A Note on Partial Insurance and the Arrow-Pratt Measure of Risk Aversion

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Abstract

Pratt [1964] establishes that a more risk-averse individual in the Arrow-Pratt sense has a higher compensating risk premium for full insurance, but no comparable result has been established for partial insurance. Ross [1981] shows that a more risk-averse individual in the Arrow-Pratt sense may not be willing to pay more for a reduction in risk in the sense of mean-preserving contraction. We show that a more risk-averse individual in the Arrow-Pratt sense has a higher compensating risk premium for all empirically relevant forms of partial insurance because they induce a reduction in risk in the stronger sense of Bickel and Lemann [1979].

Key words: partial insurance, Arrow-Pratt measure, risk aversion, Bickel-Lemann

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1. Introduction

The classic paper of Pratt [1964] establishes that a more risk-averse individual in the Arrow-Pratt sense has a higher compensating risk premium for full insurance, i.e., he is always willing to pay more to completely eliminate all the risks he is facing. There is no comparable result established, however, for partial insurance, which is virtually universal.¹ Ross [1981] gives an example showing that if individuals have the same random initial wealth, then a more risk-averse individual in the Arrow-Pratt sense may not be willing to pay more to avoid an additional risk which is uncorrelated with the initial wealth. Since adding an uncorrelated risk induces a mean-preserving spread (MPS) [Rothschild and Stiglitz, 1970], Ross's example says that a more risk-averse individual may not be willing to pay more for a reduction in risk in the sense of mean-preserving contraction. Apparently equating partial insurance to any reduction in risk, Ross [1981, p. 625] interprets it as saying that "B's premium for partial insurance exceeds A's" even though A is more risk-averse than B in the Arrow-Pratt sense. It is therefore not clear whether a more risk-averse individual might not be willing to pay more for an insurance policy with such features as a deductible or coinsurance, which is typical of policies giving partial coverage.

Landsberger and Meilijson [1994a, b] shows that a more risk-averse individual (in the Arrow-Pratt sense) is always willing to pay more for a reduction in risk in the sense that the distribution becomes "less dispersed" as defined by Bickel and Lemann [1979]. We

show that if individuals' initial wealth is non-random or if all the risks faced by individuals are (at least partially) insurable as one single composite risk, then any non-decreasing indemnity schedules for which the amount of loss uncovered is also non-decreasing induce a reduction in risk in the sense of Bickel and Lemann [1979]. Since such restrictions on the indemnity schedules are met by those used in practice, whether they incorporate a deductible or coinsurance or an upper limit or any combination of these features, a more risk-averse individual is shown to have a higher compensating risk premium for all empirically relevant forms of partial insurance.

2. Preliminaries

In what follows, $\Omega = \{\omega\}$ denotes the set of states of nature, and random variables, denoted by $\tilde{x}$, $\tilde{y}$ etc., are functions from Ω to R . The associated distribution functions are denoted by F , G , H , etc.

Define the risk premia of individuals u and v , π^U and π^V respectively, by

$$Eu(\tilde{w} - \pi^U) = Eu(\tilde{w} + \tilde{\epsilon}) \quad \text{and} \quad Ev(\tilde{w} - \pi^V) = Ev(\tilde{w} + \tilde{\epsilon}) \quad (1)$$

where $\tilde{w}$ denotes their initial wealth and $\tilde{\epsilon}$ is a zero-mean risk.

Pratt [1964] shows that if $\tilde{w}$ is degenerate, then v being more risk-averse than u in the Arrow-Pratt sense, i.e.,

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} \quad \text{for all } x$$

implies $\pi^V \geq \pi^U$. Ross [1981], on the other hand, shows that if $\tilde{w}$ is not degenerate and $\tilde{w}$ and $\tilde{\epsilon}$ are uncorrelated or equivalently $(\tilde{w} + \tilde{\epsilon})$ is an MPS from $\tilde{w}$, then the result no longer holds. Specifically if (in a lottery form)

$$\tilde{w} = [(w_1, p)(w_2, 1 - p)], \quad \tilde{w} + \tilde{\epsilon} = \left[\left(w_1 - \epsilon, \frac{1}{2}p \right), \left(w_1 + \epsilon, \frac{1}{2}p \right), \right. \\ \left. (w_2, 1 - p) \right]$$

where $w_1 > w_2$, $w_1 - w_2 > \epsilon$ and $p \in (0, 1)$ is a probability, then a more risk-averse individual in the Arrow-Pratt sense may not be willing to pay more for the elimination of $\tilde{\epsilon}$.

Other authors, on the other hand, advocate using stronger notions of increasing risk. One such stronger notion is first proposed by Bickel and Lemann [1979].

Definition 1: *The distribution F is Bickel-Lehmann less dispersed than the distribution G if for all $0 < p < q < 1$, $F^{-1}(q) - F^{-1}(p) \leq G^{-1}(q) - G^{-1}(p)$.*

Landsberger and Meilijson [1994b] shows that the notion is closely related to two random variables being "comonotonic".

Definition 2: Two random variables $\tilde{x}$ and $\tilde{z}$ are comonotonic if for any $\omega, \omega' \in \Omega$, $(\tilde{x}(\omega) - \tilde{x}(\omega'))(\tilde{z}(\omega) - \tilde{z}(\omega')) \geq 0$.

Proposition 1 (Landsberger and Meilijson [1994b]): If F and G are the distribution functions of random variables $\tilde{x}$ and $\tilde{y}$ respectively, then F is Bickel-Lehmann less dispersed than G if and only if there exists a random variable $\tilde{z}$ such that $\tilde{x} + \tilde{z} = \tilde{y}$ and $\tilde{x}$ and $\tilde{z}$ are comonotonic.

Landsberger and Meilijson [1994b] further shows that a more risk averse individual is always willing to pay more for a reduction in risk in the sense of Bickel and Lemann (1979).

Proposition 2 (Landsberger and Meilijson [1994b]): Given that v and u are increasing and π^U and π^V are as defined in (1),

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} \quad \text{for all } x \text{ implies } \pi^V \geq \pi^U$$

if and only if $\tilde{w}$ and $\tilde{\epsilon}$ are comonotonic.

It is also shown that if v and u are increasing and concave, then $\tilde{w}$ and $\tilde{\epsilon}$ being comonotonic is sufficient but not necessary.

3. Willingness to pay for partial insurance

To consider a setting similar to Raviv [1979] and others in the optimal insurance literature, denote individuals' initial wealth by w and the possible loss by a random variable $\tilde{l}$. Let F be the distribution function for $\tilde{l}$, with support in the interval $[0, \bar{l}]$. Now suppose an insurance policy becomes available for $\tilde{l}$ whose benefit (indemnity) schedule is given by $I(\tilde{l})$. Then the maximum premium an individual u is willing to pay for the policy, P^U , is given by

$$Eu(w - \tilde{l} + I(\tilde{l}) - P^U) = Eu(w - \tilde{l})$$

Is a more risk-averse individual v always willing to pay more for the same insurance policy given the same initial wealth and risk? That is, if we define P^V by

$$Ev(w - \tilde{l} + I(\tilde{l}) - P^V) = Ev(w - \tilde{l})$$

then is it true that

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} \quad \text{for all } x \text{ implies } P^V \geq P^U?$$

The exposition in Ross [1980, p. 625] and others may give the impression that unless we have full insurance (i.e., $I(\tilde{l}) = \tilde{l}$), the proposition is not true in general. The proposition is indeed untrue if the only restriction placed on the function $I()$ is that $[w - \tilde{l}]$ is an MPS

from $[w - \tilde{l} + I(\tilde{l}) - EI(\tilde{l})]$. We can easily show the proposition is true, on the other hand, if we assume both $I(l)$ and $[l - I(l)]$ are non-decreasing (or $0 \leq I'(l) \leq 1$ in the case where $I()$ is differentiable), which, as is argued below, is true of all empirically relevant forms of partial insurance. Define

$$\tilde{x} = w - \tilde{l} + I(\tilde{l}) - EI(\tilde{l}), \quad \tilde{y} = w - \tilde{l}, \quad \text{and} \quad \tilde{z} = -I(\tilde{l}) + EI(\tilde{l}).$$

Then we have $\tilde{y} = \tilde{x} + \tilde{z}$ and $Eu(\tilde{x} - P^U + EI(\tilde{l})) = Eu(\tilde{y})$ and $Ev(\tilde{x} - P^V + EI(\tilde{l})) = Ev(\tilde{y})$. Since $\tilde{x}$ and $\tilde{z}$ are non-increasing in $\tilde{l}$ under our assumptions, they are clearly comonotonic (i.e., for any $\omega, \omega' \in \Omega$, $(\tilde{x}(\omega) - \tilde{x}(\omega'))(\tilde{z}(\omega) - \tilde{z}(\omega')) \geq 0$). We thus have

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} \quad \text{for all } x \Rightarrow [P^V - EI(\tilde{l})] \geq [P^U - EI(\tilde{l})] \Leftrightarrow P^V \geq P^U$$

Now consider the three most common forms of partial insurance. A policy involving coinsurance means $I(\tilde{l}) = \alpha \tilde{l}$ for some $\alpha \in (0, 1)$. In policies with a deductible, $I(\tilde{l}) = 0$ if $\tilde{l} \leq d$ and $\tilde{l} = \tilde{l} - d$ otherwise, where $d > 0$ is the deductible. A policy with an upper limit means $I(\tilde{l}) = \tilde{l}$ if $\tilde{l} \leq L$ and $\tilde{l} = L$ otherwise, where $L \geq 0$ is the upper limit. Clearly in all these cases and indeed in cases with a combination of any of these three features, both $I(l)$ and $[l - I(l)]$ are non-decreasing. More generally it can be taken as a positive property of an insurance policy that the benefit is not lower when the loss is higher (i.e., $I(l)$ is non-decreasing) and the benefit does not over-compensate the loss (i.e., $[I(l) - l]$ is non-increasing).² Alternatively, following the analysis of optimal insurance design by Raviv [1979] and others, it is easy to establish that both $I(l)$ and $[l - I(l)]$ being non-decreasing is a necessary condition for an optimal insurance policy (and for a profit-maximizing one as well).³ That is, both normatively and positively, an insurance policy that gives partial coverage should and factually does achieve a reduction in risk in a stronger sense than that of MPS.

While we observe these positive and normative appeals of the condition that both $I(l)$ and $[l - I(l)]$ being non-decreasing, it can in fact be weakened for a more risk-averse individual to be still always willing to pay more for $I(l)$. To see that, suppose the random variable $u(w - \tilde{l})$ is an MPS from the random variable $u(w - \tilde{l} + I(\tilde{l}) - P^U)$ or equivalently $[w - \tilde{l}]$ is a “mean-utility preserving spread” from $[w - \tilde{l} + I(\tilde{l}) - P^U]$ from the viewpoint of u as is defined by Diamond and Stiglitz [1974]. Then (by Theorem 3 of Diamond and Stiglitz [1974]), v being more risk-averse than u in the Arrow-Pratt sense implies $Ev(w - \tilde{l} + I(\tilde{l}) - P^U) \geq Ev(w - \tilde{l})$. We therefore have $P^V \geq P^U$ given that $Ev(w - \tilde{l} + I(\tilde{l}) - P)$ is decreasing in P . That is, we only need $I(l)$ to be such that $u(w - \tilde{l})$ is an MPS from $u(w - \tilde{l} + I(\tilde{l}) - P^U)$, which is implied by, but does not imply, both $I(l)$ and $[l - I(l)]$ being non-decreasing. That said, a weaker and equally easy-to-interpret condition on $I(l)$ does not seem available.⁴

We have thus clarified the issue of whether a more risk-averse individual is always willing to pay more for partial insurance and put in perspective the earlier contributions of Pratt [1964] and Ross [1981]. All our arguments above clearly apply if the individuals face more than one risks, say $\tilde{l}_1$ and $\tilde{l}_2$, but the usual forms of partial insurance coverage are available for the composite risk $\tilde{l}_1 + \tilde{l}_2$.

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Notes

1. The market may offer only partial insurance for insurable risks due to either transaction costs or moral hazard.
2. Ross's counterexample can be understood in this light. Define $w \equiv w_1 + \epsilon$ and $\tilde{l} \equiv [(0, \frac{1}{2}p), (2\epsilon, \frac{1}{2}p), (\epsilon + w_1 - w_2, 1 - p)]$. Then the indemnity schedule that will eliminate $\tilde{\epsilon}$ is

$$I(l) = \begin{cases} -\epsilon, & l = 0 \\ 2\epsilon, & l = 2\epsilon \\ 0 & l = \epsilon + w_1 - w_2 \end{cases}$$

which clearly does not meet the condition that both $I(l)$ and $[l - I(l)]$ being non-decreasing. Furthermore, a negative payout is also a feature not observed in indemnity schedules used in practice. These observations of Ross's example are suggested to me by an anonymous referee.

3. See for example Theorem 1 in Raviv [1979].
4. Let $X(\tilde{l}) \equiv u(w - \tilde{l})$ and $Y(\tilde{l}) \equiv u(w - \tilde{l} + I(\tilde{l}) - P^U)$, $F_X(x)$ and $F_Y(x)$ be the cumulative distribution functions of $X(\tilde{l})$ and $Y(\tilde{l})$ respectively and $f(l)$ the density function of $\tilde{l}$. Then $I(l)$ and $[l - I(l)]$ both being non-decreasing implies the existence of $l^* \in (0, \tilde{l})$ such that $I(l^*) = P^U$ and $X(l) \geq Y(l) \geq x^* \equiv X(l^*) = Y(l^*)$ for $l \leq l^*$ and $X(l) \leq Y(l) \leq x^*$ for $l \geq l^*$. We therefore have $F_X(x) \leq F_Y(x)$ for $x \leq x^*$ and $F_X(x) \geq F_Y(x)$ for $x \geq x^*$ because $F_X(x) = \int_{\{l: X(l) \leq x\}} f(l)dl$ and $F_Y(x) = \int_{\{l: Y(l) \leq x\}} f(l)dl$ and $\{l: Y(l) \leq x\} \subseteq \{l: X(l) \leq x\}$ for $x \leq x^*$ and $\{l: X(l) \leq x\} \subseteq \{l: Y(l) \leq x\}$ for $x \geq x^*$. That is, $F_X(x)$ single-crosses $F_Y(x)$ from above. Then if $EX(\tilde{l}) = EY(\tilde{l})$, $F_X(x)$ is in fact a *simple* MPS from $F_Y(x)$ (Diamond and Stiglitz [1974, p. 338]).

Notice in particular that $[l - I(l)]$ being non-decreasing (which makes $Y(l)$ non-increasing) guarantees that $Y(l) \geq x^*$ for $l \leq l^*$ and $Y(l) \leq x^*$ for $l \geq l^*$. While this is not necessary for $X(\tilde{l})$ to be an MPS from $Y(\tilde{l})$, some form of restriction on $I(l)$ other than its being non-decreasing is needed. For illustration, if $I(l)$ is differentiable and $I'(l)$ is much larger than 1, say $I(l) = 3l$, then, though we still have $X(l) \geq Y(l)$ for $l \leq l^*$ and $X(l) \leq Y(l)$ for $l \geq l^*$, since $Y(l)$ will be increasing in l at a higher rate than the rate of decrease of $X(l)$, we will have $Y(0) < X(\tilde{l})$ and $Y(\tilde{l}) > X(0)$ and $X(\tilde{l})$ is hence clearly not an MPS from $Y(\tilde{l})$.

An alternative additional restriction on $I(l)$ would be that $I(l) \leq l$. That is, it can be shown that $I(l) \leq l$, together with $I(l)$ being non-decreasing in l , is sufficient for $X(\tilde{l})$ to be an MPS from $Y(\tilde{l})$. While this alternative set of conditions is also satisfied by practically all real-world insurance policies, it is not weaker than the condition that both $I(l)$ and $[l - I(l)]$ are non-decreasing and it does not relate immediately to an established stronger sense of risk reduction. The former point is apparent from observing that in the differentiable case $I'(l) \leq 1$ implies $I(l) \leq l$ only if $I(0) \leq 0$, which is not required for the proof above using Bickel-Lehmann's concept of increasing risk. Importantly Bickel-Lehmann's concept of increasing risk has the property of being "location-independent" and hence it does not matter if $I(l)$ has a positive or negative "intercept".

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