · Forward uniform persistence

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1 Introduction

The present paper concerns the study of systems of nonlinear parabolic boundary value problems of Kolmogorov type

$$\begin{cases} u_{1t} = \Delta u_1 + f_1(t, x, u_1, u_2)u_1, & t > 0, x \in D, \\ u_{2t} = \Delta u_2 + f_2(t, x, u_1, u_2)u_2, & t > 0, x \in D, \\ \mathcal{B}_1 u_1 = 0, & t > 0, x \in \partial D, \\ \mathcal{B}_2 u_2 = 0, & t > 0, x \in \partial D, \end{cases} \quad (1.1)$$

Dedicated to Professor Russell Johnson on the occasion of his 60th birthday.

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where $\mathbf{f} = (f_1, f_2) : [0, \infty) \times \bar{D} \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$, and $\mathcal{B}_i$ is a boundary operator of either Dirichlet or Neumann type, i.e., $\mathcal{B}_i u_i = u_i$ or $\mathcal{B}_i u_i = \frac{\partial u_i}{\partial \mathbf{v}}$ ($i = 1, 2$) ($\mathcal{B}_1$ and $\mathcal{B}_2$ are not necessarily the same).

System (1.1) arises in many applied problems, including two species competition in biology. In population dynamics, important dynamical issues associated to (1.1) include determining whether the two species will survive in the long term, which is related to the so called persistence and/or coexistence, determining whether one of the species will be driven to extinction, etc.

We are interested in the issue of the uniform persistence of the system (1.1). Uniform persistence for autonomous and time periodic nonlinear systems of parabolic partial differential equations of second order has been well studied (see [1–6, 12, 16, 28], etc.). It has also been recently intensively studied for various nonautonomous nonlinear systems of parabolic equations (see [8], [15, 22, 24, 26, 27], etc.). We point out that in [8], Hetzer and Shen gave a study for time almost periodic cases. In [22], the authors of the current paper studied the persistence for general nonautonomous and random cases by applying the principal spectral theory developed in [22]. The authors of [15] provided various sufficient conditions for uniform persistence in quite general nonautonomous cases. In [24], uniform persistence for nonautonomous n -species competition system is studied.

Systems of differential equations arisen in applied problems usually depend on time and may only be defined for positive time. We call such systems which are only defined for positive time *forward nonautonomous*. Even when a system is defined for all time, only positive time may be relevant for the associated underline problem. It is therefore important to study persistence for forward nonautonomous systems of parabolic equations.

The objective of this paper is first to introduce a proper notion of uniform persistence (i.e., *forward uniform persistence*) for forward nonautonomous systems of parabolic equations in some natural state space $\mathbf{X}$ and then to provide sufficient conditions (in terms of the principal spectrum) for those systems to be uniformly persistent under some basic assumptions.

To be more precise, let us first introduce our basic assumptions. Throughout the paper we assume

- (A1) $D \subset \mathbb{R}^N$ is a bounded domain, the boundary ∂D of which is an $(N - 1)$ -dimensional manifold of class $C^{2+\alpha}$, for some $\alpha > 0$.
- (A2) The functions f_i ($i = 1, 2$) and their first order partial derivatives are bounded and Lipschitz continuous on $[0, \infty) \times \bar{D} \times [0, M] \times [0, M]$, for any $M > 0$.
- (A3) There are $P_1 > 0$, $P_2 > 0$ and continuous functions $m_1 : [P_1, \infty) \rightarrow (0, \infty)$ and $m_2 : [P_2, \infty) \rightarrow (0, \infty)$ such that $f_1(t, x, u_1, u_2) \leq -m_1(u_1)$ for any $t \geq 0$, $x \in \bar{D}$, and any $u_1 \geq P_1$, $u_2 \geq 0$ and $f_2(t, x, u_1, u_2) \leq -m_2(u_2)$ for any $t \geq 0$, $x \in \bar{D}$, and any $u_1 \geq 0$, $u_2 \geq P_2$.
- (A4) There is a continuous function $\tilde{m} : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ such that $\partial_{u_i} f_j(t, x, u_1, u_2) \leq -\tilde{m}(u_1, u_2)$ for $i, j = 1, 2$, all $t \geq 0$, $x \in \bar{D}$, and $u_1, u_2 \in [0, \infty)$.

In the following, i denotes 1 or 2, unless otherwise specified.

Let X_i be a fractional power space of the Laplacian operator Δ in $L_p(D)$ with the boundary condition $\mathcal{B}_i u = 0$ such that X_i is compactly imbedded into $C^1(\bar{D})$. We denote the norm in X_i by $\|\cdot\|_{X_i}$.

Denote $X_i^+ := \{u \in X_i : u(x) \geq 0 \text{ for all } x \in \bar{D}\}$. The interior X_i^{++} of X_i^+ is nonempty, and is characterized in the following way: In the case of Dirichlet boundary conditions, $X_i^{++} = \{u \in X_i^+ : u(x) > 0 \text{ for all } x \in D \text{ and } (\partial u / \partial \mathbf{v})(x) < 0 \text{ for all } x \in \partial D\}$, and in the case of Neumann boundary conditions, $X_i^{++} = \{u \in X_i^+ : u(x) > 0 \text{ for all } x \in \bar{D}\}$ (see [22, Lemma 7.1.8]).

As u_i is a nonnegative quantity (usually it is the population density of the i -th species), we are interested only in nonnegative solutions of (1.1). By the theory in [7], for each $t_0 \geq 0$ and each $\mathbf{u}_0 = (u_{10}, u_{20}) \in \mathbf{X}^+ := X_1^+ \times X_2^+$ there is a (classical) solution $\mathbf{u}(\cdot; t_0, \mathbf{u}_0) = (u_1(\cdot; t_0, (u_{10}, u_{20})), u_2(\cdot; t_0, (u_{10}, u_{20})))$, defined on $[t_0, \infty)$, of (1.1) with initial condition $\mathbf{u}(t_0; t_0, \mathbf{u}_0)(x) = \mathbf{u}_0(x)$.

Let now φ_{princ}^i denote a normalized nonnegative principal eigenfunction of the Laplace operator with the corresponding boundary operator $\mathcal{B}_i$. Observe that $\varphi_{\text{princ}}^i \in X_i^{++}$. We call (1.1) *forward uniformly persistent* if there is $\eta > 0$ such that for each initial value $\mathbf{u}_0 = (u_{10}, u_{20})$ with both u_{10} and u_{20} nonnegative and not identically equal to zero there exists $\tau > 0$ such that $u_1(t; t_0, \mathbf{u}_0)(x) \geq \eta \varphi_{\text{princ}}^1(x)$ and $u_2(t; t_0, \mathbf{u}_0)(x) \geq \eta \varphi_{\text{princ}}^2(x)$ for all $t_0 \geq 0$, $t \geq t_0 + \tau$ and $x \in \bar{D}$ (see Sect. 4 for detail).

The theory of principal spectrum for linear second order parabolic equations, as presented in the authors' monograph [22] and extended to the forward nonautonomous case in [23], will be the main tool to investigate the sufficient conditions for forward uniform persistence of (1.1). The reader is also referred to [9–11, 13, 17–21, 25] for the studies of principal spectrum of various nonautonomous parabolic equations. Roughly speaking, the *principal spectrum* of the problem

$$\begin{cases} u_t = \Delta u + c(t, x)u, & t > 0, x \in D, \\ \mathcal{B}u = 0, & t > 0, x \in \partial D, \end{cases} \quad (1.2)$$

where $c: [0, \infty) \times \bar{D} \rightarrow \mathbb{R}$ and $\mathcal{B} = \mathcal{B}_1$ or $\mathcal{B}_2$, equals the set of limits of logarithmic growth rates of its nonnegative solutions (see Sect. 2 for detail). The principal spectrum is a nonempty compact interval $[\lambda_{\min}(c; \mathcal{B}), \lambda_{\max}(c; \mathcal{B})]$.

Note that when $c(t, x)$ is independent of time t or is periodic in t , then $[\lambda_{\min}(c; \mathcal{B}), \lambda_{\max}(c; \mathcal{B})]$ is a singleton (i.e., $\lambda_{\min}(c; \mathcal{B}) = \lambda_{\max}(c; \mathcal{B})$) and $\lambda_{\text{princ}}(c; \mathcal{B}) := \lambda_{\min}(c; \mathcal{B})$ is the principal eigenvalue of the associated eigenvalue problem of (1.2). More generally, if $c(t, x)$ is asymptotic to a uniquely ergodic function $c^*(t, x)$ (see Definition 2.5), then applying the theory in [14] it is proved in [23] that $[\lambda_{\min}(c; \mathcal{B}), \lambda_{\max}(c; \mathcal{B})]$ is also a singleton (see Proposition 2.3).

Let $f_{0i}(t, x) := f_i(t, x, 0, 0)$. For given $u_{10} \in X_1^+ \setminus \{0\}$ and $u_{20} \in X_2^+ \setminus \{0\}$, let $f_{12}^{u_{10}}(t, x) := f_2(t, x, u_1(t; 0, (u_{10}, 0))(x), 0)$ and $f_{21}^{u_{20}}(t, x) := f_1(t, x, 0, u_2(t; 0, (0, u_{20}))(x))$.

In [23] it is proved that if $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$, then (1.1) in the absence of the species u_2 , i.e.,

$$\begin{cases} u_{1t} = \Delta u_1 + f_1(t, x, u_1, 0)u_1, & t > 0, x \in D, \\ \mathcal{B}_1 u_1 = 0, & t > 0, x \in \partial D, \end{cases} \quad (1.3)$$

is forward uniformly persistent, and if $\lambda_{\min}(f_{02}; \mathcal{B}_2) > 0$, then (1.1) in the absence of the species u_1 , i.e.,

$$\begin{cases} u_{2t} = \Delta u_2 + f_2(t, x, 0, u_2)u_2, & t > 0, x \in D, \\ \mathcal{B}_2 u_2 = 0, & t > 0, x \in \partial D, \end{cases} \quad (1.4)$$

is forward uniformly persistent (see Proposition 3.10 for detail).

In this paper, we further prove

- (1) If $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$, then $\lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)$ are independent of $u_{10} \in X_1^+ \setminus \{0\}$ (hence we may put $\lambda_{\min}^1 := \lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\max}^1 := \lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)$) (see Theorem 4.1).

- (2) If $\lambda_{\min}(f_{02}; \mathcal{B}_2) > 0$, then $\lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1)$ and $\lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)$ are independent of $u_{20} \in X^+ \setminus \{0\}$ (hence we may put $\lambda_{\min}^2 := \lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1)$ and $\lambda_{\max}^2 = \lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)$) (see Theorem 4.1).

We then have the following general result for the forward uniform persistence of (1.1).

- (3) Assume that $\lambda_{\min}(f_{0i}; \mathcal{B}_i) > 0$ for $i = 1, 2$ and $\lambda_{\min}^i > 0$ for $i = 1, 2$. Then (1.1) is forward uniformly persistent (see Theorem 4.2).

Moreover, we get the following easily checkable sufficient conditions for forward uniform persistence of (1.1).

- (4) If $\lambda_{\min}(\tilde{f}_1; \mathcal{B}_1) > 0$ and $\lambda_{\min}(\tilde{f}_2; \mathcal{B}_2) > 0$, where $\tilde{f}_1(t, x) := f_1(t, x, 0, P_2)$ and $\tilde{f}_2(t, x) := f_2(t, x, P_1, 0)$, then (1.1) is forward uniformly persistent (see Theorem 5.1).
- (5) If $\lambda_{\min}(\hat{f}_1, \{s_n, t_n\}; \mathcal{B}_1) > 0$ and $\lambda_{\min}(\hat{f}_2, \{s_n, t_n\}; \mathcal{B}_2) > 0$ for any $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ being such that the limits $\lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \tilde{f}_1(t, x) dt$ and $\lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \tilde{f}_2(t, x) dt$ exist, where $\hat{f}_1, \{s_n, t_n\}(x) := \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \tilde{f}_1(t, x) dt$ and $\hat{f}_2, \{s_n, t_n\}(x) := \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \tilde{f}_2(t, x) dt$, then (1.1) is forward uniformly persistent (see Theorem 5.2).

The rest of the paper is organized as follows. In Sect. 2, we collect the basic properties of the principal spectrum for forward nonautonomous parabolic equations to be used in proving the main results. In Sect. 3, we explore the fundamental properties of nonnegative solutions of (1.1), which will also be used in proving the main results. The main above results (1)–(3) are stated and proved in Sect. 4 (Theorems 4.1 and 4.2). The paper concludes with Sect. 5, in which the above sufficient conditions (4) and (5) for forward uniform persistence are proved (Theorems 5.1 and 5.2).

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2 Principal Spectrum of Forward Nonautonomous Parabolic Equations

In this section, we collect some basic properties of principal spectrum of forward nonautonomous parabolic equations for the later use. The reader is referred to [23] for detail.

First, we introduce some standing notations. For B_1, B_2 Banach spaces, $\mathcal{L}(B_1, B_2)$ denotes the Banach space of all bounded linear operators from B_1 into B_2 . Instead of $\mathcal{L}(B, B)$ we write $\mathcal{L}(B)$.

We denote the norm in $L_2(D)$ by $\|\cdot\|$, where $D \subset \mathbb{R}^N$ satisfies (A1). Denote $L_2(D)^+ := \{u \in L_2(D) : u(x) \geq 0 \text{ for Lebesgue-a.e. } x \in D\}$. For $u, v \in L_2(D)$ we write

$$\begin{aligned} u &\leq v && \text{if } v - u \in L_2(D)^+, \\ u &< v && \text{if } v - u \in L_2(D)^+ \setminus \{0\}. \end{aligned}$$

The symbols $\geq$ and $>$ are used in an analogous manner.

$\mathcal{B}$ will stand for the boundary operator either of the Dirichlet type

$$\mathcal{B}u = u \quad \text{on } \partial D,$$

or of the Neumann type

$$\mathcal{B}u = \frac{\partial u}{\partial \mathbf{v}} \quad \text{on } \partial D,$$

where $\mathbf{v}$ denotes the unit normal vector pointing out of D .

Consider now the following forward nonautonomous boundary value problem (see [23]),

$$\begin{cases} u_t = \Delta u + c(t, x)u, & t > 0, x \in D, \\ \mathcal{B}u = 0, & t > 0, x \in \partial D, \end{cases} \quad (2.1)$$

where $c: [0, \infty) \times \bar{D} \rightarrow \mathbb{R}$ is a bounded and Lipschitz continuous function.

For $s \geq 0$ and $u_0 \in L_2(D)$, let

$$[s, \infty) \ni t \mapsto U_c(t, s)u_0 \in L_2(D)$$

denote the weak solution $u(t, x)$ of (2.1) satisfying the initial condition $u(s, \cdot) = u_0$ (hence $u(t, x) = U_c(t, s)u_0$). For definition and properties of weak solutions the reader is referred to [23, Subsection 2.3].

Definition 2.1 (Principal resolvent) A real number λ belongs to the *principal resolvent* $\rho(c; \mathcal{B})$ of (2.1) or $\{U_c(t, s)\}_{t \geq s \geq 0}$, if either of the following conditions holds:

- There are $\eta > 0$, $M \geq 1$, and $T > 0$ such that

$$\|U_c(t, s)\| \leq Me^{(\lambda - \eta)(t - s)} \quad \text{for } t > s \geq T$$

(in this case, we say that such λ belongs to the *upper principal resolvent* $\rho_+(c; \mathcal{B})$),

- There are $\eta > 0$, $M \in (0, 1]$, and $T > 0$ such that

$$\|U_c(t, s)\| \geq Me^{(\lambda + \eta)(t - s)} \quad \text{for } t > s \geq T$$

(in this case, we say that such λ belongs to the *lower principal resolvent* $\rho_-(c; \mathcal{B})$).

Definition 2.2 (Principal spectrum) The *principal spectrum* $\Sigma(c; \mathcal{B})$ of (2.1) or $\{U_c(t, s)\}_{t \geq s \geq 0}$ equals the complement in $\mathbb{R}$ of the principal resolvent $\rho(c; \mathcal{B})$.

Proposition 2.1 Assume that c is a bounded and Lipschitz continuous real function on $[0, \infty) \times \bar{D}$. Then there are real numbers $\lambda_{\min}(c; \mathcal{B}) \leq \lambda_{\max}(c; \mathcal{B})$ such that $[\lambda_{\min}(c; \mathcal{B}), \lambda_{\max}(c; \mathcal{B})] = \Sigma(c; \mathcal{B})$ and for any nonzero $u_0 \in L_2(D)^+$ there holds

$$\begin{aligned} \lambda_{\min}(c; \mathcal{B}) &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_c(t, s)u_0\|}{t-s} = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_c(t, s)\|}{t-s} \\ &\leq \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_c(t, s)\|}{t-s} = \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_c(t, s)u_0\|}{t-s} = \lambda_{\max}(c; \mathcal{B}). \end{aligned}$$

Proof The proposition follows from the arguments similar to those in [23, Theorem 4.7]. $\square$

The principal spectrum of (2.1) therefore describes the logarithmic growth rates of its nonnegative solutions as $t \rightarrow \infty$. It then can be characterized by the limiting equations of (2.1) as $t \rightarrow \infty$.

To see this in more detail, extend the function c to a function $\bar{c}: \mathbb{R} \times \bar{D} \rightarrow \mathbb{R}$ by putting $\bar{c}(t, x) := c(0, x)$ for $t < 0$, $x \in \bar{D}$. For any function $\bar{c}: \mathbb{R} \times \bar{D} \rightarrow \mathbb{R}$ and any $t \in \mathbb{R}$ we write $\bar{c} \cdot t(\tau, x) := \bar{c}(\tau + t, x)$, $\tau \in \mathbb{R}$, $x \in \bar{D}$. Put

$$\mathbf{C}_0(c) := \bigcap_{s \geq 0} \text{cl} \{ \bar{c} \cdot t : t \in [s, \infty) \}, \quad (2.2)$$

where the closure is taken in the open-compact topology. Note that $\mathbf{C}_0(c)$ is nonempty, compact (with respect to the open-compact topology), connected and invariant (with respect to time translation).

For given $\tilde{c} \in \mathbf{C}_0(c)$ and $s \in \mathbb{R}$, consider

$$\begin{cases} u_t = \Delta u + \tilde{c}(t, x)u, & t > s, x \in D, \\ \mathcal{B}u = 0, & t > s, x \in \partial D. \end{cases} \quad (2.3)$$

For $s \in \mathbb{R}$ and $u_0 \in L_2(D)$, let

$$[[s, \infty) \ni t \mapsto U_{\tilde{c}}(t, s)u_0 \in L_2(D)]$$

denote the weak solution $u(t, x)$ of (2.3) satisfying the initial condition $u(s, \cdot) = u_0$ (see also [23, Subsection 2.3] for the properties of weak solutions of (2.3)).

Definition 2.3 $\lambda \in \mathbb{R}$ belongs to the *principal resolvent* $\tilde{\rho}(c; \mathcal{B})$ of (2.3) over $\mathbf{C}_0(c)$, if either of the following conditions is satisfied:

- there are $\eta > 0$ and $M \geq 1$ such that

$$\|U_{\tilde{c}}(t, 0)\| \leq Me^{(\lambda-\eta)t} \quad \text{for } t > 0, \quad \tilde{c} \in \mathbf{C}_0(c)$$

(in this case, we say that λ belongs to the *upper principal resolvent* $\tilde{\rho}_+(c; \mathcal{B})$ of (2.3) over $\mathbf{C}_0(c)$),

- there are $\eta > 0$ and $M \in (0, 1]$ such that

$$\|U_{\tilde{c}}(t, 0)\| \geq Me^{(\lambda+\eta)t} \quad \text{for } t > 0, \quad \tilde{c} \in \mathbf{C}_0(c)$$

(in this case, we say that λ belongs to the *lower principal resolvent* $\tilde{\rho}_-(c; \mathcal{B})$ of (2.3) over $\mathbf{C}_0(c)$).

Definition 2.4 The *principal spectrum* $\tilde{\Sigma}(c; \mathcal{B})$ of (2.3) over $\mathbf{C}_0(c)$ equals the complement in $\mathbb{R}$ of the principal resolvent of (2.3) over $\mathbf{C}_0(c)$.

We have

Proposition 2.2 (1) $\tilde{\Sigma}(c; \mathcal{B})$ is a compact interval $[\tilde{\lambda}_{\min}(c; \mathcal{B}), \tilde{\lambda}_{\max}(c; \mathcal{B})]$.
 (2) $\Sigma(c; \mathcal{B}) = \tilde{\Sigma}(c; \mathcal{B})$, that is, $\lambda_{\min}(c; \mathcal{B}) = \tilde{\lambda}_{\min}(c; \mathcal{B})$ and $\lambda_{\max}(c; \mathcal{B}) = \tilde{\lambda}_{\max}(c; \mathcal{B})$.

Proof (1) See [22, Theorem 3.1.1].

(2) This part follows from arguments similar to those in [23, Theorem 4.3]. $\square$

Definition 2.5 (1) A real function c which is bounded and Lipschitz continuous on $\mathbb{R} \times \bar{D}$ is said to be *uniquely ergodic* if the compact flow $(H(c), \mathbb{R})$ is uniquely ergodic, where $H(c) = \text{cl}\{c \cdot \tau : \tau \in \mathbb{R}\}$ with the open-compact topology (the closure is taken in the open-compact topology), and $(\tilde{c}, t) = \tilde{c} \cdot t$ for $\tilde{c} \in H(c)$ and $t \in \mathbb{R}$.

(2) A real function c which is bounded and Lipschitz continuous on $\mathbb{R} \times \bar{D}$ is said to be *asymptotic to a uniquely ergodic* $c^*(\cdot, \cdot)$ if $c(t, x) - c^*(t, x) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $x \in \bar{D}$.

Note that if $c^*(t, x)$ is periodic or almost periodic in t , then it is uniquely ergodic. If $c(t, x)$ is asymptotic to a uniquely ergodic function $c^*(t, x)$, then $\mathbf{C}_0(c) \subset H(c^*)$ and hence $(\mathbf{C}_0(c), \mathbb{R})$ is uniquely ergodic.

Proposition 2.3 If $c(t, x)$ is asymptotic to a uniquely ergodic function, then $\lambda_{\min}(c, \mathcal{B}) = \lambda_{\max}(c, \mathcal{B})$.

Proof See [23, Corollary 4.5]. $\square$

We remark that both (2.1) and (2.3) can be considered in other state spaces. For example, let X be a fractional power space of the Laplacian operator Δ in $L_p(D)$ with the boundary condition $\mathcal{B}u = 0$ such that X is compactly imbedded into $C^1(\bar{D})$. We denote the norm in X by $\|\cdot\|_X$.

3 Fundamental Properties

In this section, we establish the fundamental properties of nonnegative solutions of (1.1), i.e.,

$$\begin{cases} u_{1t} = \Delta u_1 + f_1(t, x, u_1, u_2)u_1, & x \in D, \\ u_{2t} = \Delta u_2 + f_2(t, x, u_1, u_2)u_2, & x \in D, \end{cases} \quad (3.1)$$

endowed with the boundary conditions

$$\begin{cases} \mathcal{B}_1 u_1 = 0, & x \in \partial D, \\ \mathcal{B}_2 u_2 = 0, & x \in \partial D, \end{cases} \quad (3.2)$$

where $\mathbf{f} = (f_1, f_2): [0, \infty) \times \bar{D} \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$. Throughout the present section we assume (A1)–(A4). We focus on the proof of the properties of nonnegative solutions of (3.1)+(3.2), which will be used in the investigation of the uniform persistence of (3.1) + (3.2) in later sections.

Observe that, to investigate the uniform persistence of (3.1) + (3.2), in addition to (3.1) + (3.2), it is also important to study the limit equations of the positive time translations of (3.1) + (3.2). To this end, we extend the function $\mathbf{f}$ to a function $\bar{\mathbf{f}}: \mathbb{R} \times \bar{D} \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$ by putting $\bar{\mathbf{f}}(t, x, \mathbf{u}) := \mathbf{f}(0, x, \mathbf{u})$ for $t < 0$, $x \in \bar{D}$ and $\mathbf{u} \in [0, \infty) \times [0, \infty)$.

For any function $\mathbf{g}: \mathbb{R} \times \bar{D} \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$ and any $t \in \mathbb{R}$ we write $\mathbf{g} \cdot t(\tau, x, \mathbf{u}) := \mathbf{g}(\tau + t, x, \mathbf{u})$, $\tau \in \mathbb{R}$, $x \in \bar{D}$, $\mathbf{u} \in [0, \infty) \times [0, \infty)$.

Put

$$\mathbf{Z} := \text{cl} \{ \bar{\mathbf{f}} \cdot t : t \in \mathbb{R} \} \quad (3.3)$$

with the open-compact topology, where the closure is taken in the open-compact topology. It is a consequence of (A2) via the Ascoli–Arzelà theorem that the set $\mathbf{Z}$ is a compact metrizable space. Further, if $\mathbf{g} \in \mathbf{Z}$ and $t \in \mathbb{R}$ then $\mathbf{g} \cdot t := \boldsymbol{\zeta}_t \mathbf{g} \in \mathbf{Z}$. Hence $(\mathbf{Z}, \{\boldsymbol{\zeta}_t\}_{t \in \mathbb{R}})$ is a compact flow.

Put

$$\mathbf{Z}_0 := \bigcap_{s \geq 0} \text{cl} \{ \bar{\mathbf{f}} \cdot t : t \in [s, \infty) \}. \quad (3.4)$$

$\mathbf{Z}_0$, as the ω -limit set of a forward orbit in the compact flow $(\mathbf{Z}, \{\boldsymbol{\zeta}_t\}_{t \in \mathbb{R}})$, is nonempty, compact, connected and invariant.

Put

$$\tilde{\mathbf{Z}} := \{ \bar{\mathbf{f}} \cdot t : t \geq 0 \} \cup \mathbf{Z}_0. \quad (3.5)$$

The set $\tilde{\mathbf{Z}}$ is a closed, hence compact, subset of $\mathbf{Z}$. Further, it is *forward invariant*: for any $\mathbf{g} \in \tilde{\mathbf{Z}}$ and any $t \geq 0$ there holds $\mathbf{g} \cdot t \in \tilde{\mathbf{Z}}$.

We denote by $d(\cdot, \cdot)$ a metric on $\mathbf{Z}$ which induces the open-compact topology in $\mathbf{Z}$.

For any $\mathbf{g} = (g_1, g_2) \in \tilde{\mathbf{Z}}$, consider the following competitive Kolmogorov system of semilinear second order parabolic equations,

$$\begin{cases} \frac{\partial u_1}{\partial t} = \Delta u_1 + g_1(t, x, u_1, u_2)u_1, & t > 0, x \in D, \\ \frac{\partial u_2}{\partial t} = \Delta u_2 + g_2(t, x, u_1, u_2)u_2, & t > 0, x \in D, \\ \mathcal{B}_1 u_1 = 0, & t > 0, x \in \partial D, \\ \mathcal{B}_2 u_2 = 0, & t > 0, x \in \partial D. \end{cases} \quad (3.6)$$

To emphasize the dependence of an equation on a parameter $\mathbf{g}$ (or another) we will sometimes write (3.6) (or similar ones) as (3.6) $_{\mathbf{g}}$.

Let $\mathbf{X} := X_1 \times X_2$. We put $\mathbf{X}^+ := X_1^+ \times X_2^+$, and $\mathbf{X}^{++} := X_1^{++} \times X_2^{++}$. Observe that $\mathbf{X}^{++}$ is the interior of $\mathbf{X}^+$.

For $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbf{X}$ we write

$$\begin{aligned} \mathbf{u} \leq_2 \mathbf{v} & \quad \text{if } (v_1 - u_1, u_2 - v_2) \in \mathbf{X}^+, \\ \mathbf{u} <_2 \mathbf{v} & \quad \text{if } (v_1 - u_1, u_2 - v_2) \in \mathbf{X}^+ \setminus \{(0, 0)\}, \\ \mathbf{u} \ll_2 \mathbf{v} & \quad \text{if } (v_1 - u_1, u_2 - v_2) \in \mathbf{X}^{++}. \end{aligned}$$

The symbols $\geq_2$, $>_2$ and $\gg_2$ are used in an analogous manner.

By the theory in [7], for each $t_0 \geq 0$ and each $\mathbf{u}_0 = (u_{10}, u_{20}) \in \mathbf{X}^+$ there is a (classical) solution $\mathbf{u}(\cdot; t_0, \mathbf{u}_0) = (u_1(\cdot; t_0, (u_{10}, u_{20})), u_2(\cdot; t_0, (u_{10}, u_{20})))$, defined on $[t_0, \infty)$, of (3.1) + (3.2) with initial condition $\mathbf{u}(t_0; t_0, \mathbf{u}_0)(x) = \mathbf{u}_0(x)$.

3.1 Nonnegative Solutions

In this subsection, we present some basic properties for nonnegative solutions of (3.6) $_{\mathbf{g}}$ for $\mathbf{g} \in \tilde{\mathbf{Z}}$.

First, by the theory in [7], the following holds:

Proposition 3.1 (Existence and uniqueness of a classical solution) *For each $\mathbf{u}_0 = (u_{10}, u_{20}) \in \mathbf{X}^+$ and each $\mathbf{g} = (g_1, g_2) \in \tilde{\mathbf{Z}}$ there exists a unique solution*

$$\mathbf{u}(\cdot; \mathbf{u}_0, \mathbf{g}) = (u_1(\cdot; (u_{10}, u_{20}), (g_1, g_2)), u_2(\cdot; (u_{10}, u_{20}), (g_1, g_2)))$$

of (3.6) satisfying the initial condition $\mathbf{u}(0, \cdot) = \mathbf{u}_0$, defined on $[0, \infty)$. Moreover, the solution is classical.

Note that for any $t_0 \geq 0$ and any $\mathbf{u}_0 \in \mathbf{X}^+$ there holds

$$\mathbf{u}(t + t_0; t_0, \mathbf{u}_0) = \mathbf{u}(t; \mathbf{u}_0, \tilde{\mathbf{f}} \cdot t_0) \quad \text{for all } t \geq 0.$$

Proposition 3.2 (Continuous dependence) *The mapping*

$$[[0, \infty) \times \mathbf{X}^+ \times \tilde{\mathbf{Z}} \ni (t, \mathbf{u}_0, \mathbf{g}) \mapsto \mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) \in \mathbf{X}^+]$$

is continuous.

In view of the above result, the family (3.6) $_{\mathbf{g}}$, $\mathbf{g} \in \tilde{\mathbf{Z}}$, generates a *topological skew-product semiflow* $\tilde{\Phi}$ on the product bundle $\mathbf{X}^+ \times \tilde{\mathbf{Z}}$, covering the translation semiflow on $\tilde{\mathbf{Z}}$:

$$\tilde{\Phi}(t; \mathbf{u}_0, \mathbf{g}) := (\mathbf{u}(t; \mathbf{u}_0, \mathbf{g}), \mathbf{g} \cdot t), \quad t \geq 0, \mathbf{u}_0 \in \mathbf{X}^+, \mathbf{g} \in \tilde{\mathbf{Z}}. \quad (3.7)$$

By Φ_0 we understand the topological skew-product semiflow on the product bundle $\mathbf{X}^+ \times \mathbf{Z}_0$, covering the translation flow on $\mathbf{Z}_0$:

$$\Phi_0(t; \mathbf{u}_0, \mathbf{g}) := (\mathbf{u}(t; \mathbf{u}_0, \mathbf{g}), \mathbf{g} \cdot t), \quad t \geq 0, \mathbf{u}_0 \in \mathbf{X}^+, \mathbf{g} \in \mathbf{Z}_0. \quad (3.8)$$

Proposition 3.3 (Differentiability) *The derivative $\partial_2 \mathbf{u}$ of the mapping*

$$[[0, \infty) \times \mathbf{X}^+ \times \tilde{\mathbf{Z}} \ni (t, \mathbf{u}_0, \mathbf{g}) \mapsto \mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) \in \mathbf{X}^+]$$

with respect to the $\mathbf{u}_0$ -variable exists and is continuous on the set $(0, \infty) \times \mathbf{X}^+ \times \tilde{\mathbf{Z}}$. For $\mathbf{u}_0 = (u_{10}, u_{20}) \in \mathbf{X}^+$ and $\mathbf{v}_0 = (v_{10}, v_{20}) \in \mathbf{X}$ the function $[t \mapsto \mathbf{v}(t)]$, where $\mathbf{v}(t) = (v_1(t), v_2(t)) := \partial_2 \mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) \mathbf{v}_0$, is a classical solution of the system of parabolic equations

$$\begin{cases} \frac{\partial v_1}{\partial t} = \Delta v_1 + \frac{\partial G_1}{\partial u_1}(t, x, \mathbf{u}(t; \mathbf{u}_0, \mathbf{g})(x))v_1 \\ \quad + \frac{\partial G_1}{\partial u_2}(t, x, \mathbf{u}(t; \mathbf{u}_0, \mathbf{g})(x))v_2, & t > 0, x \in D, \\ \frac{\partial v_2}{\partial t} = \Delta v_2 + \frac{\partial G_2}{\partial u_1}(t, x, \mathbf{u}(t; \mathbf{u}_0, \mathbf{g})(x))v_1 \\ \quad + \frac{\partial G_2}{\partial u_2}(t, x, \mathbf{u}(t; \mathbf{u}_0, \mathbf{g})(x))v_2, & t > 0, x \in D, \\ \mathcal{B}_1 v_1 = 0, & t > 0, x \in \partial D, \\ \mathcal{B}_2 v_2 = 0, & t > 0, x \in \partial D, \end{cases} \quad (3.9)$$

with initial conditions $v_1(0) = v_{10}$, $v_2(0) = v_{20}$, where $G_i(t, x, u_1, u_2) := g_i(t, x, u_1, u_2)u_i$, $i = 1, 2$.

Proposition 3.4 (Order preserving) *Let $\mathbf{g} \in \tilde{\mathbf{Z}}$.*

(1) *If $(u_1, u_2), (v_1, v_2) \in \mathbf{X}^+$, $(u_1, u_2) \leq_2 (v_1, v_2)$, then*

$$\mathbf{u}(t; (u_1, u_2), \mathbf{g}) \leq_2 \mathbf{u}(t; (v_1, v_2), \mathbf{g})$$

for each $t \geq 0$.

(2) *If $(u_1, u_2), (v_1, v_2) \in \mathbf{X}^+$, $(u_1, u_2) <_2 (v_1, v_2)$ and $v_1 > 0$, $u_2 > 0$, then*

$$\mathbf{u}(t; (u_1, u_2), \mathbf{g}) \ll_2 \mathbf{u}(t; (v_1, v_2), \mathbf{g})$$

for each $t > 0$.

Proposition 3.5 (Compactness) *For any $\delta_0 > 0$ and any $\mathbf{B} \subset \mathbf{X}^+$ bounded in the $C(\bar{D}) \times C(\bar{D})$ -norm, the set $\{\mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) : t \geq \delta_0, \mathbf{u}_0 \in \mathbf{B}, \mathbf{g} \in \tilde{\mathbf{Z}}\}$ has compact closure in the $\mathbf{X}$ -norm.*

Proposition 3.6 (Semitrivial solutions)

(1) *For any $\mathbf{u}_0 \in \mathbf{X}^+ \times \{0\}$, $\mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) \in \mathbf{X}^+ \times \{0\}$ for all $t \geq 0$ and $\mathbf{g} \in \tilde{\mathbf{Z}}$ (such $\mathbf{u}(t; \mathbf{u}_0, \mathbf{g})$ is called a semitrivial solution).*

(2) *For any $\mathbf{u}_0 \in \{0\} \times \mathbf{X}^+$, $\mathbf{u}(t; \mathbf{u}_0, \mathbf{g}) \in \{0\} \times \mathbf{X}^+$ for all $t \geq 0$ and $\mathbf{g} \in \tilde{\mathbf{Z}}$ (such $\mathbf{u}(t; \mathbf{u}_0, \mathbf{g})$ is also called a semitrivial solution).*

Observe that the first coordinate of a semitrivial solution $\mathbf{u}(\cdot; \mathbf{u}_0, \mathbf{g}) = (u_1(\cdot; (u_{10}, 0), (g_1, g_2)), 0)$ is the solution of the problem

$$\begin{cases} \frac{\partial u_1}{\partial t} = \Delta u_1 + g_1(t, x, u_1, 0)u_1, & t > 0, x \in D, \\ \mathcal{B}_1 u_1 = 0, & t > 0, x \in \partial D, \end{cases} \quad (3.10)$$

satisfying the initial condition $u_1(0) = u_{10}$. Similarly, the second coordinate of a semitrivial solution $\mathbf{u}(\cdot; \mathbf{u}_0, \mathbf{g}) = (0, u_2(\cdot; 0, u_{20}, (g_1, g_2)))$ is the solution of the problem

$$\begin{cases} \frac{\partial u_2}{\partial t} = \Delta u_2 + g_2(t, x, 0, u_2)u_2, & t > 0, x \in D, \\ \mathcal{B}_2 u_2 = 0, & t > 0, x \in \partial D, \end{cases} \quad (3.11)$$

satisfying the initial condition $u_2(0) = u_{20}$.

3.2 Linearization at Trivial Solutions

In this subsection, we consider the linearization of (3.1) + (3.2) at the trivial solution $(u_1, u_2) = (0, 0)$.

Consider the problem

$$\begin{cases} \frac{\partial v_i}{\partial t} = \Delta v_i + f_{0i}(t, x)v_i, & t > 0, x \in D, \\ \mathcal{B}_i v_i = 0, & t > 0, x \in \partial D, \end{cases} \quad (3.12)$$

where $f_{0i}(t, x) = f_i(t, x, 0, 0)$, $t \geq 0$, $x \in D$, $i = 1, 2$. We write (3.12) for f_{01} (resp. f_{02}) as (3.12)₁ (resp. (3.12)₂). Recall that $[\lambda_{\min}(f_{0i}, \mathcal{B}_i), \lambda_{\max}(f_{0i}, \mathcal{B}_i)]$ denotes the principal spectrum of (3.12)_i, $i = 1, 2$.

We define mappings $p_1: \tilde{\mathbf{Z}} \rightarrow C(\mathbb{R} \times \bar{D} \times [0, \infty))$ and $p_2: \tilde{\mathbf{Z}} \rightarrow C(\mathbb{R} \times \bar{D} \times [0, \infty))$ by

$$(p_1(\mathbf{g}))(t, x, u_1) := g_1(t, x, u_1, 0) \quad \text{for } t \in \mathbb{R}, x \in \bar{D} \text{ and } u_1 \geq 0$$

and

$$(p_2(\mathbf{g}))(t, x, u_2) := g_2(t, x, 0, u_2) \quad \text{for } t \in \mathbb{R}, x \in \bar{D} \text{ and } u_2 \geq 0,$$

where $\mathbf{g} = (g_1, g_2) \in \tilde{\mathbf{Z}}$.

We write $\mathbf{p}_1(\mathbf{g}) := (p_1(\mathbf{g}), 0)$ and $\mathbf{p}_2(\mathbf{g}) := (0, p_2(\mathbf{g}))$, $\mathbf{g} \in \tilde{\mathbf{Z}}$.

Further, we introduce a mapping $\mathbf{p}_0: \tilde{\mathbf{Z}} \rightarrow C(\mathbb{R} \times \bar{D})$ by

$$\mathbf{p}_0(\mathbf{g}) := \mathbf{g}_0,$$

where $\mathbf{g}_0(t, x) := \mathbf{g}(t, x, 0, 0)$ for $\mathbf{g} \in \tilde{\mathbf{Z}}$. Finally, we introduce $p_{01}: \tilde{\mathbf{Z}} \rightarrow C(\mathbb{R} \times \bar{D})$ by

$$p_{01}(\mathbf{g}) := g_{01},$$

where $g_{01}(t, x) := g_1(t, x, 0, 0)$ for $\mathbf{g} = (g_1, g_2) \in \tilde{\mathbf{Z}}$, and $p_{02}: \tilde{\mathbf{Z}} \rightarrow L_\infty(\mathbb{R} \times D, \mathbb{R})$ by

$$p_{02}(\mathbf{g}) := g_{02},$$

where $g_{02}(t, x) := g_2(t, x, 0, 0)$.

Denote by $\tilde{\mathbf{Y}}^0$, $\tilde{\mathbf{Y}}^{01}$, and $\tilde{\mathbf{Y}}^{02}$ the images of $\mathbf{p}_0$, p_{01} , and p_{02} , respectively.

It follows from Proposition 3.3 that for any $\mathbf{v}_0 = (v_{10}, v_{20}) \in \mathbf{X}$ and any $\mathbf{g} = (g_1, g_2) \in \tilde{\mathbf{Z}}$ the function $[(0, \infty) \ni t \mapsto \partial_2 \mathbf{u}(t; \mathbf{0}, \mathbf{g})\mathbf{v}_0 \in \mathbf{X}]$ is given by the classical solution $\mathbf{v}(\cdot) = (v_1(\cdot), v_2(\cdot))$ of

$$\begin{cases} \frac{\partial v_i}{\partial t} = \Delta v_i + g_{0i}(t, x)v_i, & t > 0, x \in D, \\ \mathcal{B}_i v_i = 0, & t > 0, x \in \partial D, \end{cases} \quad (3.13)$$

with initial condition $v_i(0) = v_{i0}$, $i = 1, 2$.

For $\mathbf{g}_0 = (g_{01}, g_{02}) = \mathbf{p}_0(\mathbf{g}) \in \mathbf{p}_0(\tilde{\mathbf{Z}})$ and $v_{i0} \in L_2(D)$ consider the weak solution

$$[(0, \infty) \ni t \mapsto U_{g_{0i}}^{0i}(t, 0)v_{i0} \in L_2(D)]$$

of (3.13) _{g_{0i}} , $i = 1, 2$ (for properties of weak solutions, see [23, Subsection 2.3]). We may write $U_{g_{01}}^{01}(t, 0)$ and $U_{g_{02}}^{02}(t, 0)$ for $U_{g_{01}}^{01}(t, 0)$ and $U_{g_{02}}^{02}(t, 0)$, respectively, if no confusion occurs. Let $[\lambda_{\min}(p_{0i}(\mathbf{Z}_0); \mathcal{B}_i), \lambda_{\max}(p_{0i}(\mathbf{Z}_0); \mathcal{B}_i)]$ be the principal spectral interval of (3.13) over $p_{0i}(\mathbf{Z}_0)$.

Proposition 3.7 Assume (A1)–(A4). Then

$$\lambda_{\min}(f_{0i}; \mathcal{B}_i) = \lambda_{\min}(p_{0i}(\mathbf{Z}_0); \mathcal{B}_i) \quad \text{and} \quad \lambda_{\max}(f_{0i}; \mathcal{B}_i) = \lambda_{\max}(p_{0i}(\mathbf{Z}_0); \mathcal{B}_i)$$

for $i = 1, 2$.

Proof The proposition follows from the fact that $\mathbf{C}_0(f_{0i}) = p_{0i}(\mathbf{Z}_0)$ (see (2.2) for the definition of $\mathbf{C}_0(\cdot)$) and Proposition 2.2. $\square$

3.3 Semitrivial Solutions

In this subsection, we explore some important properties of semitrivial solutions of (3.1) + (3.2).

First, we recall the notion of part metric. For given $u_0^1, u_0^2 \in X_1^{++}$, define

$$\varrho_1(u_0^1, u_0^2) := \inf \left\{ \ln \alpha : \alpha > 1, \frac{1}{\alpha} u_0^1 \leq u_0^2 \leq \alpha u_0^1 \right\}.$$

$\varrho_1(\cdot, \cdot)$ is called the *part metric* in X_1^{++} . Similarly, for given $u_0^1, u_0^2 \in X_2^{++}$, define

$$\varrho_2(u_0^1, u_0^2) := \inf \left\{ \ln \alpha : \alpha > 1, \frac{1}{\alpha} u_0^1 \leq u_0^2 \leq \alpha u_0^1 \right\}.$$

$\varrho_2(\cdot, \cdot)$ is called the *part metric* in X_2^{++} .

Proposition 3.8 (1) For any $\mathbf{g} \in \mathbf{Z}_0$ and $u_0^1, u_0^2 \in X_1^{++}$, $u_0^1 \neq u_0^2$, the function $\varrho_1(u_1(t; (u_0^1, 0), \mathbf{g}), u_1(t; (u_0^2, 0), \mathbf{g}))$ is decreasing in $t > 0$.
 (2) For any $\mathbf{g} \in \mathbf{Z}_0$ and $u_0^1, u_0^2 \in X_2^{++}$, $u_0^1 \neq u_0^2$, the function $\varrho_2(u_2(t; (0, u_0^1), \mathbf{g}), u_2(t; (0, u_0^2), \mathbf{g}))$ is decreasing in $t > 0$.

Proof The proposition follows from the arguments in [19, Lemma 3.2]. $\square$

Proposition 3.9 (1) Assume $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$. Then there exists a continuous function $\xi_1 : \mathbf{Z}_0 \rightarrow X_1^{++}$ such that

- (i) $\Gamma_1^0 := \{ ((\xi_1(\mathbf{g}), 0), \mathbf{g}) : \mathbf{g} \in \mathbf{Z}_0 \}$ is invariant,
- (ii) for any $u^* \in X_1^+ \setminus \{0\}$ there holds

$$\sup \{ \|\mathbf{u}(t; (u^*, 0), \mathbf{g}) - \mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g})\|_{\mathbf{X}} : \mathbf{g} \in \mathbf{Z}_0 \} \rightarrow 0 \quad (3.14)$$

as $t \rightarrow \infty$,

- (iii) there is $P_1^* \in (0, P_1)$ such that $\xi_1(\mathbf{g})(x) \leq P_1^*$ for all $\mathbf{g} \in \mathbf{Z}_0$ and $x \in \bar{D}$.

(2) Assume $\lambda_{\min}(f_{02}; \mathcal{B}_2) > 0$. Then there exists a continuous function $\xi_2 : \mathbf{Z}_0 \rightarrow X_2^{++}$ such that

- (i) $\Gamma_2^0 := \{ ((0, \xi_2(\mathbf{g})), \mathbf{g}) : \mathbf{g} \in \mathbf{Z}_0 \}$ is invariant,
- (ii) for any $u^* \in X_2^+ \setminus \{0\}$ there holds

$$\sup \{ \|\mathbf{u}(t; (0, u^*), \mathbf{g}) - \mathbf{u}(t; (0, \xi_2(\mathbf{g})), \mathbf{g})\|_{\mathbf{X}} : \mathbf{g} \in \mathbf{Z}_0 \} \rightarrow 0 \quad (3.15)$$

as $t \rightarrow \infty$,

- (iii) there is $P_2^* \in (0, P_2)$ such that $\xi_2(\mathbf{g})(x) \leq P_2^*$ for all $\mathbf{g} \in \mathbf{Z}_0$ and $x \in \bar{D}$.

The invariance of Γ_1^0 is understood in the following sense: For each $\mathbf{g} \in \mathbf{Z}_0$ and each $t > 0$ there holds $\mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g}) = (\xi_1(\mathbf{g} \cdot t), 0)$, and for each $\mathbf{g} \in \mathbf{Z}_0$ and each $t < 0$ there holds $\mathbf{u}(-t; (\xi_1(\mathbf{g} \cdot t), 0), \mathbf{g} \cdot t) = (\xi_1(\mathbf{g}), 0)$.

Consequently, for each $\mathbf{g} \in \mathbf{Z}_0$ the function $[\mathbb{R} \ni t \mapsto (\xi_1(\mathbf{g} \cdot t), 0) \in \mathbf{X}]$ is an entire solution of (3.6)_g. We will denote this solution by $\mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g})$.

Proof We prove only (1), the proof of statement (2) being similar.

(1)(i) and (ii) follow from the arguments in [22, Theorem 7.1.12].

In the proof of (1)(iii) we apply an approach similar to the proof of [22, Theorem 7.1.7]. Define a function $\tilde{g}_1: [0, \infty) \rightarrow \mathbb{R}$ by $\tilde{g}_1(w) := \sup \{g_1(t, x, w, 0) : g_1 \in p_1(\mathbf{Z}_0), t \in \mathbb{R}, x \in \bar{D}\}$. The function $\tilde{g}_1$ is locally Lipschitz continuous. The (unique) solution $w(\cdot)$ of the ODE $\dot{w} = \tilde{g}_1(w)w$ satisfying the initial condition $w(0) = P_1$ is a supersolution of (3.10)_{g₁}, for any $g_1 \in p_1(\mathbf{Z}_0)$. As there are $\theta > 0$ and $P_1^* \in (0, P_1)$ such that $w(t) \leq P_1^*$ for all $t \geq \theta$, it follows that $u_1(t; (u_0, 0), g_1, 0)(x) \leq P_1^*$ for all $t \geq \theta$, $g_1 \in p_1(\mathbf{Z}_0)$ and $x \in \bar{D}$. Consequently,

$$\xi_1(\mathbf{g})(x) \leq P_1^*$$

for all $\mathbf{g} \in \mathbf{Z}_0$ and $x \in \bar{D}$. □

Recall that $\mathbf{u}(t; t_0, \mathbf{u}_0) = (u_1(t; t_0, \mathbf{u}_0), u_2(t; t_0, \mathbf{u}_0))$ denotes the solution of (3.1) + (3.2) with $\mathbf{u}(t_0; t_0, \mathbf{u}_0) = \mathbf{u}_0$.

Proposition 3.10 (1) *If $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$, then there is $\eta_{01} > 0$ such that for any $u_{10} \in X_1^+ \setminus \{0\}$ there is $\tau(u_{10}) \geq 0$ with the property that*

$$u_1(t; t_0, (u_{10}, 0)) \geq \eta_{01} \varphi_{\text{princ}}^1$$

for all $t_0 \geq 0$ and all $t \geq \tau(u_{10}) + t_0$.

(2) *If $\lambda_{\min}(f_{02}; \mathcal{B}_2) > 0$, then there is $\eta_{02} > 0$ such that for any $u_{20} \in X_2^+ \setminus \{0\}$ there is $\tau(u_{20}) \geq 0$ with the property that*

$$u_2(t; t_0, (0, u_{20})) \geq \eta_{02} \varphi_{\text{princ}}^2$$

for all $t_0 \geq 0$ and all $t \geq \tau(u_{20}) + t_0$.

Proof It follows from [23, Theorem 6.1]. □

4 Main Results

In this section, we state and prove the main results of the paper. Throughout the present section, assume (A1)–(A4).

Let φ_{princ}^i denote the unique (nonnegative) principal eigenfunction of the elliptic boundary value problem

$$\begin{cases} \Delta u = \lambda u & \text{on } D, \\ \mathcal{B}_i u = 0 & \text{on } \partial D, \end{cases} \quad (4.1)$$

normalized so that $\sup \{ \varphi_{\text{princ}}^i(x) : x \in \bar{D} \} = 1$ ($i = 1, 2$). By the elliptic strong maximum principle and the Hopf boundary point principle, in the Dirichlet case $\varphi_{\text{princ}}^i(x) > 0$ for each $x \in D$ and $(\partial \varphi_{\text{princ}}^i / \partial \nu)(x) < 0$ for each $x \in \partial D$. In the Neumann case $\varphi_{\text{princ}}^i \equiv 1$.

Definition 4.1 (3.1) + (3.2) is said to be *forward uniformly persistent* if there is $\eta > 0$ such that for any $\mathbf{u}_0 \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$ there is $\tau(\mathbf{u}_0) \geq 0$ with the property that

$$u_i(t; t_0, \mathbf{u}_0) \geq \eta \varphi_{\text{princ}}^i$$

for $i = 1, 2$, all $t_0 \geq 0$ and $t \geq t_0 + \tau(\mathbf{u}_0)$.

Let

$$f_{0i}(t, x) := f_i(t, x, 0, 0).$$

As in Sect. 1, for given $u_{10} \in X_1^+ \setminus \{0\}$ and $u_{20} \in X_2^+ \setminus \{0\}$, let

$$f_{12}^{u_{10}}(t, x) := f_2(t, x, u_1(t; 0, (u_{10}, 0))(x), 0),$$

and

$$f_{21}^{u_{20}}(t, x) := f_1(t, x, 0, u_2(t; 0, (0, u_{20}))(x)).$$

Our first result reads as follows.

Theorem 4.1 (1) If $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$, then $\lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)$ are independent of $u_{10} \in X_1^+ \setminus \{0\}$.
 (2) If $\lambda_{\min}(f_{02}; \mathcal{B}_2) > 0$, then $\lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1)$ and $\lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)$ are independent of $u_{20} \in X_2^+ \setminus \{0\}$.

Assume

(A5) $\lambda_{\min}(f_{0i}; \mathcal{B}_i) > 0$ for $i = 1, 2$.

By Theorem 4.1, if we assume (A1)–(A5) then $\lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)$ are independent of $u_{10} \in X_1^+ \setminus \{0\}$ and $\lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1)$ and $\lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)$ are independent of $u_{20} \in X_2^+ \setminus \{0\}$. In such a case, we put $\lambda_{\min}^1 := \lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\max}^1 := \lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)$ and $\lambda_{\min}^2 := \lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1)$ and $\lambda_{\max}^2 := \lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)$. Our second result then reads as follows.

Theorem 4.2 Assume (A1)–(A5) and $\lambda_{\min}^i > 0$ for $i = 1, 2$. Then (3.1) + (3.2) is forward uniformly persistent.

Proof of Theorem 4.1 (1) First assume $\lambda_{\min}(f_{01}; \mathcal{B}_1) > 0$. Let ξ_1 be as in Proposition 3.9.

For $\mathbf{g} \in \mathbf{Z}_0$ denote

$$p_{12}(\mathbf{g})(t, x) := g_2(t, x, \xi_1(\mathbf{g} \cdot t)(x), 0), \quad t \in \mathbb{R}, x \in \bar{D}.$$

Let $Y_0^1 := p_{12}(\mathbf{Z}_0)$. The set Y_0^1 is considered endowed with the open-compact topology. As a continuous image of a compact set, it is compact, too. Denote a generic element of Y_0^1 by g_{12} .

For $g_{12} \in Y_0^1$ and $s \in \mathbb{R}$ we put $(g_{12} \cdot s)(t, x) := g_2(t + s, x, \xi_1(\mathbf{g} \cdot (t + s))(x), 0)$, $t \in \mathbb{R}$, $x \in \bar{D}$. The translation flow on Y_0^1 is a compact flow.

Consider the equation

$$\begin{cases} \frac{\partial v_2}{\partial t} = \Delta v_2 + g_{12}(t, x)v_2, & x \in D, \\ \mathcal{B}_2 v_2 = 0, & x \in \partial D, \end{cases} \quad (4.2)$$

where $g_{12} \in Y_0^1$. We write sometimes (4.2) as $(4.2)_{g_{12}}$.

For $\mathbf{g} \in \mathbf{Z}_0$ the equation $(4.2)_{p_{12}(\mathbf{g})}$ equals the second coordinate of the linearization (3.9) of (3.6) along the semitrivial solution $\mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g})$.

For $g_{12} \in Y_0^1$, $t \geq 0$ and $v_{20} \in L_2(D)$ denote by $U_{g_{12}}^{12}(t, 0)v_{20}$ the weak solution operator of (4.2). Put

$$\Pi_2^1(t; v_{20}, g_{12}) := (U_{g_{12}}^{12}(t, 0)v_{20}, g_{12} \cdot t), \quad t \geq 0, v_{20} \in L_2(D), g_{12} \in Y_0^1. \quad (4.3)$$

Denote by $[\lambda_{\min}^1, \lambda_{\max}^1]$ the principal spectrum of (4.2) over Y_0^1 .

We claim that $[\lambda_{\min}(f_1^{u_{10}}; \mathcal{B}_2), \lambda_{\max}(f_1^{u_{10}}; \mathcal{B}_2)] = [\lambda_{\min}^1, \lambda_{\max}^1]$ for any $u_{10} \in X_1^+ \setminus \{0\}$. To this end, it suffices to prove that $\mathbf{C}_0(f_{12}^{u_{10}}) = Y_0^1$ for any $u_{10} \in X_1^+ \setminus \{0\}$ (see (2.2) for the definition of $\mathbf{C}_0(\cdot)$).

For any given $u_{10} \in X^+ \setminus \{0\}$, take $\tilde{c} \in \mathbf{C}_0(f_{12}^{u_{10}})$. Then there is $t_n \rightarrow \infty$ such that $f_2(t + t_n, x, u_1(t + t_n; 0, \mathbf{u}_0)(x), 0) \rightarrow \tilde{c}(t, x)$ in the open-compact topology, where $\mathbf{u}_0 = (u_{10}, 0)$. Without loss of generality we may assume that $\tilde{\mathbf{f}} \cdot t_n \rightarrow \mathbf{g} = (g_1, g_2)$ as $n \rightarrow \infty$ in $\tilde{\mathbf{Z}}$ (that is, in the open-compact topology).

By applying the diagonal procedure, we can assume that for each $k = 0, 1, 2, 3, \dots$ the sequence $\mathbf{u}(t_n - k; 0, \mathbf{u}_0) = \mathbf{u}(t_n - k; \mathbf{u}_0, \tilde{\mathbf{f}})$ converges in $\mathbf{X}$, as $n \rightarrow \infty$, to some $\mathbf{u}^*(-k)$. Observe that for any $k \in \mathbb{N} \cup \{0\}$ there holds

$$\begin{aligned} \mathbf{u}(1; \mathbf{u}^*(-k-1), \mathbf{g} \cdot (-k-1)) &= \lim_{n \rightarrow \infty} \mathbf{u}(1; \mathbf{u}(t_n - k - 1; \mathbf{u}_0, \tilde{\mathbf{f}}), \tilde{\mathbf{f}} \cdot (t_n - k - 1)) \\ &= \lim_{n \rightarrow \infty} \mathbf{u}(t_n - k; \mathbf{u}_0, \tilde{\mathbf{f}}) = \mathbf{u}^*(-k). \end{aligned}$$

By putting $\mathbf{u}^*(t) := \mathbf{u}(t - \lfloor t \rfloor; \mathbf{u}^*(\lfloor t \rfloor), \mathbf{g} \cdot (\lfloor t \rfloor))$ for $t < 0$ and $\mathbf{u}^*(t) := \mathbf{u}(t; \mathbf{u}_0^*, \mathbf{g})$ for $t \geq 0$, we get an entire solution $\mathbf{u}^* = (u_1^*, 0): \mathbb{R} \rightarrow \mathbf{X}$ of (3.6)_g. It follows from Proposition 3.5 that the set $\{\mathbf{u}^*(t) : t \in \mathbb{R}\}$ has compact closure in $\mathbf{X}$.

Observe that $[\mathbb{R} \ni t \mapsto \mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g})]$ is also an entire solution of (3.6)_g, and its image has compact closure. We will prove that $\mathbf{u}^*(t) = \mathbf{u}(t; (\xi_1(\mathbf{g}), 0), \mathbf{g})$ for all $t \in \mathbb{R}$.

First, observe that there is $\eta_{01}^* > 0$ such that

$$u_1^*(t) \geq \eta_{01}^* \varphi_{\text{princ}}^1 \quad \text{and} \quad u_1(t; (\xi_1(\mathbf{g}), 0), \mathbf{g}) \geq \eta_{01}^* \varphi_{\text{princ}}^1 \quad (4.4)$$

for $t \in \mathbb{R}$. Indeed, the first inequality follows from the construction of $\mathbf{u}^*$ via Proposition 3.10(1), the other one is a consequence of Proposition 3.9(1).

By Proposition 3.8, $\varrho_1(u_1^*(t), u_1(t; (\xi_1(\mathbf{g}), 0), \mathbf{g}))$ is decreasing in $t \in \mathbb{R}$.

Suppose to the contrary that $u_1^*(\tau) \neq \xi_1(\mathbf{g} \cdot \tau)$ for some $\tau \in \mathbb{R}$. Take a sequence $s_n \rightarrow -\infty$ such that $\mathbf{u}^*(\tau + s_n) \rightarrow \mathbf{u}^{**}$ (in $\mathbf{X}$), $\mathbf{u}(s_n; (\xi_1(\mathbf{g} \cdot \tau), 0), \mathbf{g} \cdot \tau) \rightarrow \tilde{\mathbf{u}}$ (in $\mathbf{X}$), and $\mathbf{g} \cdot (\tau + s_n) \rightarrow \tilde{\mathbf{g}}$ (in $\mathbf{Z}_0$) as $n \rightarrow \infty$. Repeating the above construction we can obtain two entire semitrivial solutions $\mathbf{u}(t; \mathbf{u}^{**}, \tilde{\mathbf{g}}) = (u_1(t; \mathbf{u}^{**}, \tilde{\mathbf{g}}), 0)$ and $\mathbf{u}(t; \tilde{\mathbf{u}}, \tilde{\mathbf{g}}) = (u_1(t; \tilde{\mathbf{u}}, \tilde{\mathbf{g}}), 0)$ of (3.6)_g. They both have precompact images. Moreover,

$$u_1(t; \mathbf{u}^{**}, \tilde{\mathbf{g}}) \geq \eta_{01}^* \varphi_{\text{princ}}^1 \quad \text{and} \quad u_1(t; \tilde{\mathbf{u}}, \tilde{\mathbf{g}}) \geq \eta_{01}^* \varphi_{\text{princ}}^1 \quad (4.5)$$

for $t \in \mathbb{R}$. By the strict decrease of $\varrho_1(u_1^*(t), u_1(t; (\xi_1(\mathbf{g}), 0), \mathbf{g}))$ in t , which follows from Proposition 3.8, we have $\mathbf{u}^{**} \neq \tilde{\mathbf{u}}$ and $\varrho_1(u_1(t; \mathbf{u}^{**}, \tilde{\mathbf{g}}), u_1(t; \tilde{\mathbf{u}}, \tilde{\mathbf{g}}))$ is independent of t . On the other hand, by Proposition 3.8, $\varrho_1(u_1(t; \mathbf{u}^{**}, \tilde{\mathbf{g}}), u_1(t; \tilde{\mathbf{u}}, \tilde{\mathbf{g}}))$ is decreasing in t . This is a contradiction. Therefore $u_1^*(t) = \xi_1(\mathbf{g} \cdot t)$ for all $t \in \mathbb{R}$.

Take $\tilde{c} \in Y_0^1$. There is a sequence $t_n \rightarrow \infty$ such that $\tilde{\mathbf{f}} \cdot t_n \rightarrow \mathbf{g} = (g_1, g_2)$ in $\tilde{\mathbf{Z}}$ as $n \rightarrow \infty$, where $\tilde{c}(t, x) = g_2(t, x, \xi_1(\mathbf{g} \cdot t)(x), 0)$ for all $t \in \mathbb{R}$ and all $x \in D$. Pick any $u_{10} \in X^+ \setminus \{0\}$. By repeating the above construction we can show that (after possibly taking a subsequence) $f_2(t + t_n, x, u_1(t + t_n; 0, \mathbf{u}_0)(x), 0) \rightarrow \tilde{c}(t, x)$ in the open-compact topology, where $\mathbf{u}_0 = (u_{10}, 0)$. This proves the inclusion $Y_0^1 \subset \mathbf{C}_0(f_{12}^{u_{10}})$.

We then have $\mathbf{C}_0(f_{12}^{u_{10}}) = Y_0^1$ for any $u_{10} \in X_1^+ \setminus \{0\}$. Therefore $[\lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2), \lambda_{\max}(f_{12}^{u_{10}}; \mathcal{B}_2)] = [\lambda_{\min}^1, \lambda_{\max}^1]$ for any $u_{10} \in X_1^+ \setminus \{0\}$.

(2) For $\mathbf{g} \in \mathbf{Z}_0$ denote

$$p_{21}(\mathbf{g})(t, x) := g_1(t, x, 0, \xi_2(\mathbf{g} \cdot t)(x)), \quad t \in \mathbb{R}, x \in \bar{D}.$$

Let $Y_0^2 := p_{21}(\mathbf{Z}_0)$. The set Y_0^2 is considered endowed with the open-compact topology. As a continuous image of a compact set, it is compact, too. Denote a generic element of Y_0^2 by g_{21} .

For $g_{21} \in Y_0^2$ and $s \in \mathbb{R}$ we put $(g_{21} \cdot s)(t, x) := g_2(t + s, x, 0, \xi_2(\mathbf{g} \cdot (t + s))(x))$, $t \in \mathbb{R}$, $x \in \bar{D}$. The translation flow on Y_0^2 is a compact flow.

Consider the equation

$$\begin{cases} \frac{\partial v_1}{\partial t} = \Delta v_1 + g_{21}(t, x)v_1, & x \in D, \\ \mathcal{B}_1 v_1 = 0, & x \in \partial D, \end{cases} \quad (4.6)$$

where $g_{21} \in Y_0^2$. We write sometimes (4.6) as $(4.6)_{g_{21}}$.

For $\mathbf{g} \in \mathbf{Z}_0$ the equation $(4.6)_{p_{21}(\mathbf{g})}$ equals the first coordinate of the linearization (3.9) of (3.6) along the semitrivial solution $\mathbf{u}(t; (0, \xi_2(\mathbf{g})), \mathbf{g})$.

For $g_{21} \in Y_0^2$, $t \geq 0$ and $v_{10} \in L_2(D)$ denote by $U_{g_{21}}^{21}(t, 0)v_{10}$ the weak solution operator of (4.6). Put

$$\Pi_1^2(t; v_{10}, g_{21}) := (U_{g_{21}}^{21}(t, 0)v_{10}, g_{21} \cdot t), \quad t \geq 0, v_{10} \in L_2(D), g_{21} \in Y_0^2.$$

Denote by $[\lambda_{\min}^2, \lambda_{\max}^2]$ the principal spectrum of (4.6) over Y_0^2 .

Similarly, we can prove that $[\lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1), \lambda_{\max}(f_{21}^{u_{20}}; \mathcal{B}_1)] = [\lambda_{\min}^2, \lambda_{\max}^2]$.

Theorem 4.1 is thus proved. $\square$

Next we prove Theorem 4.2. We will prove only that there is $\eta_1 > 0$ such that for any $\mathbf{u}_0 \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$ there is $\tau_1(\mathbf{u}_0) \geq 0$ such that

$$u_2(t; t_0, \mathbf{u}_0) \geq \eta_1 \varphi_{\text{princ}}^2$$

for any $t_0 \geq 0$ and any $t \geq t_0 + \tau(\mathbf{u}_0)$, the other part being similar.

To do so, we first prove some lemmas. For $u_{10} \in X_1^+ \setminus \{0\}$ and $\mathbf{g} \in \tilde{\mathbf{Z}}$ denote

$$\tilde{p}_{12}(u_{10}, \mathbf{g})(t, x) := g_2(t, x, \mathbf{u}(t; (u_{10}, 0), \mathbf{g}))(x), \quad t \geq 0, x \in \bar{D}.$$

For $u_{10} \in X_1^+ \setminus \{0\}$ and $\mathbf{g} \in \tilde{\mathbf{Z}}$ consider the equation

$$\begin{cases} \frac{\partial v_2}{\partial t} = \Delta v_2 + g_{12}^{u_{10}}(t, x)v_2, & x \in D, \\ \mathcal{B}_2 v_2 = 0, & x \in \partial D, \end{cases} \quad (4.7)$$

where $g_{12}^{u_{10}} = \tilde{p}_{12}(u_{10}, \mathbf{g})$. We write sometimes (4.7) as $(4.7)_{g_{12}^{u_{10}}}$.

Equation $(4.7)_{g_{12}^{u_{10}}}$ equals the second coordinate of the linearization (3.9) of (3.6) along the semitrivial solution $\mathbf{u}(t; (u_{10}, 0), \mathbf{g})$.

For $g_{12}^{u_{10}} \in \tilde{p}_{12}(X^+ \times \tilde{\mathbf{Z}})$, $t \geq 0$ and $v_{20} \in L_2(D)$ denote by $U_{g_{12}^{u_{10}}}^{12}(t, 0)v_{20}$ the weak solution operator of $(4.7)_{g_{12}^{u_{10}}}$.

Arguing as in the proof of [22, Lemma 7.1.16] we fix $T > 0$ such that

$$U_{g_{12}^{u_{10}}}^{12}(T, 0)\varphi_{\text{princ}}^2 \gg 3\varphi_{\text{princ}}^2$$

for any $\mathbf{g} \in \mathbf{Z}_0$. Moreover, as X_2^{++} is open, the function

$$[\mathbf{Z}_0 \ni \mathbf{g} \mapsto \sup \{ r > 0 : U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 \gg r \varphi_{\text{princ}}^2 \}]$$

is lower semicontinuous. Since it takes values in $(3, \infty)$, its infimum (denoted by m) is, by the compactness of $\mathbf{Z}_0$, attained, hence is larger than 3.

Fix $\delta_0 > 0$ so small that

$$\delta_0 < \min \left\{ \frac{\ln m - \ln 3}{T}, \frac{P_1 - \sup \{ \xi_1(\mathbf{g})(x) : \mathbf{g} \in \mathbf{Z}_0, x \in \bar{D} \}}{2} \right\}.$$

Lemma 4.3 *Let T and δ_0 be as above. If $u_{10} \in X_1^+ \setminus \{0\}$, $\tilde{\mathbf{g}} \in \tilde{\mathbf{Z}}$ and $\mathbf{g} \in \mathbf{Z}_0$ are such that*

$$\tilde{g}_{12}^{u_{10}}(t, x) \geq g_{12}(t, x) - \delta_0 \quad \text{for } t \in [0, T] \text{ and } x \in \bar{D},$$

where $\tilde{g}_{12}^{u_{10}} = \tilde{p}_{12}(u_{10}, \tilde{\mathbf{g}})$ and $g_{12} = p_{12}(\mathbf{g})$, then $U_{\tilde{g}_{12}^{u_{10}}}^{12}(T, 0) \varphi_{\text{princ}}^2 \gg 3 \varphi_{\text{princ}}^2$.

Proof Denote by $[[0, T] \ni t \mapsto \tilde{v}_2(t) \in X_2]$ the solution of the problem

$$\begin{cases} \frac{\partial v_2}{\partial t} = \Delta v_2 + (g_{12}(t, x) - \delta_0) v_2, & t \in [0, T], x \in D, \\ \mathcal{B}_2 v_2 = 0, & t \in [0, T], x \in \partial D, \end{cases}$$

satisfying $\tilde{v}_2(0) = \varphi_{\text{princ}}^2$. There holds $\tilde{v}_2(t) = e^{-t\delta_0} U_{g_{12}}^{12}(t, 0) \varphi_{\text{princ}}^2$ for all $t \in [0, T]$. It follows from the monotone dependence of solutions on the zero-order coefficients (compare, for instance, [22, Proposition 2.2.10(1)–(2)]) that $U_{\tilde{g}_{12}^{u_{10}}}^{12}(t, 0) \varphi_{\text{princ}}^2 \geq \tilde{v}_2(t)$ for all $t \in [0, T]$. Consequently,

$$U_{\tilde{g}_{12}^{u_{10}}}^{12}(T, 0) \varphi_{\text{princ}}^2 \geq \tilde{v}_2(T) = e^{-T\delta_0} U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 \gg 3 \varphi_{\text{princ}}^2. \quad \square$$

As Γ_1^0 is compact, it follows from uniform continuity on compact sets that there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ with the property that if for any $\mathbf{g} = (g_1, g_2) \in \mathbf{Z}_0$ and $\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2) \in \tilde{\mathbf{Z}}$ we have $d(\mathbf{g}, \tilde{\mathbf{g}}) \leq \epsilon_1$ then

$$\tilde{g}_2(t, x, \xi_1(\mathbf{g})(x) + \epsilon_2, 0) \geq g_2(t, x, \xi_1(\mathbf{g})(x), 0) - \delta_0, \quad t \in [0, T], x \in \bar{D}. \quad (4.8)$$

Fix ϵ_1 and ϵ_2 as above. Denote

$$\begin{aligned} \mathbf{E} := \{ ((u_{10}, 0), \tilde{\mathbf{g}}) \in (X_1^+ \times \{0\}) \times \tilde{\mathbf{Z}} : & \text{there is } \mathbf{g} \in \mathbf{Z}_0 \\ & \text{such that } d(\mathbf{g}, \tilde{\mathbf{g}}) \leq \epsilon_1 \text{ and } u_{10} \leq \xi_1(\mathbf{g}) + \epsilon_2 \}. \end{aligned}$$

$\mathbf{E}$ is a closed subset of $\mathbf{X} \times \tilde{\mathbf{Z}}$.

It follows from Lemma 4.3 that $U_{\tilde{g}_{12}^{u_{10}}}^{12}(T, 0) \varphi_{\text{princ}}^2 \gg 3 \varphi_{\text{princ}}^2$ for all $((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E}$. The closure in the X_2 -topology of the set $\{ U_{\tilde{g}_{12}^{u_{10}}}^{12}(T, 0) \varphi_{\text{princ}}^2 : (u_{10}, \tilde{\mathbf{g}}) \in \mathbf{E} \}$ is contained in $3 \varphi_{\text{princ}}^2 + X_2^+ \subset 2 \varphi_{\text{princ}}^2 + X_2^{++}$ and is, by Proposition 3.5, compact. Therefore

$$\begin{aligned} & \text{cl} \{ U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 - 2\varphi_{\text{princ}}^2 : ((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E} \} \\ & = \text{cl} \{ U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 : ((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E} \} - 2\varphi_{\text{princ}}^2 \subset X_2^{++} \end{aligned}$$

is compact, hence there is $\epsilon_0 > 0$ such that

$$\inf \{ \| (\varrho U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 - 2\varphi_{\text{princ}}^2) - v \|_{X_2} : ((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E}, v \in \partial X_2^+ \} \geq \varrho \epsilon_0 \quad (4.9)$$

for any $\varrho > 0$.

Lemma 4.4 *Let T and $\mathbf{E}$ be as above. There is $\varrho_0 > 0$ such that*

$$u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) \gg 2\varrho \varphi_{\text{princ}}^2$$

for each $((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E}$ and each $\varrho \in (0, \varrho_0]$.

Proof First, let

$$\begin{aligned} \tilde{\mathbf{E}} := \{ ((u_{10}, 0), \tilde{\mathbf{g}}) \in (X_1^+ \times \{0\}) \times \tilde{\mathbf{Z}} : & \text{there exists } \mathbf{g} \in \mathbf{Z}_0 \\ & \text{such that } d(\mathbf{g}, \tilde{\mathbf{g}}) \leq \epsilon_1 \text{ and } u_{10} = \xi_1(\mathbf{g}) + \epsilon_2 \}. \end{aligned}$$

It is clear that $\tilde{\mathbf{E}} \subset \mathbf{E}$ and $\text{cl } \tilde{\mathbf{E}}$ is a compact subset of $\mathbf{E}$. Note that for any $((u_{10}, 0), \tilde{\mathbf{g}}) \in \mathbf{E}$, there is $\mathbf{g} \in \mathbf{Z}_0$ such that $d(\tilde{\mathbf{g}}, \mathbf{g}) \leq \epsilon_1$ and $u_{10} \leq \xi_1(\mathbf{g}) + \epsilon_2$. Hence $((\xi_1(\mathbf{g}) + \epsilon_2, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$ and

$$u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) \geq u_2(T; (\xi_1(\mathbf{g}) + \epsilon_2, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}).$$

It then suffices to prove that there is $\varrho_0 > 0$ such that

$$u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) \gg 2\varrho \varphi_{\text{princ}}^2$$

for each $((u_{10}, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$ and each $\varrho \in (0, \varrho_0]$.

By the compactness of $\text{cl } \tilde{\mathbf{E}}$ and the arguments similar to those in the proof of [22, Theorem 7.1.5], we can prove that

$$\frac{\| u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) - \varrho U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 \|_{X_2}}{\varrho} \rightarrow 0$$

as $\varrho \rightarrow 0^+$, uniformly in $((u_{10}, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$. Consequently, there is $\varrho_0 > 0$ such that

$$\| u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) - \varrho U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 \|_{X_2} \leq \frac{\varrho \epsilon_0}{2}$$

for all $((u_{10}, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$ and $\varrho \in (0, \varrho_0]$. We estimate

$$\begin{aligned} & \| (u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) - 2\varrho \varphi_{\text{princ}}^2) - (\varrho U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 - 2\varrho \varphi_{\text{princ}}^2) \|_{X_2} \\ & = \| u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) - \varrho U_{g_{12}}^{12}(T, 0) \varphi_{\text{princ}}^2 \|_{X_2} \leq \frac{\varrho \epsilon_0}{2} \end{aligned}$$

for any $((u_{10}, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$ and any $\varrho \in (0, \varrho_0]$. It follows from (4.9) that $u_2(T; (u_{10}, \varrho \varphi_{\text{princ}}^2), \tilde{\mathbf{g}}) - 2\varrho \varphi_{\text{princ}}^2 \in X_2^{++}$ for any $((u_{10}, 0), \tilde{\mathbf{g}}) \in \tilde{\mathbf{E}}$ and any $\varrho \in (0, \varrho_0]$. $\square$

Proof of Theorem 4.2 Let $\mathbf{u}_0 = (u_{10}, u_{20}) \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$ be fixed. Denote $\mathbf{u}_0^* := (u_{10}, 0)$. For $t_0 \geq 0$ we write

$$\begin{aligned} \mathbf{u}(t; t_0, \mathbf{u}_0) &= \mathbf{u}(t; t_0) = (u_1(t; t_0), u_2(t; t_0)), \\ \mathbf{u}(t; t_0, \mathbf{u}_0^*) &= \mathbf{u}^*(t; t_0) = (u_1^*(t; t_0), 0). \end{aligned}$$

By Proposition 3.4(2),

$$u_1(t; t_0) \ll u_1^*(t; t_0) \quad \text{for all } t > t_0. \quad (4.10)$$

We claim that there is $\Theta \geq 0$ such that for each $t_0 \geq 0$ and each $t \geq t_0 + \Theta$ we can find $\mathbf{g} \in \mathbf{Z}_0$ with the property that $d(\bar{\mathbf{f}} \cdot t, \mathbf{g}) < \epsilon_1$ and $u_1^*(t)(x) \leq \xi_1(\mathbf{g})(x) + \epsilon_2$ for all $x \in \bar{D}$. The fact that for each sufficiently large $t \geq 0$ there is $\mathbf{g} \in \mathbf{Z}_0$ such that $d(\bar{\mathbf{f}} \cdot t, \mathbf{g}) < \epsilon_1$ follows from the definition of $\mathbf{Z}_0$. Suppose to the contrary that there are sequences $(t_0^{(n)})_{n=1}^\infty \subset [0, \infty)$, $(t^{(n)})_{n=1}^\infty \subset (0, \infty)$ with $\lim_{t \rightarrow \infty} t^{(n)} = \infty$, $(x^{(n)})_{n=1}^\infty \subset \bar{D}$ and $(\mathbf{g}^{(n)})_{n=1}^\infty \subset \mathbf{Z}_0$ such that $d(\bar{\mathbf{f}} \cdot (t_0^{(n)} + t^{(n)}), \mathbf{g}^{(n)}) < \epsilon_1$ but $u_1^*(t^{(n)}; t_0^{(n)})(x^{(n)}) > \xi_1(\mathbf{g}^{(n)})(x^{(n)}) + \epsilon_2$. Proposition 3.5 gives that the set $\{u_1^*(t^{(n)}; t_0^{(n)}) : n \in \mathbb{N}\}$ has compact closure in X_1 . By passing to a subsequence, if necessary, we can assume that $\lim_{n \rightarrow \infty} \bar{\mathbf{f}} \cdot (t_0^{(n)} + t^{(n)}) = \lim_{n \rightarrow \infty} \mathbf{g}^{(n)} = \hat{\mathbf{g}}$ in $\bar{\mathbf{Z}}$, $\lim_{n \rightarrow \infty} u_1^*(t^{(n)}; t_0^{(n)}) = \hat{u}_1$ in X_1 and $\lim_{n \rightarrow \infty} x^{(n)} = \hat{x}$ in $\bar{D}$. Consequently we have $\hat{u}_1(\hat{x}) \geq \xi_1(\hat{\mathbf{g}})(\hat{x}) + \epsilon_2$. By arguments as in the proof of Theorem 4.1, $\hat{u}_1(\hat{x}) = \xi_1(\hat{\mathbf{g}})(\hat{x})$, a contradiction.

We have proved that $((u_1^*(t; t_0), 0), \bar{\mathbf{f}} \cdot t) \in \mathbf{E}$ for any $t_0 \geq 0$ and any $t \geq t_0 + \Theta$. Lemma 4.4 states that there is $\varrho_0 > 0$ such that

$$u_2(T; (u_1^*(t; t_0), \varrho\varphi_{\text{princ}}^2), \bar{\mathbf{f}} \cdot t) \gg 2\varrho\varphi_{\text{princ}}^2 \quad (4.11)$$

for all $t_0 \geq 0$, all $t \geq t_0 + \Theta + 1$ and all $\varrho \in (0, \varrho_0]$.

Since $[\Theta + 1, \Theta + T + 1] \times \{\mathbf{u}_0\} \times \bar{\mathbf{Z}}$ is compact, Proposition 3.2 implies that the set $\{\mathbf{u}(t; \mathbf{u}_0, \tilde{\mathbf{g}}) : t \in [\Theta + 1, \Theta + T + 1], \tilde{\mathbf{g}} \in \bar{\mathbf{Z}}\}$ is compact, too. Consequently, the closure (in X) of the set $\{u_2(t; t_0) : t_0 \geq 0, t \in [t_0 + \Theta + 1, t_0 + \Theta + T + 1]\}$, as contained in $p_2(\{\mathbf{u}(t; \mathbf{u}_0, \tilde{\mathbf{g}}) : t \in [\Theta + 1, \Theta + T + 1], \tilde{\mathbf{g}} \in \bar{\mathbf{Z}}\})$, is compact. Further, by Proposition 3.4(2), $p_2(\{\mathbf{u}(t; \mathbf{u}_0, \tilde{\mathbf{g}}) : t \in [\Theta + 1, \Theta + T + 1], \tilde{\mathbf{g}} \in \bar{\mathbf{Z}}\}) \subset X^{++}$. Applying arguments as in the proof of [22, Theorem 7.1.6] we obtain the existence of $\tilde{\varrho} > 0$ such that $u_2(t; t_0) \geq \tilde{\varrho}\varphi_{\text{princ}}$ for all $t_0 \geq 0$ and all $t \in [t_0 + \Theta + 1, t_0 + \Theta + T + 1]$.

Assume $\tilde{\varrho} \geq \varrho_0$. Then for each $t_0 \geq 0$ and each $t \in [t_0 + \Theta + 1, t_0 + \Theta + T + 1]$ we have $(u_1(t; t_0), u_2(t; t_0)) \leq 2(u_1^*(t; t_0), \varrho_0\varphi_{\text{princ}})$, which yields with the help of Proposition 3.4(1) and (4.11) that

$$\begin{aligned} u_2(t + T; t_0) &= u_2(T; (u_1(t; t_0), u_2(t; t_0)), \bar{\mathbf{f}} \cdot t) \\ &\geq u_2(T; (u_1^*(t; t_0), \varrho_0\varphi_{\text{princ}}^2), \bar{\mathbf{f}} \cdot t) \\ &\gg 2\varrho_0\varphi_{\text{princ}}^2. \end{aligned}$$

By induction, we have $u_2(t + nT; t_0) \gg 2\varrho_0\varphi_{\text{princ}}^2$ for all $n = 1, 2, \dots$. Therefore we can take $\tau_1(\mathbf{u}_0) = \Theta + T + 1$.

Assume $\tilde{\varrho} < \varrho_0$. Then for each $t_0 \geq 0$ and each $t \in [t_0 + \Theta + 1, t_0 + \Theta + T + 1]$ such that $u_2(t; t_0) \geq \varrho\varphi_{\text{princ}}^2$ for some $\varrho < \varrho_0$ we have $(u_1(t; t_0), u_2(t; t_0)) \leq 2(u_1^*(t; t_0), \varrho\varphi_{\text{princ}}^2)$, which yields with the help of Proposition 3.4(1) and (4.11) that

$$\begin{aligned} u_2(t + T; t_0) &= u_2(T; (u_1(t; t_0), u_2(t; t_0)), \bar{\mathbf{f}} \cdot t) \\ &\geq u_2(T; (u_1^*(t; t_0), \varrho\varphi_{\text{princ}}^2), \bar{\mathbf{f}} \cdot t) \\ &\gg 2\varrho\varphi_{\text{princ}}^2. \end{aligned}$$

Repeating this procedure sufficiently many times we obtain that $u_2(t + nT; t_0) \gg 2^n\varrho\varphi_{\text{princ}}$ as long as $2^{n-1}\varrho \leq \varrho_0$. After some calculation we conclude that we can take $\tau_1(\mathbf{u}_0) = \left(\left\lfloor \frac{\ln \varrho_0 - \ln \tilde{\varrho}}{\ln 2} \right\rfloor + 2\right)T + \Theta + 1$.

In both cases, $\eta_1 = 2\varrho_0$. □

5 Sufficient Conditions for Forward Uniform Persistence

As checking the assumptions in Theorem 4.2 requires actually some knowledge of the principal spectrum over the attracting sets Γ_1^0 and Γ_2^0 (which can be difficult to obtain), in the present section we give some easy to check sufficient conditions.

In the present section we assume (A1) through (A4).

Denote

$$\begin{aligned}\tilde{f}_1(t, x) &:= f_1(t, x, 0, P_2), & t \geq 0, x \in \bar{D}, \\ \tilde{f}_2(t, x) &:= f_2(t, x, P_1, 0), & t \geq 0, x \in \bar{D}.\end{aligned}$$

Consider the boundary value problems

$$\begin{cases} \frac{\partial v_1}{\partial t} = \Delta v_1 + \tilde{f}_1(t, x)v_1, & t > 0, x \in D \\ \mathcal{B}_1 v_1 = 0, & t > 0, x \in \partial D \end{cases} \quad (5.1)$$

and

$$\begin{cases} \frac{\partial v_2}{\partial t} = \Delta v_2 + \tilde{f}_2(t, x)v_2, & t > 0, x \in D \\ \mathcal{B}_2 v_2 = 0, & t > 0, x \in \partial D. \end{cases} \quad (5.2)$$

Recall that $[\lambda_{\min}(\tilde{f}_1; \mathcal{B}_1), \lambda_{\max}(\tilde{f}_1; \mathcal{B}_1)]$ (resp. $[\lambda_{\min}(\tilde{f}_2; \mathcal{B}_2), \lambda_{\max}(\tilde{f}_2; \mathcal{B}_2)]$) denotes the principal spectrum of (5.1) (resp. of (5.2)).

Theorem 5.1 *Assume that $\lambda_{\min}(\tilde{f}_1; \mathcal{B}_1) > 0$ and $\lambda_{\min}(\tilde{f}_2; \mathcal{B}_2) > 0$. Then (3.1) + (3.2) is forward uniformly persistent.*

Proof First, note that

$$f_{0i}(t, x) = f_i(t, x, 0, 0) \geq \tilde{f}_i(t, x), \quad t \geq 0, x \in D,$$

hence, by the monotone dependence of the principal spectrum on the zero-order coefficients (see [23, Theorem 3.8]),

$$\lambda_{\min}(f_{0i}; \mathcal{B}_i) \geq \lambda_{\min}(\tilde{f}_i; \mathcal{B}_i) > 0,$$

for $i = 1, 2$.

Next, note that for any $u_{10} \in X_1^+ \setminus \{0\}$ with $u_{10} \leq P_1$,

$$u_1(t; 0, (u_{10}, 0)) \leq P_1 \quad \text{for } t \geq 0.$$

This implies that

$$f_2(t, x, u_1(t; 0, (u_{10}, 0))(x), 0) \geq f_2(t, x, P_1, 0) = \tilde{f}_2(t, x), \quad t \geq 0, x \in D.$$

Hence, by the monotone dependence of the principal spectrum on the zero-order coefficients,

$$\lambda_{\min}(f_{12}^{u_{10}}; \mathcal{B}_2) \geq \lambda_{\min}(\tilde{f}_2; \mathcal{B}_2) > 0.$$

Similarly, we have

$$\lambda_{\min}(f_{21}^{u_{20}}; \mathcal{B}_1) \geq \lambda_{\min}(\tilde{f}_1; \mathcal{B}_1)$$

for any $u_{20} \in X_2^+ \setminus \{0\}$ with $u_{20} \leq P_2$. Then by Theorems (4.1) and (4.2), (3.1) + (3.2) is forward uniformly persistent. $\square$

A function $\hat{f}_1 \in C(\bar{D})$ belongs to $\hat{Y}_1$ if there are subsequences $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$, with $0 < s_n < t_n$ for all $n = 1, 2, \dots$, $\lim_{n \rightarrow \infty} s_n = \infty$, $\lim_{n \rightarrow \infty} (t_n - s_n) \rightarrow \infty$, such that

$$\hat{f}_1(x) = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} f_1(t, x, 0, P_2) dt$$

uniformly for $x \in \bar{D}$. Similarly, a function $\hat{f}_2 \in C(\bar{D})$ belongs to $\hat{Y}_2$ if there are subsequences $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$, with $0 < s_n < t_n$ for all $n = 1, 2, \dots$, $\lim_{n \rightarrow \infty} s_n = \infty$, $\lim_{n \rightarrow \infty} (t_n - s_n) \rightarrow \infty$, such that

$$\hat{f}_2(x) = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} f_2(t, x, P_1, 0) dt$$

uniformly for $x \in \bar{D}$.

For a given $\hat{f}_i \in \hat{Y}_i$, denote by $\lambda_{\text{princ}}(\hat{f}_i; B_i)$ the principal eigenvalue of

$$\begin{cases} \Delta u + \hat{f}_i(x)u = \lambda u, & x \in D, \\ B_i u = 0, & x \in \partial D. \end{cases} \quad (5.3)$$

Theorem 5.2 *If $\lambda_{\text{princ}}(\hat{f}_i; B_i) > 0$ for any $\hat{f}_i \in \hat{Y}_i$, $i = 1, 2$, then (3.1) + (3.2) is forward uniformly persistent.*

Proof By [23, Theorem 5.3(1)], there are $\hat{f}_1 \in \hat{Y}_1$ such that $\lambda_{\min}(\tilde{f}_1; B_1) \geq \lambda_{\text{princ}}(\hat{f}_1; B_1)$, and $\hat{f}_2 \in \hat{Y}_2$ such that $\lambda_{\min}(\tilde{f}_2; B_2) \geq \lambda_{\text{princ}}(\hat{f}_2; B_2)$. An application of Theorem 5.1 concludes the proof. $\square$

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