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NONSINGULAR α-RIGID MAPS

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ABSTRACT. It is shown that for any $\alpha \in [0, 1/2]$, there exists an α-rigid transformation whose spectrum has a Lebesgue component. This answers the question posed by Klemes and Reinhold in [7]. We apply a certain correspondence between weak limits of powers of a transformation and their skew products.

INTRODUCTION

Let T be a measure-preserving transformation defined on a nonatomic standard Borel probability space $(X, \mathcal{F}, \mu)$. The *spectral* properties of T are those of the induced (Koopman's) unitary operator on $L^2(\mu)$ defined by

$$\widehat{T} : L^2(\mu) \to L^2(\mu); \quad \widehat{T} f(x) = f(Tx).$$

The transformation T is said to be α-rigid on a sequence of integers k_i, where $\alpha \in [0; 1]$, if for any measurable A

$$\lim_i \mu(T^{k_i} A \cap A) \geq \alpha \mu(A). \tag{1}$$

If we vary the sequence k_i, all these α's form a closed subset in $[0; 1]$. The transformation T is said to be *α-rigid* if α is the right point of this subset.

By *mixing component* in the spectrum of the transformation T we mean a nonzero $\widehat{T}$-invariant subspace, say H, of $L^2(\mu)$ such that $\langle \widehat{T}^k f, g \rangle \to 0$ as $k \to \infty$ for any $f, g \in H$.

We say that a transformation T is a *skew product* over T_0 if T acts on a product probability Borel space $(X \times Y, \mathcal{F} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$T(x, y) = (T_0 x, S_x(y)),$$

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where $(S_x)_{x\in X}$ is a family of transformations of $(Y, \mathcal{C}, \nu)$ such that the map $(x, y) \mapsto S_x y$ is measurable. A particular case of it is any *G-extension* over T_0; that is, $(Y, \mathcal{C}, \nu)$ is a group G with the Haar measure ν, and $S_x y = \phi(x) + y$ for some measurable map (*cocycle*) $\phi : X \to G$. The restriction of $\widehat{T}$ to an invariant space $H_0 = \{f(x, y) \in L^2 : \exists g(x)[f(x, y) = g(x)\ \mu\otimes\nu-\text{a.e.}]\}$ is unitary equivalent to $\widehat{T_0}$. We say that T is *relatively mixing* if $H_0^{\perp}$ is its mixing component.

It is well known (see, e.g., [4, 8]) that any α-rigid transformation has no mixing components if $\alpha > 1/2$ and is not mixing if $\alpha > 0$. In this brief note, we prove that for any $\alpha \in [0, 1/2]$, there exists an α-rigid transformation whose spectrum has a Lebesgue component. This paper is based on well-known facts to the author, and we offer a written version of that because of the current interest to this theme (see [1]). Recently, the author has been informed about [1], where, actually, the main result (as it is stated in the abstract) is that there exists an α-rigid transformation with $\alpha \leq 1/2$ whose spectrum has a Lebesgue component. We note that the author of [1] knew that there exists an alternative proof of this theorem. This proof written below is of the independent interest for possible applications, because, in particular, of its large constructive potential.

It is well known that the notion to be α-rigid transformation T can be rewritten in terms of the powers weak operator closure $PCL(T)$ of $\widehat{T}$. Indeed, (0.1) is equivalent to

$$\forall f \geq 0 \quad \lim_i \langle \widehat{T}^{k_i} f, f\rangle \geq \alpha \langle f, f\rangle.$$

Taking a subsequence of k_i, we can assume that there exist $Q = \lim_i \widehat{T}^{k_i}$. Obviously, $Q \geq 0$, i.e., $Qf \geq 0$ for any $f \geq 0$.

Let E be the identity map. If for some $f > 0$ a.e., $Q\prime f = (Q - \alpha\widehat{E})f < 0$ on some measurable A, then

$$0 \leq \langle Q(f\chi_{\overline{A}}), f\chi_A\rangle = \langle Q\prime(f\chi_{\overline{A}}), f\chi_A\rangle \leq \langle Q\prime f, f\chi_A\rangle \leq 0.$$

Thus, A has a zero measure and $Q\prime \geq 0$. This implies that

$$\alpha = \max_{P\in PCL(T)} \{\beta \in [0, 1] : \exists Q \geq 0\ [P = \beta\widehat{E} + Q]\}.$$

1. The basic result, applications, and comments

It is convenient to think that any $P \in PCL(T_0)$ is also an operator $P \otimes \widehat{E}$ acting on $L^2(X \times Y, \mu \otimes \nu)$.

In this paper, we prove the following theorem.

Theorem 1.1. *Let T be a skew product over T_0 and P_{H_0} be the orthogonal projection onto H_0. If T is relatively mixing, then*

$$PCL(T) = PCL(T_0)P_{H_0}.$$

Proof. Take $P \in PCL(T)$ and $f, g \in L^2$. Denote by f_0 and $f_0^\perp$ the corresponding parts of f in H_0 and $H_0^\perp$, respectively. Then

$$\langle Pf, g\rangle = \lim_i \langle \widehat{T}^{k_i} f, g\rangle = \lim_i \langle \widehat{T}^{k_i}(f_0 + f_0^\perp), g_0 + g_0^\perp\rangle = \lim_i \langle \widehat{T}^{k_i} f_0, g_0\rangle$$
$$= \lim_i \langle \widehat{T}_0^{k_i} f_0, g_0\rangle = \lim_i \langle \widehat{T}_0^{k_i} f_0, g\rangle = \lim_i \langle \widehat{T}_0^{k_i} P_{H_0} f, g\rangle.$$

Thus,

$$PCL(T) \subseteq PCL(T_0)P_{H_0}.$$

Along the same line, we also obtain

$$PCL(T_0)P_{H_0} \subseteq PCL(T).$$

The theorem is proved □

Corollary 1.2. *Let T be a relatively mixing $\boldsymbol{Z}_2$-extension over T_0. Then T_0 is α-rigid if and only if T is $\alpha/2$-rigid.*

Proof. First, we note that in this case $P_{H_0} = (\widehat{E} + \widehat{S})/2$, where $S(x, y) = (x, y + 1 \bmod 1)$. Applying the theorem, we must show that if $\alpha\widehat{E} + P = 1/2Q + 1/2Q\widehat{S}$, where $P \geq 0$ and $Q \in PCL(T_0)$, then $Q = 2\alpha\widehat{E} + Q\prime$ for some $Q\prime \geq 0$. Taking $f \geq 0$, supp $f \subseteq \{(x, y) : y = 0\}$, $(f \in L^2)$, we see that $\langle 1/2Qf, f\rangle = \langle 1/2Q + 1/2Q\widehat{S}f, f\rangle \geq \alpha\langle f, f\rangle$. This implies $\langle 1/2Qf, f\rangle \geq \alpha\langle f, f\rangle$ for any $f \geq 0$. Thus, $Q\prime \geq 0$ (see the discussion near definitions of the α-rigidity). □

Corollary 1.3. *For any $\alpha \in [0, 1/2]$, there exists an α-rigid transformation whose spectrum has a Lebesgue component.*[1] *In fact, for any such α, we can take continuum mutually nonisomorphic metrically or even spectrally α-rigid transformations with a Lebesgue component in their spectrum.*

Proof. Following Helson and Parry (see [6]), any transformation admits a $\mathbf{Z}_2$-extension such that $H_0^\perp$ is a Lebesgue component in the spectrum. By the Riemann–Lebesgue lemma, any Lebesgue component is mixing. It remains to note that for any $\alpha \in [0, 1]$, we can construct different α-rigid transformations applying the technique developed in [2, 7]. □

Remark 1.4. In [3], a family of different α-rigid transformations admitting relatively mixing $\mathbf{Z}_2$-extensions with twofold Lebesque component, where $\alpha = 1$ or less, was constructed. This implies that there exist 1/2(or less)-rigid transformations with a twofold Lebesque component in its spectrum.

[1] So, it is nonsingular in the sense that its maximal spectral type is nonsingular.

Since Mathew–Nadkarni transformations are included in this family and they are $\mathbf{Z}_2$-extensions over the *adic* shift, they are 1/2-rigid transformations with a twofold Lebesque component. Indeed, any adic shift has the discrete spectrum, therefore it is just (1-)rigid, and we apply Theorem 1.1.

Remark 1.5. All the below notes admit natural extensions to both (non) relatively mixing G-extensions if we can calculate powers limits on $H_0^{\perp}$ and group actions.

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