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Arcs in $\mathbb{Z}_{2p}^2$

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Abstract An arc in $\mathbb{Z}_n^2$ is defined to be a set of points no three of which are collinear. We describe some properties of arcs and determine the maximum size of arcs for some small n.

Keywords No-three-in-line problem · Arc · Collinearity

Mathematics Subject Classification 05B99

1 Introduction

The no-three-in-line problem is an old question (see Dudeney 1917) which asks for the maximum number of points that can be placed in the $n \times n$ grid with no three points collinear. This question has been widely studied (see Erdös 1951; Flammenkamp 1992, 1998; Guy and Kelly 1968; Hall et al. 1975), but is still not resolved.

In this paper we consider the no-three-in-line problem over the $\mathbb{Z}_n^2$. This modified problem is still interesting and was investigated in Huizenga (2006), Kurz (2009) and Misiak et al. (2016). Many authors considered arcs in the context of projective geometry, see e.g. Bose (1947), Segre (1955) and Hirschfeld (1979) for further reference, and in a context of Hjelmslev geometry, see e.g. Honold and Landjev (2001), Honold

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and Landjev (2005), Kiermaier et al. (2011) and Kleinfeld (1959) for definition of abstract Hjelmslev plane.

Four integers a, b, u, v with $\gcd(u, v) = 1$ correspond to the line $\{(a + uk, b + vk) : k \in \mathbb{Z}\}$ on $\mathbb{Z}^2$. We define lines in $\mathbb{Z}_n^2$ to be images, by natural projection, of lines in the $\mathbb{Z}^2$ (see Misiak et al. 2016). We say that points $a_1, \ldots, a_k$ are in line on $\mathbb{Z}_n^2$ (or collinear) if there exists a line l in $\mathbb{Z}_n^2$ such that $a_1, \ldots, a_n \in l$. We would like to remark that one could also define a line as a coset of a (maximal) cyclic subgroup of $\mathbb{Z}_n^2$ (see Huizenga 2006; Kurz 2009).

If $n = p$ is a prime the resulting space is just $\mathrm{AG}(2, p)$ where every two distinct lines intersect in either 0 or 1 point. In contrast, for arbitrary n it may happen that two distinct lines meet each other in more than a single point.

We call $X \subset \mathbb{Z}_n^2$ an arc if it does not contain any three collinear points, and we call X complete if it is maximal with respect to the set-theoretical inclusion. We denote by $\tau(\mathbb{Z}_n^2)$ the maximum possible size of an arc in $\mathbb{Z}_n^2$.

Kurz (2009) uses the integer linear programming to determine numbers $\tau(\mathbb{Z}_n^2)$. He identifies values of $\tau(\mathbb{Z}_n^2)$ for $n \leq 21$ and, after transformations, the value of $\tau(\mathbb{Z}_{25}^2)$ (see also Kiermaier et al. 2011 for this case). We were interested in finding the lacking values of $\tau(\mathbb{Z}_{22}^2)$ and $\tau(\mathbb{Z}_{24}^2)$. Our first try was using the mathematical programming solver Gurobi. We succeed with computing $\tau(\mathbb{Z}_{24}^2)$ but were not able to determine $\tau(\mathbb{Z}_{22}^2)$ in this way. Therefore, we investigated some properties of arcs in $\mathbb{Z}_{2p}^2$ for p prime, which allowed us to compute $\tau(\mathbb{Z}_{22}^2)$.

We will use the following notation. Denote by π_n the natural projection from $\mathbb{Z}$ to $\mathbb{Z}_n$. If the context is clear, we omit the index n. For $u = (u_x, u_y)$ we write $\pi(u)$ for $(\pi(u_x), \pi(u_y))$. If $p|n$ then by ϕ_p, we denote the projection $\phi_p : \mathbb{Z}_n \to \mathbb{Z}_p$ defined by $\phi_p \circ \pi_n = \pi_p$. Similarly, we write $\phi_p(u)$ for $(\phi_p(u_x), \phi_p(u_y))$.

By automorphism of $\mathbb{Z}_n^2$ we mean mapping from $\mathbb{Z}_n^2$ to $\mathbb{Z}_n^2$ which preserves arcs. The group formed by these elements is denoted by $\mathcal{G}_n$. Let $GL_2(\mathbb{Z}_{2p})$ denote the group of invertible 2×2 matrices with coefficients in $\mathbb{Z}_{2p}$. We denote by f_A the automorphism of $\mathbb{Z}_{2p}^2$ corresponding to the matrix A. Another class of automorphisms are translations. In this paper t_v denotes the translation by a vector v. More explicitly, $t_{(v_x, v_y)}(u_x, u_y) = (u_x + v_x, u_y + v_y)$.

By $D(u, v, w)$ we denote

$$\begin{vmatrix} 1 & 1 & 1 \\ u_x & v_x & w_x \\ u_y & v_y & w_y \end{vmatrix}$$

where $u = (u_x, u_y)$, $v = (v_x, v_y)$ and $w = (w_x, w_y)$.

2 Basic facts

Recall that the maximum size $m(2, q)$ of a complete arc in projective spaces $\mathrm{PG}(2, q)$ is known (see Bose 1947; Hirschfeld 1979):

Theorem 2.1 $m(2, q) = \begin{cases} q + 2 & for\ q\ even, \\ q + 1 & for\ q\ odd. \end{cases}$

It is easy to see that $m(2, q)$ is an upper bound for $\tau(\mathbb{Z}_p^2)$ in the case of a prime $q = p$. To establish the lower bound for $\tau(\mathbb{Z}_p^2)$, recall that there are arcs of $\mathbb{Z}_p^2$ of cardinality $p + 1$ (see for instance Misiak et al. 2016 for more details). Therefore, the following lemma is true.

Lemma 2.2 *Let p be an odd prime, then* $\tau\left(\mathbb{Z}_p^2\right) = p + 1$.

The following bound was proven in Kurz (2009).

Lemma 2.3 $\tau\left(\mathbb{Z}_{mn}^2\right) \leq \min\left\{m \cdot \tau\left(\mathbb{Z}_n^2\right), n \cdot \tau\left(\mathbb{Z}_m^2\right)\right\}$ *for coprime integers $m, n > 1$.*

Recall the well-known test to check whether three points are collinear or not.

Lemma 2.4 *Points $a, b, c \in \mathbb{Z}_p^2$ for p prime are in a line if and only if $D(a, b, c) = 0$.*

The following lemma is adapted from Huizenga (2006).

Lemma 2.5 *Let $N = m \cdot n$ with coprime m and n. Then any three points in $\mathbb{Z}_N^2$ are collinear if and only if both projections onto $\mathbb{Z}_m^2$ and $\mathbb{Z}_n^2$ give a collinear point set.*

Theorem 2.6 *Let p_1, p_2 be primes such that $p_1 \neq p_2$. Then points $a, b, c \in \mathbb{Z}_{p_1 \cdot p_2}^2$ are in a line if and only if $D(a, b, c) = 0$.*

Proof Assume that $D(a, b, c) = 0 \in \mathbb{Z}_{p_1 \cdot p_2}$. Consequently, $D(\phi_{p_i}(a), \phi_{p_i}(b), \phi_{p_i}(c)) = 0 \in \mathbb{Z}_{p_i}$ and by Lemma 2.4 $\phi_{p_i}(a), \phi_{p_i}(b), \phi_{p_i}(c)$ are in a line on $\mathbb{Z}_{p_i}^2$ for $i = 1, 2$. By Lemma 2.5 a, b, c are in a line on $Z_{p_1 \cdot p_2}^2$.

Conversely, assume that $a, b, c \in \mathbb{Z}_{p_1 \cdot p_2}^2$ are in a line. There exists $A, B, C \in \mathbb{Z}^2$ in a line such that $\pi(A) = a$, $\pi(B) = b$ and $\pi(C) = c$. Hence, $D(A, B, C) = 0 \in \mathbb{Z}$ and consequently $D(a, b, c) = 0 \in \mathbb{Z}_{p_1 \cdot p_2}$. □

Remark 2.7 Generally, zeroing of determinant is necessary but not sufficient for three points to be collinear. Let $p^2 | m$ for some prime p. Then for the points $(0, 0), (p, 0), (0, p)$ we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & p & 0 \\ 0 & 0 & p \end{vmatrix} = 0$$

but these point are not collinear.

3 Arcs in $\mathbb{Z}_{2p}^2$

Theorem 3.1 *We have* $\tau\left(\mathbb{Z}_{2p}^2\right) \leq 2p + 2$ *and* $\tau\left(\mathbb{Z}_{2p}^2\right) = 2p + 2$ *for* $p = 3, 5$.

Proof It is an immediate consequence of Lemmas 2.3 and 2.2, Figs. 1 and 2. □

Remark 3.2 We conjecture that $p = 3$ and 5 are the only values for which the equality holds.

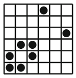

Fig. 1 A complete arc of cardinality 8 over $\mathbb{Z}_6^2$

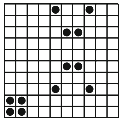

Fig. 2 A complete arc of cardinality 12 over $\mathbb{Z}_{10}^2$

Define the maps $\alpha_2 : \mathbb{Z}_p \to \mathbb{Z}_{2p}$, and $\alpha_p : \mathbb{Z}_2 \to \mathbb{Z}_{2p}$ by

$$\alpha_2(i + p\mathbb{Z}, j + p\mathbb{Z}) = (2i + 2p\mathbb{Z}, 2j + 2p\mathbb{Z}),$$
$$\alpha_p(i + 2\mathbb{Z}, j + 2\mathbb{Z}) = (pi + 2p\mathbb{Z}, pj + 2p\mathbb{Z}).$$

Theorem 3.3 *Let X be a complete arc of $\mathbb{Z}_p^2$. Then $\alpha_2(X)$ is the complete arc of $\mathbb{Z}_{2p}^2$.*

Proof Let X be a complete arc of $\mathbb{Z}_p^2$. Then $\alpha_2(X)$ is obviously the arc of $\mathbb{Z}_{2p}^2$. Let $c \in \mathbb{Z}_{2p}^2$. We will show that there are $a, b \in \alpha_2(X)$ such that a, b, c are collinear.

Assume first that $c \in \phi_2^{-1}(0, 0)$. Then there is $c' \in Z_p^2$ such that $\alpha_2(c') = c$. Since X is a complete arc then there exist $a', b' \in X$ such that a', b', c' are collinear. By Theorem 2.4, $D(a', b', c') = 0 \in \mathbb{Z}_p$. Then $D(\alpha_2(a'), \alpha_2(b'), c) = 0 \in \mathbb{Z}_{2p}$. By Theorem 2.6, $a = \alpha_2(a')$, $b = \alpha_2(b')$, c are collinear.

Now assume that $c \notin \phi_2^{-1}(0, 0)$. Then there is $v \in \{[0, p], [p, 0], [p, p]\}$ such that $t_v(c) \in \phi_2^{-1}(0, 0)$. Hence the first part of the proof shows that there are $a, b \in \alpha_2(X)$ such that a, b, $t_v(c)$ are collinear (i.e. $D(a, b, t_v(c)) = 0$). Because $\alpha_2(X) \subset \phi_2^{-1}(0, 0)$, a straightforward calculation shows that $D(a, b, c) = D(a, b, t_v(c))$ for $a, b \in \alpha_2(X)$. Hence $D(a, b, c) = 0$ and a, b, c are collinear, by Theorem 2.6. This completes the proof. □

Lemma 3.4 *Let X be a complete arc of $\mathbb{Z}_2^2$. Then $\alpha_p(X)$ is the complete arc of $\mathbb{Z}_{2p}^2$.*

Proof Let X be a complete arc in $\mathbb{Z}_2^2$. Then $X = \mathbb{Z}_2^2$ and $\alpha_p(X)$ is obviously the arc of $\mathbb{Z}_{2p}^2$. It is easy to check that for every $c \in Z_{2p}^2$ there is $b \in \alpha_p(X)\setminus(0, 0)$ such that $D((0, 0), b, c) = 0$. By Theorem 2.6, $(0, 0)$, b, c are collinear. □

The proof of the Theorem 3.9 takes up the rest of this section. We prepare for the proof by collecting together some useful technical results.

Lemma 3.5 *Let σ be an arbitrary permutation of the set $\mathbb{Z}_2^2$. Then there is $f \in \mathcal{G}_{2p}$ such that $\phi_2 \circ f = \sigma \circ \phi_2$.*

Proof Let f_A, f_B be linear transformations determined by matrices $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, respectively. Note that the group of permutations of the set $\mathbb{Z}_2^2$ is generated by transpositions $\sigma_1 = ((0,0), (0,1))$, $\sigma_2 = ((0,1), (1,0))$, $\sigma_3 = ((1,0), (1,1))$.

One can verify by a straightforward calculation that $\phi_2 \circ f_A \circ t_{[1,1]} = \sigma_1 \circ \phi_2$, $\phi_2 \circ f_B = \sigma_2 \circ \phi_2$ and $\phi_2 \circ f_A = \sigma_3 \circ \phi_2$. □

Lemma 3.6 *Let $X \subset \mathbb{Z}_{2p}^2$ be an arc. If $|\phi_2(X)| \leq 2$, then $|X| \leq p + 1$.*

Proof The condition $|\phi_2(X)| \leq 2$ means that $\phi_2(X)$ is collinear in $\mathbb{Z}_2^2$. Since X is an arc then, by Lemma 2.5, $\phi_p(X) \subset \mathbb{Z}_p^2$ is an arc and $|\phi_p(X)| = |X|$. By Lemma 2.2, $|X| \leq p + 1$. □

Lemma 3.7 *Let $X \subset \mathbb{Z}_{2p}^2$ be an arc and $a \in X$. If $t_v(a) \in X$ for some $v \in \{[0, p], [p, 0], [p, p]\}$, then $|X| \leq p + 3$.*

Proof Consider first the case that $v = [0, p]$. Let $l_0 = \pi(L_0)$ and $l_1 = \pi(L_1)$ denote the lines in $\mathbb{Z}_2^2$, where L_0 and L_1 are given by the equations $x = 0$, $x = 1$, respectively. Assume without loss of generality that $a \in \phi_2^{-1}(l_0)$. A straightforward calculation shows that $D(a, t_{[0,p]}(a), b) = 0$ for all $b \in \phi_2^{-1}(l_0)$. Hence, by Theorem 2.6, $X \subseteq \left(\phi_2^{-1}(l_1) \cup \left\{a, t_{[0,p]}(a)\right\}\right)$. The same argument as in Lemma 3.6 shows that $\left|X \cap \phi_2^{-1}(l_1)\right| \leq p + 1$. The remaining two cases are dealt with similarly. □

Lemma 3.8 *Let $(a_x, a_y) \in \mathbb{Z}_{2p}^2$ and a_y is invertible in $\mathbb{Z}_{2p}$. If $a_y b_x^1 - b_y^1 a_x = a_y b_x^2 - b_y^1 a_x = a_y b_x^3 - b_y^1 a_x = A$ then b^1, b^2, b^3 are on line.*

Proof We have

$$D\left(b^1, b^2, b^3\right) = a_y^{-1} D\left(\left(A, b_y^1\right), \left(A, b_y^2\right), \left(A, b_y^3\right)\right) = 0.$$

The result now follows from Theorem 2.6. □

Theorem 3.9 *Let $p \geq 5$ and $X \subset \mathbb{Z}_{2p}^2$ be an arc. If $|X| > p+3$, then there is $f \in \mathcal{G}_{2p}$ such that $(0, 0)$, $(1, 0)$, $(0, 1) \in f(X)$.*

Proof By Lemmas 3.6 and 3.5 we may assume that $\phi_2^{-1}(0, 0) \cap X \neq \emptyset$, $\phi_2^{-1}(0, 1) \cap X \neq \emptyset$ and $\phi_2^{-1}(1, 0) \cap X \neq \emptyset$. Since $|X| > 8$ we may assume that $|\phi_2^{-1}(1, 0) \cap X| \geq 3$, by Lemma 3.5. We may also assume that $(0, 0) \in X$. For if not, then we may translate X by a vector $[-u, -v]$ for some $(u, v) \in \phi_2^{-1}(0, 0) \cap X$. Let $(a_x, a_y) \in \phi_2^{-1}(0, 1) \cap X$. Then possibilities for a_y are: (i) $a_y = p$, (ii) a_y is invertible in $\mathbb{Z}_{2p}$.

In the case (i), the assumption that $|X| > p + 3$ and by Lemma 3.7 imply that $a_x \neq 0$. Hence $a_x b_y - b_x a_y \neq p$ for all $(b_x, b_y) \in \phi_2^{-1}(1, 0) \cap X$.

In the case (ii), the assumption that $|\phi_2^{-1}(1, 0) \cap X| \geq 3$ and Lemma 3.8 imply that there is $(b_x, b_y) \in \phi_2^{-1}(1, 0) \cap X$ such that $a_x b_y - b_x a_y \neq p$.

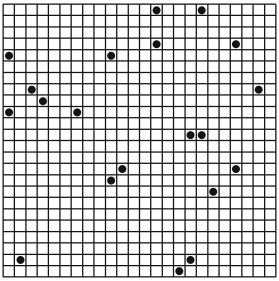

Fig. 3 A complete arc of cardinality 20 over $\mathbb{Z}_{24}^2$

```
// GLOBALS: N - size of the torus, best_solution
void select_point(int x, int y, Torus& torus, Solution& solution) {
    torus.set_IN(x,y);
    solution.add_point(x,y);
    if(solution.size() > best_solution.size()) {
        best_solution = solution;
    }
    int x2 = (x+N/2)%N, y2 = (y+N/2)%N;
    torus.set_OUT(x,y2);
    torus.set_OUT(x2,y);
    torus.set_OUT(x2,y2);
    for(int i=0; i<solution.size()-1; ++i) {
        torus.mark_all_lines_through_two_points(x, y,
                    solution.get(i).getx(), solution.get(i).gety());
    }
}
void attempt(int x, int y, Torus torus, Solution solution) {
    select_point(x,y,torus,solution);
    if(solution.size()+torus.number_of_FREE()<=best_solution.size()) {
        return; // backtrack
    }
    while(torus.find_next_FREE(x,y)) {
        attempt(x,y,torus,solution);
        torus.set_OUT(x,y);
    }
}
int main() {
    Torus torus; Solution solution;
    select_point(0,0,torus,solution);
    select_point(1,0,torus,solution);
    select_point(0,1,torus,solution);
    int x = 1, y = 1;
    while(torus.find_next_FREE(x,y)) {
        attempt(x,y,torus,solution);
        torus.set_OUT(x,y);
        torus.set_OUT(y,x);
    }
}
```

Fig. 4 Backtracking search

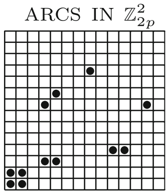

Fig. 5 A complete arc of cardinality 12 over $\mathbb{Z}_{14}^2$

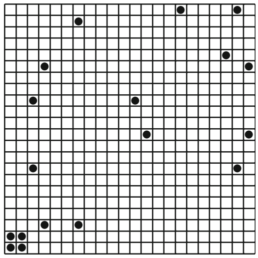

Fig. 6 A complete arc of cardinality 18 over $\mathbb{Z}_{22}^2$

Table 1 Values for $\tau(\mathbb{Z}_m)$

m	4	6	8	9	10	12	14	15	16	18	20	21	22	24	25
$\tau(m)$	6	8	8	9	12	12	12	15	14	17	18	18	**18**	**20**	20

In both cases, there is $(b_x, b_y) \in \phi_2^{-1}(1, 0) \cap X$ such that $a_x b_y - b_x a_y$ is invertible in $\mathbb{Z}_{2p}$. Then the map f_A with matrix $A = \frac{1}{a_x \cdot b_y - b_x \cdot a_y} \begin{pmatrix} b_y & -b_x \\ -a_y & a_x \end{pmatrix}$ maps

$$\begin{aligned} (a_x, a_y) &\longrightarrow (1, 0), \\ (b_x, b_y) &\longrightarrow (0, 1), \\ (0, 0) &\longrightarrow (0, 0). \end{aligned}$$

□

4 Numerical results

Computing $\tau(\mathbb{Z}_{24}^2)$ was quite easy. We used the mathematical programming solver Gurobi. After expanding 12946227 nodes (602109089 simplex iterations) the solver found the solution depicted in Fig. 3. On the other hand, using the solver to compute $\tau(\mathbb{Z}_{22}^2)$ was not successful.

Consequently, we used a more direct approach. For finding the value of $\tau(\mathbb{Z}_{2p}^2)$ we implemented backtracking search algorithm depicted in Fig. 4. Figures 2, 5 and 6 show results of our search. Note that it is easy to find these complete arcs. The difficult part is showing that there are no bigger ones.

To limit searching space we made some optimizations. Thanks to Theorem 3.9 we start with the set (0, 0), (1, 0), (0, 1) (lines 29–31). Since the initial set is symmetric,

Table 2 Bounds for $\tau(\mathbb{Z}_m)$

m	26	27	28	30	32	33	34	35	36	38	39	40
$\tau(m)$	20–28	20–28	22–32	23–30	23–35	23–36	22–36	26–40	25–36	23–40	25–42	26–40

Table 3 Examples of arcs

m	k	An arc of cardinality k over $\mathbb{Z}_m^2$
26	20	(0, 16) (0, 25) (2, 19) (2, 22) (4, 4) (4, 20) (5, 4) (7, 22) (8, 10) (9, 0) (11, 7) (13, 18) (13, 23) (15, 11) (16, 23) (18, 7) (21, 0) (22, 9) (25, 9) (25, 11)
27	20	(0,1) (0, 9) (2, 0) (2, 1) (3, 3) (6, 3) (6, 11) (7, 10) (13, 9) (13, 24) (15, 22) (16, 14) (16, 16) (18, 8) (20, 4) (20, 6) (21, 10) (23, 2) (23, 5) (25, 8)
28	22	(0, 21) (1, 22) (4, 23) (9, 13) (10, 7) (11, 8) (11, 17) (14, 22) (14, 27) (15, 20) (15, 23) (16, 14) (17, 2) (17, 20) (22, 12) (23, 17) (25, 4) (25, 7) (26, 12) (26, 13) (27, 2) (27, 18)
30	23	(4, 12) (5, 12) (5, 15) (6, 1) (6, 8) (7, 5) (7, 18) (9, 3) (13, 4) (13, 15) (14, 25) (17, 14) (18, 27) (19, 1) (19, 14) (20, 11) (20, 18) (21, 4) (21, 7) (22, 2) (23, 23) (26, 10) (27, 6)
32	23	(1, 22) (3, 13) (4, 15) (5, 12) (5, 22) (6,0) (6, 26) (7, 16) (11, 15) (12, 8) (14, 1) (16, 29) (16, 31) (18, 8) (19, 18) (24, 26) (25, 12) (26, 2) (26, 19) (28, 13) (29, 31) (31, 21) (31, 23)
33	23	(3, 7) (5, 22) (9, 15) (10, 27) (12, 7) (12, 26) (15, 27) (17, 25) (17, 26) (19, 10) (21, 0) (22, 5) (22, 32) (23, 5) (25, 12) (28, 0) (29, 9) (29, 21) (30, 22) (30, 23) (31,8) (31,31) (32,32)
34	22	(4,1) (4,19) (7,4) (7,8) (9,8) (12,7) (12,27) (16,3) (16, 12) (17, 11) (18, 0) (18, 14) (19, 10) (25, 5) (25, 19) (27, 30) (28, 30) (30, 10) (31, 3) (31, 13) (33, 2) (33, 7)
35	26	(0, 6) (6, 1) (6, 18) (8, 26) (9, 19) (9, 20) (12, 25) (17, 8) (17, 17) (18, 33) (19, 16) (19, 29) (21, 30) (23, 2) (23, 4) (25, 18) (25, 34) (26, 19) (27, 6) (29, 2) (30, 7) (30, 10) (31, 32) (32, 24) (33, 20) (34, 33)
36	25	(5, 24) (6, 9) (6, 32) (9, 25) (11, 24) (12, 21) (15, 7) (15, 13) (16, 2) (16, 25) (17, 30) (19, 27) (20, 35) (22, 1) (24, 8) (25, 9) (25, 15) (28, 32) (29, 4) (30, 2) (30, 27) (34, 19) (34, 26) (35, 4) (35, 18)
38	23	(1, 1) (6, 26) (6, 31) (7, 10) (7, 17) (8, 16) (8, 24) (9, 12) (10, 15) (10, 33) (11, 0) (11, 9) (12, 28) (14, 7) (16, 9) (21, 6) (21, 22) (23, 3) (26, 15) (26, 16) (33, 6) (33, 31) (35, 7)
39	25	(0, 12) (0, 31) (1, 9) (4, 13) (4, 32) (5, 10) (6, 14) (7, 32) (8, 0) (9, 33) (11, 11) (11, 14) (12, 30) (13, 30) (14, 1) (14, 24) (17, 2) (18, 17) (20, 10) (21, 38) (22, 9) (23, 33) (28, 4) (28, 37) (33, 28)
40	26	(0, 1) (0, 10) (1, 0) (2, 6) (3, 8) (3, 11) (4, 15) (7, 0) (7, 12) (9, 21) (10, 29) (10, 32) (11, 34) (12, 21) (21, 18) (23, 25) (24, 14) (26, 27) (29, 38) (32, 28) (32, 29) (33, 27) (36, 8) (38, 6) (39, 19) (39, 37)

after checking the fourth point (x, y) (line 34) we can exclude (y, x) from further search (line 35 and 36). After choosing a point (x, y) (line 3 and 4) we can also exclude points $t_{[p,0]}(a)$, $t_{[0,p]}(a)$ and $t_{[p,p]}(a)$ from further search (lines 8–11 and Lemma 3.7). The program presented here is simplified, the real computations were performed in parallel.

For reference we present in Table 1 all known values of $\tau(\mathbb{Z}_m^2)$ for nonprime m (bold numbers are computed by us). Recall that $\tau(\mathbb{Z}_2^2) = 4$ and $\tau(\mathbb{Z}_p^2) = p + 1$ for primes $p > 2$. We also attach a table with some lower and upper bounds for nonprime $26 \leq n \leq 40$ found by computer search or application of Lemma 2.3 (Table 2). Note that with the exception of $n = 3^3$ and $n = 2^5$, upper bounds are derived from Lemma 2.3. We were not able to improve them using numerical computations. The corresponding examples for the lower bounds can be found in Table 3.

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