A classification of pentavalent arc-transitive bicirculants

Iva Antončič · Ademir Hujdurović ·
Klavdija Kutnar

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Abstract A *bicirculant* is a graph admitting an automorphism with two cycles of equal length in its cycle decomposition. A graph is said to be *arc-transitive* if its automorphism group acts transitively on the set of its arcs. All cubic and tetravalent arc-transitive bicirculants are known, and this paper gives a complete classification of connected pentavalent arc-transitive bicirculants. In particular, it is shown that, with the exception of seven particular graphs, a connected pentavalent bicirculant is arc-transitive if and only if it is isomorphic to a Cayley graph $\text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ on the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$, where $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Keywords Bicirculant · Vertex-transitive · Edge-transitive · Arc-transitive · Automorphism group

I. Antončič · A. Hujdurović · K. Kutnar (✉)
University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
e-mail: klavdija.kutnar@upr.si

I. Antončič
e-mail: iva.antoncic@upr.si

A. Hujdurović
e-mail: ademir.hujdurovic@upr.si

A. Hujdurović · K. Kutnar
University of Primorska, IAM, Muzejski trg 2, 6000 Koper, Slovenia

1 Introductory remarks

For a finite simple graph X let $V(X)$, $E(X)$, $A(X)$, and $\text{Aut}(X)$ be its vertex set, its edge set, its arc set, and its automorphism group, respectively. A graph is said to be *arc-transitive* if its automorphism group acts transitively on the set of arcs of the graph. A non-identity automorphism is *semiregular*, in particular (m, n) -*semiregular* if it has m cycles of equal length n in its cycle decomposition. An n -*bicirculant* (*bicirculant*, in short) is a graph admitting a $(2, n)$ -semiregular automorphism.

The classification of connected cubic arc-transitive bicirculants follows from results obtained in [6, 23, 26] whereas the classification of connected tetravalent arc-transitive bicirculants follows from results obtained in [15, 16, 18]. Motivated by the recent classification of arc-transitive Tabačjn graphs [1] this paper aims to give a complete classification of all connected pentavalent arc-transitive bicirculants, see Theorem 1.1. (Tabačjn graphs are a natural generalization of the generalized Petersen graphs and the rose window graphs [29] to graphs of valency 5; precise definition is given below.)

The existence of a $(2, n)$ -semiregular automorphism in a bicirculant enables us to label its vertex set and edge set in the following way. Let X be a connected n -*bicirculant*. Then there exists a $(2, n)$ -semiregular automorphism $\rho \in \text{Aut}(X)$ and the vertices of X can be labeled by x_i and y_i with $i \in \mathbb{Z}_n$, such that $\rho = (x_0 x_1 \cdots x_{n-1})(y_0 y_1 \cdots y_{n-1})$. Moreover, the edge set $E(X)$ can be partitioned into three subsets

$$\begin{aligned}\mathcal{L} &= \bigcup_{i \in \mathbb{Z}_n} \{\{x_i, x_{i+l}\} \mid l \in L\} \quad (\text{left edges}), \\ \mathcal{M} &= \bigcup_{i \in \mathbb{Z}_n} \{\{x_i, y_{i+m}\} \mid m \in M\} \quad (\text{middle (or spoke) edges}), \\ \mathcal{R} &= \bigcup_{i \in \mathbb{Z}_n} \{\{y_i, y_{i+r}\} \mid r \in R\} \quad (\text{right edges}),\end{aligned}$$

where L , M , and R are subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$, and $0 \notin L \cup R$. We shall denote this graph by $BC_n[L, M, R]$ (this notation has been introduced in [18]). The vertices x_i , $i \in \mathbb{Z}_n$, will be referred to as *left vertices* and vertices y_i , $i \in \mathbb{Z}_n$, will be referred to as *right vertices*. Observe that the generalized Petersen graph $GP(n, r)$ and the rose window graph $R_n(a, r)$ are isomorphic to $BC_n[\{\pm 1\}, \{0\}, \{\pm r\}]$ and $BC_n[\{\pm 1\}, \{0, -a\}, \{\pm r\}]$, respectively. Since the Tabačjn graph $T(n, a, b, r)$ mentioned above is isomorphic to $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$ we will say that a bicirculant $BC_n[L, M, R]$ with $|M| = 3$ is a *generalized Tabačjn graph*. (Note that in a generalized Tabačjn graphs the cycles induced by the edges from $\mathcal{L}$ and $\mathcal{R}$ are consistent cycles [24, 25].) In a pentavalent bicirculant $X = BC_n[L, M, R]$ with $|M| = 5$ a mapping τ defined with $\tau(x_i) = x_{-i}$ and $\tau(y_i) = y_{-i}$ is an automorphism of X , and the group $G = \langle \tau, \rho \rangle \cong D_{2n}$ acts regularly on $V(X)$, implying that X is a Cayley graph on the group G . Such graphs are called *dihedrants*.

We can now state the main theorem of this paper.

Theorem 1.1 *A connected pentavalent bicirculant $BC_n[L, M, R]$ is arc-transitive if and only if it is isomorphic to one of the following graphs:*

- (i) $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$, $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$;
- (ii) $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$;
- (iii) $BC_6[\emptyset, \{0, 1, 2, 3, 4\}, \emptyset]$, $BC_{12}[\emptyset, \{0, 1, 2, 4, 9\}, \emptyset]$, $BC_{21}[\emptyset, \{0, 1, 4, 14, 16\}, \emptyset]$, $BC_{24}[\emptyset, \{0, 1, 3, 11, 20\}, \emptyset]$, or $\text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$, where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$, and $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Remark 1.2 There are no connected pentavalent arc-transitive bicirculants with $|M| \in \{1, 4\}$. Theorem 1.1(i), (ii), and (iii) give pentavalent arc-transitive bicirculants with $|M| = 2$, $|M| = 3$, and $|M| = 5$, respectively. The graphs given in Theorem 1.1(ii) are Tabačjn graphs isomorphic to the complete graph K_6 , the complete bipartite graph minus a matching $K_{6,6} - 6K_2$ and the icosahedron, respectively. The graph $K_{6,6} - 6K_2$ occurs three times as this graph has three different bicirculant representations $K_{6,6} - 6K_2 \cong BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}] \cong BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}] \cong BC_6[\emptyset, \{0, 1, 2, 3, 4\}, \emptyset]$. The third graph in Theorem 1.1(iii) is isomorphic to the incidence graph $B(PG(2, 4))$ of the projective space $PG(2, 4)$, that is, the graph whose vertices are all 1- and 2-dimensional subspaces of the 3-dimensional vector space over the finite field $GF(4)$, and adjacencies are given by the inclusion.

The proof of Theorem 1.1 is carried out over Sect. 3, considering separately several cases depending on the number of the spoke edges. First, it is shown that no connected pentavalent arc-transitive bicirculant $BC_n[L, M, R]$ with $|M| = 1$ or $|M| = 4$ exists (see Theorem 3.2). Next, in Theorem 3.3, it is shown that there are only two connected pentavalent arc-transitive bicirculants with $|M| = 2$ (the graphs in Theorem 1.1(i)). Bicirculants with $|M| = 3$ and $|M| = 5$ (that is, the generalized Tabačjn graphs and the dihedrants) are considered using the same approach as it was used in [5, 17] and [1] in the framework of cubic arc-transitive graphs admitting (m, n) -semiregular automorphisms for $m \in \{3, 4, 5\}$ and arc-transitive Tabačjn graphs, respectively. In particular, pentavalent arc-transitive bicirculants with $|M| = 3$ and $|M| = 5$ are classified by considering first the so-called *core-free bicirculants*, that is, the bicirculants in which the subgroup generated by a $(2, n)$ -semiregular automorphism ρ is core-free in the full automorphism group of the graph. A group-theoretic result of Lucchini [13, Theorem 2.20], which says that “sufficiently large” cyclic subgroups are never core-free (see Proposition 2.2), combined together with a result which gives the upper bounds for the order of the automorphism group of a pentavalent arc-transitive graph (see Proposition 2.1), enable us to determine all core-free pentavalent arc-transitive bicirculants with $|M| \in \{3, 5\}$. As for non-core-free pentavalent arc-transitive bicirculants, we use the fact that any such graph is a regular cyclic cover either of a core-free pentavalent arc-transitive bicirculant or of a dipole with five parallel edges. This then enables us to use graph covering techniques to classify all pentavalent arc-transitive bicirculants.

2 Preliminaries

In this section, some notation and terminology is introduced and certain results, essential to the strategy of the proof of Theorem 1.1, are gathered.

Throughout this paper graphs are simple, finite, undirected, and connected. A sequence of $s + 1$ not necessarily distinct vertices of a graph X such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct is called an s -arc. If $v \in V(X)$ then $N(v)$ denotes the set of neighbors of v . The *girth* of X is the length of a shortest cycle contained in X .

2.1 Groups and graphs

Let X be a graph. A subgroup $G \leq \text{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* provided it acts transitively on the sets of vertices, edges, and arcs of X , respectively. In this case, the graph X is said to be G -*vertex-transitive*, G -*edge-transitive*, and G -*arc-transitive*, respectively. In case of $G = \text{Aut}(X)$, the symbol G is omitted. An arc-transitive graph is also called *symmetric*. A subgroup $G \leq \text{Aut}(X)$ is said to be s -*arc-transitive* if it acts transitively on the set of s -arcs of X , and it is said to be s -*regular* if it is s -arc-transitive and the stabilizer of an s -arc in G is trivial. A graph X is said to be (G, s) -*arc-transitive* and (G, s) -*regular* if G is transitive and regular on the set of s -arcs of X , respectively. A (G, s) -arc-transitive graph is said to be (G, s) -*transitive* if the graph is not $(G, s + 1)$ -arc-transitive. By Weiss [27, 28], for a pentavalent (G, s) -transitive graph, $s \geq 1$, the order of the vertex stabilizer G_v in G is a divisor of $2^{17} \cdot 3^2 \cdot 5$. In addition, the following result can be deduced from his work, as was recently observed by Guo and Feng [10].

Proposition 2.1 [10, Theorem 1.1] *Let X be a connected pentavalent (G, s) -transitive graph for some $G \leq \text{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:*

- (i) For $s = 1$, $G_v \cong \mathbb{Z}_5$, D_{10} or D_{20} ;
- (ii) For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5 ;
- (iii) For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$, or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;
- (iv) For $s = 4$, $G_v \cong \text{ASL}(2, 4)$, $\text{AGL}(2, 4)$, $\text{A}\Sigma\text{L}(2, 4)$ or $\text{A}\Gamma\text{L}(2, 4)$;
- (v) For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$.

Recall that the *core* of a subgroup K in a group G (denoted by $\text{core}_G(K)$) is the largest normal subgroup of G contained in K . The following group-theoretic result of Lucchini [13, Theorem 2.20] will be needed in Sect. 3 (see also [11, Theorem B]).

Proposition 2.2 *If H is a cyclic subgroup of a finite group G with $|H| \geq \sqrt{|G|}$, then H contains a non-trivial normal subgroup of G .*

Fig. 1 Multigraphs that can occur as quotient multigraphs of pentavalent bicirculants with respect to a $(2, n)$ -semiregular automorphism ρ

2.2 Bicirculants

We first introduce a nice tool that visualizes the structure of a bicirculant. (This tool can be used for any graph admitting a (m, n) -semiregular automorphism, but we will only define it in the framework of bicirculants.) Let X be a bicirculant $BC_n[L, M, R]$ with a $(2, n)$ -semiregular automorphism $\rho \in \text{Aut}(X)$. Let $\mathcal{W} = \{W, W'\}$ be the set of orbits of $\langle \rho \rangle$. Then clearly the subgraph of X induced on W (as well as the subgraph induced on W') and the bipartite subgraph of X induced by the edges having one endvertex in W and the other endvertex in W' are regular. We let $d(W)$ and $d(W, W')$ denote the valency of these subgraphs. (Observe that $d(W, W') = |M|$.) We let the *quotient multigraph* X_ρ corresponding to ρ be the multigraph whose vertex set consists of $\mathcal{W}$, the two vertices $W, W' \in \mathcal{W}$ are joined by $d(W, W')$ edges, and at a vertex $W \in \mathcal{W}$ there are $d(W)/2$ loops if $d(W)$ is even and $(d(W) - 1)/2$ loops and one semiedge if $d(W)$ is odd. In case X is a connected pentavalent bicirculant, there are five different multigraphs on two vertices that potentially can occur as quotient multigraph of X . These five possibilities are shown in Fig. 1. However, we will prove in Theorem 1.1 that the multigraphs shown in Fig. 1a, d cannot occur as quotient multigraphs of arc-transitive bicirculants.

For a partition $\mathcal{W}$ of $V(X)$, we let $X_{\mathcal{W}}$ be the associated *quotient graph* of X relative to $\mathcal{W}$, that is, the graph with vertex set $\mathcal{W}$ and edge set induced naturally by the edge set $E(X)$. In the case when $\mathcal{W}$ corresponds to the set of orbits of a subgroup N of $\text{Aut}(X)$, the symbol $X_{\mathcal{W}}$ will be replaced by X_N . Observe that in case $\mathcal{W}$ is the set of orbits of a $(2, n)$ -semiregular automorphism ρ the quotient graph $X_{\mathcal{W}}$ is precisely the underlying graph of X_ρ (ignoring loops and semiedges).

A bicirculant $X = BC_n[L, M, R]$ of order $2n$ is said to be *core-free* if there exists a $(2, n)$ -semiregular automorphism $\rho \in \text{Aut}(X)$ such that the cyclic subgroup $\langle \rho \rangle$ has trivial core in $\text{Aut}(X)$. The following lemma is a straightforward generalization of [19, Theorem 9].

Lemma 2.3 *Let $X = BC_n[L, M, R]$ be a pentavalent arc-transitive bicirculant with a $(2, n)$ -semiregular automorphism $\rho \in \text{Aut}(X)$, and let N be the core of $\langle \rho \rangle$ in $\text{Aut}(X)$. Then N is the kernel of $\text{Aut}(X)$ acting on the set of orbits of N , $\text{Aut}(X)/N$ acts arc-transitively on X_N , and either*

- (i) X_N is a core-free pentavalent arc-transitive bicirculant with a $(2, n/|N|)$ -semiregular automorphism $\bar{\rho}$ and $X_\rho \cong (X_N)_{\bar{\rho}}$, or
- (ii) X_N is isomorphic to the dipole $\mathcal{D}_5$ with five parallel edges and $X_\rho \cong \mathcal{D}_5$.

We end this subsection with some obvious isomorphisms between bicirculants, which will be used in our further analysis.

Proposition 2.4 [18] *Let L , M , and R be subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$, and $0 \notin L \cap R$. Then*

$$BC_n[L, M, R] \cong BC_n[\lambda L, \lambda M + \mu, \lambda R] \quad (\lambda \in \mathbb{Z}_n^*, \mu \in \mathbb{Z}_n)$$

with the isomorphism $\phi_{\lambda, \mu}$ given by $\phi_{\lambda, \mu}(x_i) = x_{\lambda i + \mu}$ and $\phi_{\lambda, \mu}(y_i) = y_{\lambda i}$.

2.3 Graph covers

A *covering projection* of a graph $\tilde{X}$ is a surjective mapping $p: \tilde{X} \rightarrow X$ such that for each $\tilde{u} \in V(\tilde{X})$ the set of arcs emanating from $\tilde{u}$ is mapped bijectively onto the set of arcs emanating from $u = p(\tilde{u})$. The graph $\tilde{X}$ is called a *covering graph* of the *base graph* X . The set $\text{fib}_u = p^{-1}(u)$ is the *fiber* of the vertex $u \in V(X)$. The subgroup K of all automorphisms of $\tilde{X}$ which fix each of the fibers setwise is called the *group of covering transformations*. The graph $\tilde{X}$ is also called a *K-cover* of X . It is a simple observation that the group of covering transformations of a connected covering graph acts semiregularly on each of the fibers. In particular, if the group of covering transformations is regular on the fibers of $\tilde{X}$, we say that $\tilde{X}$ is a *regular K-cover*. We say that $\alpha \in \text{Aut}(X)$ *lifts* to an automorphism of $\tilde{X}$ if there exists an automorphism $\tilde{\alpha} \in \text{Aut}(\tilde{X})$, called a *lift* of α , such that $\tilde{\alpha}p = p\alpha$. If the covering graph $\tilde{X}$ is connected then K is the lift of the trivial subgroup of $\text{Aut}(X)$. Note that a subgroup $G \leq \text{Aut}(\tilde{X})$ projects if and only if the partition of $V(\tilde{X})$ into the orbits of K is G -invariant.

A combinatorial description of a K -cover was introduced through so-called voltages by Gross and Tucker [9] as follows. Let X be a graph and K be a finite group. A *voltage assignment* on X is a mapping $\zeta: A(X) \rightarrow K$ with the property that $\zeta(u, v) = \zeta(v, u)^{-1}$ for any arc $(u, v) \in A(X)$ (here, and in the rest of the paper, $\zeta(u, v)$ is written instead of $\zeta((u, v))$ for the sake of brevity). The voltage assignment ζ extends to walks in X in a natural way. In particular, for any walk $D = u_1 u_2 \cdots u_t$ of X we let $\zeta(D)$ to denote the product $\zeta(u_1, u_2)\zeta(u_2, u_3) \cdots \zeta(u_{t-1}, u_t)$, that is, the ζ -voltage of D .

The values of ζ are called *voltages*, and K is the *voltage group*. The *voltage graph* $X \times_\zeta K$ derived from a voltage assignment $\zeta: A(X) \rightarrow K$ has vertex set $V(X) \times K$, and edges of the form $\{(u, g), (v, \zeta(x)g)\}$, where $x = (u, v) \in A(X)$. Clearly, $X \times_\zeta K$ is a covering graph of X with respect to the projection to the first coordinate. By letting K act on $V(X \times_\zeta K)$ as $(u, g)g' = (u, gg')$, $(u, g) \in V(X \times_\zeta K)$, $g' \in K$, one obtains a semiregular group of automorphisms of $X \times_\zeta K$, showing that $X \times_\zeta K$ can in fact be viewed as a K -cover of X .

Given a spanning tree T of X , the voltage assignment $\zeta : A(X) \rightarrow K$ is said to be T -reduced if the voltages on the tree arcs are trivial, that is, if they equal the identity element in K . In [8], it is shown that every regular covering graph $\tilde{X}$ of a graph X can be derived from a T -reduced voltage assignment ζ with respect to an arbitrary fixed spanning tree T of X .

The problem of whether an automorphism α of X lifts or not is expressed in terms of voltages as follows (see Proposition 2.5). Given $\alpha \in \text{Aut}(X)$ and the set of fundamental closed walks $\mathcal{C}$ based at a fixed vertex $v \in V(X)$, we define $\bar{\alpha} = \{(\zeta(C), \zeta(C^\alpha)) \mid C \in \mathcal{C}\} \subseteq K \times K$. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X . Also, from the definition, it is clear that for a T -reduced voltage assignment ζ the derived graph $X \times_\zeta K$ is connected if and only if the voltages of the cotree arcs generate the voltage group K .

We conclude this section with four propositions dealing with lifting of automorphisms in graph covers. The first one may be deduced from [20, Theorem 4.2], the second one from [12] whereas the third one is taken from [3, Proposition 2.2], but it may also be deduced from [21, Corollaries 9.4, 9.7, 9.8].

Proposition 2.5 [20] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K . Then an automorphism α of X lifts if and only if $\bar{\alpha}$ is a function which extends to an automorphism α^* of K .*

For a connected regular cover $X \times_\zeta K$ of a graph X derived from a T -reduced voltage assignment ζ with an abelian voltage group K and an automorphism $\alpha \in \text{Aut}(X)$ that lifts, $\bar{\alpha}$ will always denote the mapping from the set of voltages of the fundamental cycles on X to the voltage group K and α^* will denote the automorphism of K arising from $\bar{\alpha}$.

Two coverings $p_i : \tilde{X}_i \rightarrow X$, $i \in \{1, 2\}$, are said to be *isomorphic* if there exists a graph isomorphism $\phi : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $\phi p_2 = p_1$.

Proposition 2.6 [12] *Let K be a finite group. Two connected regular covers $X \times_\zeta K$ and $X \times_\varphi K$, where ζ and φ are T -reduced, are isomorphic if and only if there exists an automorphism $\sigma \in \text{Aut}(K)$ such that $\zeta(u, v)^\sigma = \varphi(u, v)$ for any cotree arc (u, v) of X .*

Proposition 2.7 [3] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K , and let the lifts of $\alpha \in \text{Aut}(X)$ centralize K , considered as the group of covering transformations. Then for any closed walk W in X , there exists $k \in K$ such that $\zeta(W^\alpha) = k\zeta(W)k^{-1}$. In particular, if K is abelian, $\zeta(W^\alpha) = \zeta(W)$ for any closed walk W of X .*

Given a voltage assignment ζ on X and $\beta \in \text{Aut}(X)$, we let ζ^β be the voltage assignment on X given by $\zeta^\beta(u, v) = \zeta(u^{\beta^{-1}}, v^{\beta^{-1}})$, $(u, v) \in A(X)$; and we let $\tilde{\beta}$ be the permutation of $V(X) \times K$ acting as $(u, k)^{\tilde{\beta}} = (u^\beta, k)$. Our last proposition is straightforward.

Proposition 2.8 *Let K be a finite group, and let $\tilde{X} = X \times_{\zeta} K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K , and let $\beta \in \text{Aut}(X)$. Then the following holds.*

- (i) $\tilde{\beta}$ is an isomorphism from $\tilde{X}$ to $X \times_{\zeta\beta} K$.
- (ii) If $\tilde{\alpha}$ is in $\text{Aut}(\tilde{X})$ which projects to α , then $\tilde{\beta}^{-1}\tilde{\alpha}\tilde{\beta}$ is in $\text{Aut}(X \times_{\zeta\beta} K)$, and it projects to $\beta^{-1}\alpha\beta$.
- (iii) If $\tilde{\alpha} \in \text{Aut}(\tilde{X})$ centralizes the group K of covering transformations, then also $\beta^{-1}\tilde{\alpha}\beta$ centralizes K .

3 Classification of pentavalent arc-transitive bicirculants

Throughout this section let X be a connected pentavalent bicirculant $BC_n[L, M, R]$, where $n \geq 5$. The following proposition about arc-transitivity of pentavalent bicirculants of small order, obtained using the computer algebra system MAGMA [2], will be useful throughout this section. (In the computer work, the methods explained in [1] were used.)

Proposition 3.1 *Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant. Then:*

- (i) If $|M| = 4$ and $n \leq 14$ then X is not arc-transitive;
- (ii) If $|M| = 2$ and $n \leq 12$ then X is arc-transitive if and only if it is either isomorphic to $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$ or to $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$;
- (iii) If $|M| = 3$ and $n < 240$ then X is arc-transitive if and only if it is isomorphic to one of the following bicirculants: $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$, and $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. In addition, in the first two cases X is 2-arc-transitive, and in the third case X is 1-arc-transitive (but not 2-arc-transitive).

By the following theorem in a pentavalent arc-transitive bicirculant depending on the size of the set M only three different cases can occur, which will be considered in separate subsections.

Theorem 3.2 *If $X = BC_n[L, M, R]$ is a connected pentavalent arc-transitive bicirculant, then $|M| \in \{2, 3, 5\}$.*

Proof Let $X = BC_n[L, M, R]$ be a connected pentavalent arc-transitive bicirculant. By [14, Theorem 1.1], $|M| \neq 1$. We thus only need to show that $|M| \neq 4$. By way of contradiction suppose that $|M| = 4$. Then n is even and $L = R = \{\frac{n}{2}\}$. Also, we can, without loss of generality, assume that $M = \{0, a, b, c\}$, where $a, b, c \in \mathbb{Z}_n \setminus \{0\}$ are pairwise distinct.

Suppose first that $a = \frac{n}{2}$. Then the subgraph of X induced by the vertices $x_0, x_{\frac{n}{2}}, y_0$, and $y_{\frac{n}{2}}$ is isomorphic to the complete graph K_4 . Consequently also the edge $\{x_0, y_b\}$ is contained in a subgraph of X isomorphic to K_4 , in particular it belongs to at least two 3-cycle. Thus, x_0 and y_b have at least two common neighbors. Since $R = L = \{\frac{n}{2}\}$ and $b \notin \{0, a\}$, the common neighbors of x_0 and y_b are y_c and $x_{\frac{n}{2}}$, implying that $c = b + \frac{n}{2}$. Applying the automorphism $\rho \in \text{Aut}(X)$ one can now easily see that $X \cong C_n[K_2]$,

which is clearly not arc-transitive. We can conclude that none of a , b , and c is equal to $\frac{n}{2}$. Notice that the assumption that $\frac{n}{2}$ is not contained in M is equivalent to the assumption that for $i, j \in M$, the difference $i - j$ is not equal to $\frac{n}{2}$.

Now consider the edge $\{x_0, x_{\frac{n}{2}}\}$. Observe that every 4-cycle that contains this edge also contains exactly one right edge $\{y_i, y_{i+\frac{n}{2}}\}$ ($i \in \{0, a, b, c\}$), and thus it is contained on exactly four 4-cycle. Any two of these 4-cycle intersect only in $\{x_0, x_{\frac{n}{2}}\}$. The same must hold for the four 4-cycle containing the edge $\{x_0, y_0\}$, and consequently the path

$$(y_0, x_0, y_a) \text{ is contained on exactly one 4-cycle.} \quad (1)$$

Therefore, we may assume that exactly one of the following holds: $b = 2a$ or $a + b = 0$ or $a + c = b$.

If $a + b = 0$ then three 4-cycle containing the edge $\{x_0, y_0\}$ are exactly determined: $(x_0, y_0, y_{\frac{n}{2}}, x_{\frac{n}{2}}, x_0)$, $(x_0, y_0, x_a, y_a, x_0)$, and $(x_0, y_0, x_b, y_b, x_0)$. Hence, the remaining 4-cycle containing $\{x_0, y_0\}$ must be $(x_0, y_0, x_{-c}, y_c, x_0)$, which implies that either $a = 2c$ or $b = 2c$.

If $a + c = b$ then again three of the four 4-cycle containing the edge $\{x_0, y_0\}$ are exactly determined: $(x_0, y_0, y_{\frac{n}{2}}, x_{\frac{n}{2}}, x_0)$, $(x_0, y_0, x_{a-b}, y_a, x_0)$, and $(x_0, y_0, x_{c-b}, y_c, x_0)$. Therefore, the remaining 4-cycle containing $\{x_0, y_0\}$ is $(x_0, y_0, x_{-b}, y_b, x_0)$ which implies that either $a = 2b$ or $c = 2b$.

This implies that we may, without loss of generality, assume that $b = 2a$. Clearly

$$(x_0, y_0, y_{\frac{n}{2}}, x_{\frac{n}{2}}, x_0) \text{ and } (x_0, y_0, x_{a-b}, y_a, x_0)$$

are two 4-cycle containing the edge $\{x_0, y_0\}$, and the edges $\{x_0, y_b\}$, $\{x_0, y_c\}$, $\{y_0, x_{-b}\}$, and $\{y_0, x_{-c}\}$ must be contained in the remaining two 4-cycle containing the edge $\{x_0, y_0\}$. Therefore, either $\{y_b, x_{-c}\}$ and $\{y_c, x_{-b}\}$ are edges in X , or $\{y_b, x_{-b}\}$ and $\{y_c, x_{-c}\}$ are edges in X . Inspecting the possibilities for these edges the following cases need to be considered. (Recall that $\frac{n}{2} \notin \{a, b, c\}$.)

Case 1 $\{y_b, x_{-c}\}$ is a 0-spoke.

Then $b + c = 0$, and consequently $M = \{0, a, 2a, -2a\}$. This implies that $(x_0, y_0, x_{-2a}, y_{-2a}, x_0)$ is the only 4-cycle containing the edge $\{x_0, y_{-2a}\}$ (and not containing the edge $\{x_0, x_{\frac{n}{2}}\}$) that exists regardless of the order of a . However, since there must exist another two 4-cycle, it follows that $4a = 0$, $5a = 0$, $6a = 0$, or $7a = 0$. Since $\langle M, \frac{n}{2} \rangle \cong \mathbb{Z}_n$ we can conclude that $n \leq 14$, contradicting Proposition 3.1.

Case 2 $\{y_b, x_{-c}\}$ is an a -spoke.

Then $b + c = a$, $M = \{0, a, 2a, -a\}$, and consequently $(x_0, y_0, x_a, y_a, x_0)$ and $(x_0, y_0, x_a, y_{2a}, x_0)$ are two distinct 4-cycle containing the 2-arc (x_0, y_0, x_a) , contradicting (1).

Case 3 $\{y_b, x_{-b}\}$ is an a -spoke.

Then $2b = a$, and since $2a = b$ we have $a + b = 0$. Moreover, since $\{y_c, x_{-c}\}$ is an a -spoke or a b -spoke, either $a = 2c$ or $b = 2c$. In both cases, we obtain that

$M = \{0, c, 2c, -2c\}$ and that $6c = 0$. Since $\langle M, \frac{n}{2} \rangle \cong \mathbb{Z}_n$ we can conclude that $n \leq 12$, contradicting Proposition 3.1.

Case 4 $\{y_b, x_{-b}\}$ is a c -spoke.

Then $2b = c$, and consequently $M = \{0, a, 2a, 4a\}$. Similarly as in Case 1, by considering the 4-cycle containing the edge $\{x_0, y_{4a}\}$ one can obtain that $n \leq 14$, contradicting Proposition 3.1. $\square$

3.1 Pentavalent bicirculants with $|M| = 2$

The following theorem gives the graphs appearing in Theorem 1.1(i).

Theorem 3.3 *Let X be a connected pentavalent arc-transitive bicirculant $X = BC_n[L, M, R]$ with $|M| = 2$. Then either*

- (i) $X \cong BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$, or
- (ii) $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$.

Proof Without loss of generality, we can assume that $L = \{\frac{n}{2}, \pm l\}$, $M = \{0, a\}$ and $R = \{\frac{n}{2}, \pm r\}$, where $a \in \mathbb{Z}_n \setminus \{0\}$ and $l, r \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$. Observe that the edge $\{x_0, x_{\frac{n}{2}}\}$ is contained on at least four 4-cycle. Namely,

$$(x_0, x_{\frac{n}{2}}, y_0, x_0), (x_0, x_{\frac{n}{2}}, y_{\frac{n}{2}+a}, y_a, x_0), (x_0, x_{\frac{n}{2}}, x_{\frac{n}{2}+l}, x_l, x_0), \\ \text{and } (x_0, x_{\frac{n}{2}}, x_{\frac{n}{2}-l}, x_{-l}, x_0) \quad (2)$$

are all 4-cycle containing the edge $\{x_0, x_{\frac{n}{2}}\}$. Moreover, observe that every 2-arc containing the arc $(x_0, x_{\frac{n}{2}})$ lies on a 4-cycle, and by arc-transitivity of X this must then hold for every 2-arc in X . The existence of a 4-cycle containing the 2-arc (y_0, x_0, y_a) implies that $2a = 0$, $a = \pm 2r$, or $a = \pm r + \frac{n}{2}$.

If $2a = 0$ then $a = \frac{n}{2}$ and the edge $\{x_0, y_0\}$ is contained in a subgraph of X isomorphic to K_4 . Consequently, by arc-transitivity, any edge in X is contained in such a subgraph. In particular, the edges $\{x_0, x_l\}$ and $\{y_0, y_r\}$ must lie on two 3-cycle in X , implying that $2l = \frac{n}{2}$ and $2r = \frac{n}{2}$. By connectivity, it follows that $\mathbb{Z}_n = \langle \frac{n}{2}, l, r \rangle$, where $4l = 4r = 0$, implying that $n = 4$ and $X \cong BC_4[\{\pm 1, 2\}, \{0, 2\}, \{\pm 1, 2\}]$, contradicting Proposition 3.1.

Therefore, either $a = \pm 2r$ or $a = \pm r + \frac{n}{2}$. Moreover, replacing r by $-r$ if necessary, we may assume that $a = 2r$ or $a = r + \frac{n}{2}$. The existence of a 4-cycle containing the 2-arc (x_0, y_0, x_{-a}) implies that we may also assume that either $a = 2l$ or $a = l + \frac{n}{2}$. In total, there are therefore four possibilities which we consider in the four cases below. In the case analysis, it will be helpful to know that the existence of 4-cycle containing the 2-arc (x_0, x_l, x_{2l}) and the 2-arc (y_0, y_r, x_{2r}) implies, respectively, that

$$4l = 0 \text{ or } 3l + \frac{n}{2} = 0 \text{ or } 2l = a \text{ or } 2l = -a \quad (3)$$

and

$$4r = 0 \text{ or } 3r + \frac{n}{2} = 0 \text{ or } 2r = a \text{ or } 2r = -a. \quad (4)$$

Case 1 $a = 2r = 2l$.

Then either $l = r$ or $l = r + \frac{n}{2}$, and the connectedness of X implies that $\langle r, \frac{n}{2} \rangle = \mathbb{Z}_n$. If $l = r$ then the edge $\{x_0, y_a\}$ lies on at least five 4-cycle: $(x_0, y_a, y_r, y_0, x_0)$, $(x_0, y_a, x_{2l}, x_l, x_0)$, $(x_0, y_a, y_{a+\frac{n}{2}}, x_{\frac{n}{2}}, x_0)$, $(x_0, y_a, y_{3l}, x_l, x_0)$, and $(x_0, y_a, y_l, x_{-l}, x_0)$. Hence also the edge $\{x_0, x_{\frac{n}{2}}\}$ lies on at least five 4-cycle. Four of them are listed in (2) whereas the existence of the fifth 4-cycle combined together with (3) and (4) gives $3l = \frac{n}{2}$. This implies that $\mathbb{Z}_n = \langle r = l \rangle = \mathbb{Z}_6$, and thus, by Proposition 3.1, X is isomorphic to $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$. If, however, $l = r + \frac{n}{2}$ then

$$(x_0, x_{\frac{n}{2}}, y_{\frac{n}{2}+a}, y_a, x_0), (x_0, x_l, x_{2l}, y_a, x_0), \text{ and } (x_0, y_a, y_r, y_0, x_0)$$

are 4-cycle containing the edge $\{x_0, y_a\}$. Since there also exists a 4-cycle containing the 2-arc (x_{-l}, x_0, y_a) , one of the neighbors of x_{-l} , that is one of $x_{-l+\frac{n}{2}}, x_{-2l}, y_l, y_{-l}$, is adjacent to the vertex y_a . Since $a \neq \frac{n}{2}$ it follows that either $4r = 0$ or $3r = 0$ or $8r = 0$, giving that $n \leq 8$, and thus, by Proposition 3.1, $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$.

Case 2 $a = 2r = l + \frac{n}{2}$.

Then $l = 2r + \frac{n}{2}$ and $\langle r, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (3) yields $n \in \{6, 8, 12\}$, and thus Proposition 3.1 applies.

Case 3 $a = r + \frac{n}{2} = 2l$.

Similarly as in Case 2, $\langle l, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (4) yields $n \in \{6, 8, 12\}$, and thus Proposition 3.1 applies again.

Case 4 $a = r + \frac{n}{2} = l + \frac{n}{2}$.

Then $l = r$ and $\langle r, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (4) yields $n \in \{4, 6, 8, 12\}$, and thus Proposition 3.1 applies also in this case. $\square$

3.2 Pentavalent bicirculants with $|M| = 3$: Generalized Tabačjn graphs

Let X be a generalized Tabačjn graph, that is, a bicirculant $BC_n[L, M, R]$ with $|M| = 3$. Then we may, without loss of generality, assume that $L = \{\pm l\}$, $M = \{0, a, b\}$ and $R = \{\pm r\}$, where $l, r \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$ and $a, b \in \mathbb{Z}_n \setminus \{0\}$. We first show that X is not 3-arc-transitive.

Theorem 3.4 *There exists no 3-arc-transitive generalized Tabačjn graph.*

Proof Suppose on the contrary that for some $n \geq 3$ and $1 \leq l, r, a, b \leq n-1$ the bicirculant $X = BC_n[\{\pm l\}, \{0, a, b\}, \{\pm r\}]$ is 3-arc-transitive. Clearly, a 3-arc-transitive graph cannot be of girth 3, in particular $\text{girth}(X) \geq 4$. Moreover, if $\text{girth}(X) = 4$ then, by [7, Lemma 4.1.4], X is bipartite of diameter 2, which implies that $n = 5$. However, since X is bipartite, l must be of even order in $\mathbb{Z}_5$ which is clearly impossible.

Suppose now that $\text{girth}(X) = 5$. Then the 3-arc $A = (x_0, x_l, x_{2l}, x_{3l})$ lies on a 5-cycle in X . Suppose first that $5l \neq 0$. Then the 5-cycle containing A must contain one of y_i vertices, and we may assume that this 5-cycle is $(x_0, x_l, x_{2l}, x_{3l}, y_{3l}, x_0)$. That is $a = 3l$. Similarly, the 3-arc (y_0, x_0, x_l, y_l) lies on a 5-cycle of X , and so the fact that $b - a \neq \pm l$ (since $\text{girth}(X) = 5$) implies that $l = \pm 2r$. Without loss of generality, we can assume that $l = 2r$. Observe that $5r \neq 0$, since otherwise we would have $5l = 0$.

Therefore, the 3-arc $(y_0, y_r, y_{2r}, y_{3r})$ lies on a 5-cycle containing a left vertex x_i . This further implies that one of the following holds:

$$a = \pm 3r \text{ or } a - b = \pm 3r \text{ or } b = \pm 3r.$$

If $a = 3r$, then since $a = 3l = 6r$, we obtain $3r = 0$ which is in contradiction with $\text{girth}(X) = 5$. If $a = -3r$ then since $l = 2r$ it follows that $(x_0, x_{2r}, y_{-r}, y_0, x_0)$ is a 4-cycle in X , a contradiction. Both $a - b = \pm 3r$ and $b = \pm 3r$ imply that $b \in \langle r \rangle$, and since also $a, l \in \langle r \rangle$, in both cases r generates $\mathbb{Z}_n$, and therefore X is isomorphic to a Tabačjn graph. But in view of [1, Lemma 3.2] this is impossible.

Suppose now that $5l = 0$. Then, by symmetry, also $5r = 0$. Namely, since the 3-arc (x_0, y_0, y_r, x_r) lies on a 5-cycle of X , and $b - a \neq \pm r$ (since $\text{girth}(X) = 5$) we have $r = \pm 2l$, and thus $5r = \pm 10l = 0$. We can, without loss of generality, assume that $l = r$ or $l = 2r$. However, if $l = r$ then $(x_0, y_0, y_l, x_l, x_0)$ is a 4-cycle in X , a contradiction. Therefore, $l = 2r$. Since X is 3-arc-transitive also the 3-arc (y_0, x_0, x_l, y_{l+a}) lies on a 5-cycle, and thus $N(y_0) \cap N(y_{l+a}) \neq \emptyset$. If $y_r \in N(y_0) \cap N(y_{l+a})$ then $2r - l - a = 0$, and so $a = 0$, a contradiction. If $y_{-r} \in N(y_0) \cap N(y_{l+a})$ then $2r + l + a = 0$, and so $(x_0, x_l, x_{2l}, x_{2l+a}, x_0)$ is a 4-cycle in X , again a contradiction. It follows that $y_{\pm r} \notin N(y_0) \cap N(y_{l+a})$, and thus at least one of x_{l+a} and x_{l+a-b} belongs to $N(y_0) \cap N(y_{l+a})$, implying that at least one of following holds:

$$l + 2a = 0, \quad (5)$$

$$l + a + b = 0, \quad (6)$$

$$l + 2a - b = 0. \quad (7)$$

Similarly, considering the possibilities for 5-cycle containing the 3-arc (y_0, x_0, x_l, y_{l+b}) one can see that at least one of the following also holds:

$$l + 2b = 0, \quad (8)$$

$$l + a + b = 0, \quad (9)$$

$$l + 2b - a = 0. \quad (10)$$

If (5) holds then $l + 2a = 0$, $10a = -5l = 0$, and $r = 4a$, since $2(2r + 2a) = 4r + 4a = -r + 4a = 0$. Thus $\langle a, b \rangle = \mathbb{Z}_n$, and considering Eqs. (8)–(10), we can conclude that either $\mathbb{Z}_n = \langle a \rangle$ and $n \leq 10$, or $\mathbb{Z}_n = \langle b \rangle$ and $n \leq 20$. In both cases, Proposition 3.1 gives a contradiction. If $l + a + b \neq 0$ then both $l + 2a - b = 0$ and $l + 2b - a = 0$ must hold. Using the fact that $5l = 0$ we get $10a = 5b$ and $10b = 5a$, and so $15a = 15b = 0$. This implies that $l, a \in \langle b \rangle$. Since $l = 2r$ and $5l = 5r = 0$ we also have $3l = 6r = r$, implying that $n \leq 15$, and Proposition 3.1 applies. Therefore, we can assume that $l + a + b = 0$. But then since the two 3-arc (x_0, y_0, y_r, y_{r-a}) and (x_0, y_0, y_r, y_{r-b}) both belong to a 5-cycle one can see that $r = a + b$, and thus $l = -r$, which is impossible since $\text{girth}(X) = 5$.

We can conclude that $\text{girth}(X) > 5$. This implies that

$$r \notin \{\pm l, \pm 2l\} \text{ and } a, b, b - a \notin \{\pm l, \pm 2l, \pm 3l\}. \quad (11)$$

Moreover, since there exists a 6-cycle in X (one of such 6-cycle is $(x_0, y_0, x_{-a}, x_{l-a}, y_l, x_l, x_0)$), we have $\text{girth}(X) = 6$. Note that $\text{girth}(X) = 6$ implies that no two distinct 6-cycle contain a common 4-arc. Consider the 3-arc $A_0 = (y_0, x_0, x_l, y_l)$. This 3-arc lies on two 6-cycle $(y_0, x_0, x_l, y_l, x_{l-a}, x_{-a}, y_0)$ and $(y_0, x_0, x_l, y_l, x_{l-b}, x_{-b}, y_0)$.

Claim Each 3-arc of X lies on exactly two 6-cycle.

To prove the claim suppose that there exists an additional 6-cycle C containing the 3-arc A_0 . (Therefore, we are supposing that any 3-arc in X lies on at least three 6-cycle.) None of the edges $e \in E(C) \setminus \{\{x_0, y_0\}, \{x_l, y_l\}\}$ containing y_0 or y_l , can be a spoke edge (since no two different 6-cycle in X have a common 4-arc). The only possibility is thus $l = \pm 3r$. Without loss of generality, we may assume that $l = 3r$. Similarly (inspecting the 3-arc (x_0, y_0, y_r, x_r)), we obtain $r = \pm 3l$. Therefore, we have $8l = 0$ or $10l = 0$. Also, since $\text{girth}(X) = 6$, l and r are elements of the same order in $\mathbb{Z}_n$, and they induce cycles of the same length k ($k = 8$ or $k = 10$). Consider two such cycles $C_l = (x_0, x_l, \dots, x_{(k-1)l}, x_0)$ and $C_r = (y_0, y_r, \dots, y_{(k-1)r}, y_0)$. Observe that 0-spoke induce a matching between C_l and C_r . If there is an edge joining a vertex from C_l and a vertex from C_r “different from a 0-spoke”, then there is another matching between C_l and C_r , and it is not difficult to see that in this case there exists a cycle in X of length less than 6. Therefore, two different spoke edges with the same endvertex cannot occur between C_l and C_r . The same holds for arbitrary two cycles of length k induced by l and r , respectively.

Suppose first that $r = 3l$. Consider the 3-arc $B_0 = (x_0, x_l, y_l, y_{4l})$. Clearly $(x_0, x_l, y_l, y_{4l}, y_{7l}, x_{-l})$ is a 6-cycle containing B_0 , and thus there are at least two other 6-cycle containing B_0 , none of which contains edges $\{x_0, x_{-l}\}$ or $\{y_{4l}, y_{7l}\}$. If there exists a 6-cycle containing B_0 and the edge $\{x_0, y_0\}$, then if the remaining vertex of this 6-cycle is x_t then x_t is adjacent to both y_0 and y_{4l} . However, y_0 and y_{4l} lie on the same cycle of length 8 induced by $r = 3l$, and we have a contradiction with the argument from the previous paragraph. Similarly, there cannot exist a 6-cycle containing B_0 and the edge $\{y_{4l}, x_{4l}\}$. Therefore, either $\{x_{4l-a}, y_a\}$ and $\{x_{4l-b}, y_b\}$ are the edges which lie on 6-cycle containing B_0 , or $\{x_{4l-a}, y_b\}$ and $\{x_{4l-b}, y_a\}$ are the edges which lie on 6-cycle containing B_0 . In the former case, we have $2a - 4l \in \{b, 0\}$ and $2b - 4l \in \{a, 0\}$. If $2a - 4l = b$ and $2b - 4l = a$, then $b = -a$, and we obtain a 4-cycle $(x_0, y_a, x_a, y_0, x_0)$, a contradiction. If $2a - 4l = 0$ and $2b - 4l = a$, then $4a = 0$ and $8b = 0$, therefore $n = 8$, contradicting Proposition 3.1. Similarly, if $2a - 4l = 0$ and $2b - 4l = 0$, or $2a - 4l = b$ and $2b - 4l = 0$, we obtain $n = 8$, again contradicting Proposition 3.1. If the latter holds, then $\{x_{4l-a}, y_b\}$ must be a 0-spoke, hence $a + b = 4l$. Considering possible 6-cycle containing 3-arc (x_0, x_l, y_l, y_{-2l}) , a contradiction follows straightforward. This shows that $r = 3l$ is impossible.

Suppose now that $r = -3l$. Consider the 3-arc $B = (y_0, x_0, x_l, x_{2l})$. It lies on at least three 6-cycle of X . Therefore, besides $(y_0, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_0)$ there are at least two other 6-cycle, say D_1 and D_2 , containing B . Moreover, since no two different 6-cycle in X have a common 4-arc none of D_1 and D_2 contains x_{3l} . It is not difficult to see that y_{2l} is not contained in neither of D_1 and D_2 . Namely, if this was the case, say that D_1 contains y_{2l} , then the remaining vertex on D_1 cannot be a right vertex, since the distance from y_0 and y_{2l} using only right edges is 4. On the other hand, if the remaining vertex on D_1 is a left vertex, say x_t , then we obtain two matchings between the cycle

induced by l which contains x_l and the cycle induced by r which contains y_0 . Hence, we can assume that D_1 contains y_{2l+a} , and that D_2 contains y_{2l+b} . Observe that none of D_1 and D_2 contains right edges, since otherwise there would be a 0-spoke and an a -spoke between two cycles induced by l and r . Therefore, $D_1 = (y_0, x_0, x_l, x_{2l}, y_{2l+a}, x_i, y_0)$ and $D_2 = (y_0, x_0, x_l, x_{2l}, y_{2l+b}, x_j, y_0)$. Applying the same argument to the 3-arc $B' = (y_0, x_0, x_{-l}, x_{-2l})$, we see that $D'_1 = (y_0, x_0, x_{-l}, x_{-2l}, y_{-2l+a}, x_{i'}, y_0)$ and $D'_2 = (y_0, x_0, x_{-l}, x_{-2l}, y_{-2l+b}, x_{j'}, y_0)$ are 6-cycle in X , where $\{i', j'\} = \{i, j\}$. If $i' = i$ and $j' = j$, then we obtain that y_{-2l+a} and y_{2l+a} have the common neighbor x_i . This further implies that $b = \pm 4l$. Similarly, y_{-2l+b} and y_{2l+b} have the common neighbor x_j , which implies that $a = \pm 4l$. It follows that $\mathbb{Z}_n = \langle l \rangle$, implying that X is isomorphic to a Tabačjn graph. But this is impossible, since, by [1, Lemma 3.2], there are no 3-arc-transitive Tabačjn graphs. If, however, $i' = j$ and $j' = i$ then the vertices y_0, y_{-2l+b} , and y_{2l+a} have a common neighbor x_i , while the vertices y_0, y_{-2l+a} , and y_{2l+b} have a common neighbor x_j . The vertex y_0 can be adjacent to x_i via a -spoke or via b -spoke. If y_0 is adjacent to x_i via a -spoke, then we have the following: $\{x_i, y_{2l+a}\}$ and $\{x_j, y_0\}$ are b -spokes, $\{x_i, y_{-2l+b}\}$ and $\{x_j, y_{-2l+a}\}$ are 0-spokes and $\{x_j, y_{2l+b}\}$ is an a -spoke. Considering 2-arcs (y_{2l+a}, x_i, y_0) and $(y_{-2l+a}, x_j, y_{2l+b})$ gives $2l + a - b + a = 0$ and $-2l + a - 0 + a = 2l + b$. These two equations combined together imply that $6l = 0$, which is impossible, since l is of order 10 in $\mathbb{Z}_n$. The same argument gives a contradiction if $\{y_0, x_i\}$ is a b -spoke. Therefore, we can conclude that each 3-arc lies on precisely two 6-cycle as claimed.

Now again consider the 3-arc $A = (x_0, x_l, x_{2l}, x_{3l})$. It is contained in exactly two 6-cycle, say C_1 and C_2 . This implies that $r \neq \pm 3l$ (otherwise A would be contained on at least three different 6-cycle, namely the ones using two 0-spokes, two a -spokes, and two b -spokes, respectively). We distinguish three cases depending on whether the vertices x_{-l} and x_{4l} are contained on one of C_1 and C_2 or not.

Case 1 C_1 or C_2 contains both of x_{-l} and x_{4l} .

Then we may assume that $C_1 = (x_{-l}, x_0, x_l, x_{2l}, x_{3l}, x_{4l}, x_{-l})$, and therefore $6l = 0$. Since no 4-arc in X belongs to two different 6-cycle the 6-cycle C_2 must contain two right vertices. In particular, we may, without loss of generality, assume that $C_2 = (x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_a, x_0)$, and so $a = 3l - r$. However, since $6l = 0$, we obtain a third 6-cycle containing A , namely $(x_0, x_l, x_{2l}, x_{3l}, y_{-r}, y_0, x_0)$, contradicting Claim 1.

Case 2 C_1 or C_2 contains one of x_{-l} and x_{4l} .

Then we may, without loss of generality, assume that $C_1 = (x_0, x_l, x_{2l}, x_{3l}, x_{4l}, y_{4l}, x_0)$, and thus $4l = a$. It follows that $C_2 = (x_{-l}, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, x_{-l})$. Recall also that this implies that there exists no 6-cycle containing A and two right vertices. In particular, the equation $3l + i \pm r - j = 0$, where $i, j \in \{0, 4l, b\}$, thus has no solution. In addition to (11), we thus also have

$$r \notin \{\pm 3l, \pm 7l, \pm(b-l), \pm(b+3l), \pm(b-3l), \pm(b-7l)\}. \quad (12)$$

Consider now the 3-arc

$$B' = (x_0, x_l, y_l, y_{l+r}),$$

and let C'_1 and C'_2 be the two 6-cycle in X containing B' . Since none of the two 6-cycle containing $(x_l, y_l, y_{r+l}, x_{r+l})$ contains x_0 , none of C'_1 and C'_2 contains x_{r+l} . Thus, two of the vertices $x_{r+l-a} = x_{r-3l}$, x_{r+l-b} and y_{2r+l} belong to C'_1 and C'_2 , one to C'_1 and one to C'_2 . Similarly, none of C'_1 and C'_2 can contain y_0 , and so two of the vertices x_{-l} , y_{4l} and y_b belong to C'_1 and C'_2 , one to each.

We first show that $x_{r-3l} \notin V(C'_1) \cup V(C'_2)$. Suppose on the contrary that, say C'_1 contains x_{r-3l} . If the remaining vertex v of C'_1 is x_{-l} , then $-2l = r - 3l$, contradicting (11). If $v = y_{4l}$, then x_{r-3l} and y_{4l} are either adjacent by a 0-spoke or by a b -spoke. In the former case $r - 3l = 4l$ and in the latter case $r - 3l + b = 4l$, both contradicting (12). Finally, if $v = y_b$, then x_{r-3l} and y_b are adjacent by a 0-spoke, and so $r - 3l = b$, contradicting (12). Therefore $x_{r-3l} \notin V(C'_1)$. Clearly, the same argument can be used to show that we also have $x_{r-3l} \notin V(C'_2)$. Therefore, we may, without loss generality, assume that C'_1 contains the vertex x_{r+l-b} . Now, if the remaining vertex v of C'_1 is x_{-l} , then $-2l = r + l - b$, contradicting (12). Similarly, if $v = y_{4l}$, then x_{r+l-b} and y_{4l} are adjacent via a 0-spoke, so that $r + l - b = 4l$, again contradicting (12). Thus $v = y_b$, and so either $r + l - b = b$ or $r + l - b + 4l = b$. In other words, we have that either

$$r = 2b - l \quad \text{or} \quad r = 2b - 5l.$$

We can now repeat the argument for the 3-arc (x_0, x_l, y_l, y_{l-r}) to find that either $-r = 2b - l$ or $-r = 2b - 5l$ holds. Since $2r \neq 0$, we thus must have that $2b - l = 5l - 2b$, that is $4b = 6l$. Thus either $2r = 4b - 2l = 4l = a$ or $-2r = 4l = a$, both contradicting $\text{girth}(X) = 6$. This completes the analysis of Case 2.

Case 3 None of C_1 and C_2 contains x_{-l} or x_{4l} .

Then, since $r \neq \pm 3l$, we can assume that C_1 is $(y_0, x_0, x_l, x_{2l}, x_{3l}, y_{a+3l}, y_0)$ and

$$r = a + 3l. \quad (13)$$

This of course implies that C_2 contains one of y_a and y_b , and one of y_{3l} and y_{b+3l} . If it contains y_b then, as $r \neq \pm 3l$, we have $C_2 = (y_b, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_b)$, which forces $3l \pm r = b$. Moreover (13) applies that we cannot have $3l - r = b$, since in this case $b = -a$, and so $(x_0, y_{-a}, x_{-a}, y_0, x_0)$ is a 4-cycle, contradicting $\text{girth}(X) = 6$. Thus $3l + r = b$, and so $b - a = 6l$. If, however, C_2 contains y_a , then it cannot contain y_{3l} . Namely, in this case $a \pm r = 3l$ would have to hold. But (13) combined together with $a - r = 3l$ implies $6l = 0$, while combined together with $a + r = 3l$ implies $2a = 0$, the former contradicts the assumption of this case and the latter contradicts $\text{girth}(X) = 6$ (as $(x_0, y_a, x_a, y_0, x_0)$ would be a 4-cycle). Thus $C_2 = (y_a, x_0, x_l, x_{2l}, x_{3l}, y_{b+3l}, y_a)$, and so $a \pm r = b + 3l$. If $a + r = b + 3l$ then (13) implies $2a = b$, contradicting $\text{girth}(X) = 6$ (as $(x_0, y_a, x_a, y_b, x_0)$ would be a 4-cycle). Hence $a - r = b + 3l$, and so $b + 6l = 0$. Therefore, either $b = a + 6l$ or $b = -6l$. However, by Proposition 2.4, $BC_n[\{\pm l\}, \{0, a, a + 6l\}, \{\pm r\}] \cong BC_n[\{\pm l\}, \{0, a, -6l\}, \{\pm r\}]$, and thus we can assume that $C_2 = (y_a, x_0, x_l, x_{2l}, x_{3l}, y_{b+3l})$ and

$$b + 6l = 0. \quad (14)$$

Consider now the 3-arc $F = (y_0, y_{3l+a}, y_{6l+2a}, y_{9l+3a})$. Since F lies on exactly two 6-cycle, two of the following six equations must hold:

$$\pm l = 3l + 2a, \quad (15)$$

$$\pm l = 3l + 3a, \quad (16)$$

$$\pm l = 15l + 4a, \quad (17)$$

$$\pm l = 9l + 4a, \quad (18)$$

$$\pm l = 9l + 2a, \quad (19)$$

$$\pm l = 15l + 3a. \quad (20)$$

If (15) holds then $(x_0, y_a, y_{3l+2a}, x_{3l+2a}, x_0)$ is a 4-cycle in X , a contradiction. If (16) and one of (17)–(19) hold then $a = kl$, and therefore $\mathbb{Z}_n = \langle l \rangle$. But then X is isomorphic to a Tabačjn graph, contradicting [1, Lemma 3.2]. The same contradiction is obtained if (20) and one of (17)–(19) hold. Suppose now that (16) and (20) hold. Then $12l \in \{2l, -2l, 0\}$, and so $10l = 0$, $14l = 0$ or $12l = 0$. Since $3a = -3l \pm l$, and $\mathbb{Z}_n = \langle l, a \rangle$ we can conclude that $n \leq 42$, contradicting Proposition 3.1. The same contradiction is obtained when (17) and (18) hold. Suppose next that (17) and (19) hold. Then $2a \in \{-8l, -6l, -4l\}$. If $2a = -8l$ then since by (11) $r \neq \pm l$ we have $a = -4l + n/2$ and $r = -l + n/2$, and so $(x_0, y_{-4l+n/2}, y_{-5l}, x_l, x_0)$ is a 4-cycle in X , a contradiction. If $2a = -6l$ then $2r = 2(3l + a) = 0$, which is clearly impossible. If $2a = -4l$ then since by (11) $r \neq \pm l$ we have $a = -2l + n/2$ and $r = l + n/2$, and so $(x_0, y_{-2l+n/2}, y_{-l}, x_{-l}, x_0)$ is a 4-cycle in X , a contradiction. Therefore, the only case left to consider is when (18) and (19) hold. In this case, we have $2a = \pm l \pm l$. If $2a = -2l$ then $(x_0, y_a, y_{3l+2a}, x_{3l+2a}, x_0)$ is a 4-cycle in X , a contradiction. If $2a = 0$ then $(x_0, y_a, x_a, y_0, x_0)$ is a 4-cycle in X , a contradiction. If, however, $2a = 2l$ then $11l = \pm l$, and hence $n \leq 12$, contradicting Proposition 3.1. This completes the proof that there are no 3-arc-transitive generalized Tabačjn graphs. $\square$

Theorem 3.5 *Let X be an arc-transitive generalized Tabačjn graph. Then X is core-free if and only if $X \cong BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$.*

Proof Let X be an arc-transitive generalized Tabačjn graph. Let $2n$ and $m = |\text{Aut}(X)_{x_0}|$ be the order of X and the order of the vertex stabilizer of x_0 in $\text{Aut}(X)$, respectively. Since X is core-free, Proposition 2.2 implies that $n^2 < |\text{Aut}(X)| = 2nm$, and consequently $n < 2m$. Next, by Theorem 3.4, we have that X is s -transitive, for some $s \in \{1, 2\}$. By Proposition 2.1, for $s = 2$ we have $m \leq 120$, while for $s = 1$ we have $m \leq 20$. Therefore, if X is 2-transitive then $n < 240$, and if X is 1-transitive then $n < 40$. Using Proposition 3.1, we can conclude that X is isomorphic to $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$, or $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. However, one can easily see that between these three graphs only the graph $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$ is core-free. $\square$

We are now ready to classify arc-transitive generalized Tabačjn graphs. The classification gives the graphs appearing in Theorem 1.1(ii), and shows that every arc-transitive generalized Tabačjn graph is in fact a Tabačjn graph.

Fig. 2 The voltage assignment ζ on $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$. The spanning tree consists of undirected bold edges, all carrying trivial voltage

Theorem 3.6 A bicirculant $BC_n[L, M, R]$ with $|M| = 3$ is arc-transitive if and only if it is isomorphic to one of the graphs $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}] \cong K_{6,6} - 6K_2$ and $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. Moreover, the first two graphs are 2-transitive and the third graph is 1-transitive.

Proof Let X be an arc-transitive generalized Tabačjn graph $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$ with a $(2, n)$ -semiregular automorphism ρ giving the prescribed bicirculant structure. If X is core-free then, by Theorem 3.5, X is isomorphic to $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$.

Suppose now that X is not core-free. Then there exists a non-trivial subgroup N of $\langle \rho \rangle$ which is normal in $\text{Aut}(X)$. By Lemma 2.3, the quotient graph X_N is still a generalized Tabačjn graph. In particular, it is a connected core-free arc-transitive generalized Tabačjn graph, and hence, by Theorem 3.5, it is isomorphic to $X_N \cong BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$. In fact, X is isomorphic to a regular $\mathbb{Z}_m$ -cover of this graph, where $|N| = m$. Note also that ρ projects to a $(2, n/m)$ -semiregular automorphism of X_N giving the generalized Tabačjn structure. (Below, all arithmetic operations are to be taken modulo m if at least one argument is from $\mathbb{Z}_m$ and the symbol mod m is always omitted.)

The graph $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$ is illustrated in Fig. 2. Let us choose the following automorphisms of $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$

$$\alpha = (y_0 y_2 x_2 x_0 x_1)(y_1) \text{ and } \beta = (y_0 y_1 y_2)(x_0 x_1 x_2),$$

and let $G = \langle \alpha, \beta \rangle$. Since the automorphism group of $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$ is isomorphic to the symmetric group S_6 , we can conclude that every $(2, 3)$ -semiregular automorphism of $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$ is conjugate to β , and that every arc-transitive subgroup of its automorphism group contains the subgroup G . Because of Proposition 2.8 we may assume, without loss of generality, that ρ projects to β (therefore, the lifts of β centralize the group N of covering transformations) and that G lifts to a subgroup of $\text{Aut}(X)$.

Any such cover X can be derived from K_6 through a suitable voltage assignment $\zeta : A(K_6) \rightarrow \mathbb{Z}_m$. To find this voltage assignment ζ fix the spanning tree T of K_6 as the one consisting of the edges

Table 1 Fundamental cycles and their images with corresponding voltages in K_6

	C	$\zeta(C)$	C^α	$\zeta(C^\alpha)$	
1	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_2, y_0, x_1, y_2)	$t_1 + t_{10}$
2	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_2, x_1, y_1, y_2)	$-t_{10} - t_3 + t_5$
3	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_2, y_1, y_0, x_1, y_2)$	$-t_5 + t_1 + t_{10}$
4	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_2, y_1, x_0, x_1, y_2)$	$-t_5 - t_2 + t_{10}$
5	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_2, y_1, x_2, y_2)	$-t_5 + t_4 - t_6$
6	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_2, x_2, x_0, x_1, y_2)$	$t_6 + t_{10}$
7	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_2, x_2, x_1, y_2)	$t_6 + t_9 + t_{10}$
8	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_2, x_1, x_0, y_2)	$-t_{10} - t_7$
9	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, y_0, x_1)	t_1
10	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_2, x_1, y_0, x_2, y_2)$	$-t_{10} - t_1 - t_8 - t_6$
	C	$\zeta(C)$	C^β	$\zeta(C^\beta)$	
11	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_1, x_2, x_1, y_1)	$t_4 + t_9 - t_3$
12	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_1, x_1, y_2, y_1)	$t_3 + t_{10} - t_5$
13	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_1, y_2, x_2, x_1, y_1)$	$t_5 + t_6 + t_9 - t_3$
14	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_1, y_2, x_0, x_1, y_1)$	$t_5 + t_7 - t_3$
15	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_1, y_2, y_0, y_1)	t_5
16	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_1, y_0, x_0, x_1, y_1)$	$-t_3$
17	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_1, y_0, x_1, y_1)	$t_1 - t_3$
18	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_1, x_1, x_0, y_1)	$t_3 + t_2$
19	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, x_2, x_1)	t_9
20	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_1, x_1, x_2, y_0, y_1)$	$t_3 - t_9 + t_8$

$$\{y_0, y_1\}, \{y_0, y_2\}, \{y_0, x_0\}, \{x_0, x_1\}, \{x_0, x_2\}$$

(see also Fig. 2).

There are ten fundamental cycles in K_6 , which are generated, respectively, by ten cotree arcs

$$(y_0, x_1), (x_0, y_1), (y_1, x_1), (y_1, x_2), (y_1, y_2), (y_2, x_2), (y_2, x_0), (x_2, y_0), (x_2, x_1), \text{ and } (x_1, y_2).$$

Since X is connected, we have $\mathbb{Z}_m = \langle t_i \mid i \in \{1, \dots, 10\} \rangle$. In addition, by Proposition 2.5, α extends to an automorphism α^* of $\mathbb{Z}_m$, and, by Proposition 2.7, β^* is the identity automorphism of $\mathbb{Z}_m$. This implies that for $i \in \{1, 2, \dots, 10\}$ we have $\zeta(C_i^\alpha) = \alpha^*(\zeta(C_i)) = \alpha^*(t_i)$ (Rows 1–10 of Table 1) and $\zeta(C_i^\beta) = \zeta(C_i) = t_i$ (Rows 11–20 of Table 1). We get from Table 1, where all fundamental cycles and their voltages are listed, that $\mathbb{Z}_m = \langle t_3, t_9 \rangle$. In particular, if $t_1 = t$, $t_3 = r$, and $t_9 = k$ then

$$t_1 = t = \alpha^*(k) \in \langle k \rangle \text{ (by Row 9 of Table 1),}$$

$$t_2 = -\alpha^*(t) \text{ (by applying } \alpha^* \text{ to Row 11 of Table 1 we have } \alpha^*(t) = \alpha^*(t_4) + \alpha^*(t_9)$$

Table 2 Voltages $t_i = \zeta(C_i)$ of the ten fundamental cycles C_i listed in Table 1 given by t, k, r , and their images under α^*

		x	$\alpha^*(x)$
1	t_1	t	$2r - t$
2	t_2	t	$-t$
3	t_3	r	$-r + 2t$
4	t_4	$r - 2t$	$-r$
5	t_5	$3r - 3t$	$t - r$
6	t_6	$-r$	$r - 2t$
7	t_7	$t - r$	$r + t$
8	t_8	$r + t$	$t - r$
9	t_9	$3t$	t
10	t_{10}	$2t$	$-4t$

$-\alpha^*(t_3)$ and then by using Rows 3, 4, and 9 of Table 1),

$t_4 = t - k + r$ (by Row 11 of Table 1),

$t_5 = 3r - k$ (combining together Rows 11, 14, and 17 of Table 1),

$t_6 = -r$ (by Row 16 of Table 1),

$t_7 = t - r$ (by Row 17 of Table 1),

$t_8 = -\alpha^*(t) + r$ (by Row 18 of Table 1),

$t_{10} = 2r - k - \alpha^*(t) = 2r - 2t$ (by Row 20 and applying α^* to Row 18 of Table 1),

$t_2 = -\alpha^*(t) = k - 2t$,

$2t_3 = 2r = 2k - 2t \in \langle k \rangle$ (applying α^* to Row 17 of Table 1).

Further, by Row 5 of Table 1 we have $\alpha^*(t_5) = -t_5 + t_4 - t_6$, and then by using the obtained results for the values of t_4 , t_5 , and t_6 we have $\alpha^*(t_5) = t - r$. On the other hand, we have $t_5 = 3t_3 - t_9$, and then from Rows 3 and 9 of Table 1 we get $\alpha^*(t_5) = 3\alpha^*(t_3) - \alpha^*(t_9) = -3r + 3k - 4t$. Combining the two obtained values for $\alpha^*(t_5)$ we conclude that $2r = -5t + 3k$. This further implies that $3t = k$ and $2r = 4t$. The voltages of the fundamental cycles using these equalities are given in Table 2.

Combining together Rows 1 and 2 of Table 2 we obtain $2r = 0$. This implies that $4t = 0$, and then by Row 10 of Table 2 we conclude that $2t = 0$, and therefore we have $k = 3t = t$. Hence, $\mathbb{Z}_m = \langle k, r \rangle$, where $2k = 0$ and $2r = 0$, and therefore $m = 2$ and $k, r \in \{0, 1\}$. One can now easily see that for $(k, r) = (1, 0)$ we have $X \cong BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$, while for $(k, r) \in \{(0, 1), (1, 1)\}$ we have $X \cong BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. $\square$

3.3 Pentavalent bicirculants with $|M| = 5$: bipartite dihedrants

As observed in Sect. 1, a bicirculant $BC_n[\emptyset, M, \emptyset]$ is a bipartite Cayley graph on the dihedral group D_{2n} , that is, a bipartite dihedrant. To obtain the classification of pentavalent arc-transitive bipartite dihedrants the following lemma (which considers prime-valent bipartite dihedrants) will be useful.

Fig. 3 The dipole graph with p parallel edges

Lemma 3.7 Let X be a p -valent arc-transitive bipartite dihedrant $X = \text{Cay}(D_{2n}, S)$, where p is a prime and $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$. If $H = \langle a \rangle$ is normal in $\text{Aut}(X)$ then $X \cong \text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, \dots, ba^{r^{p-2}+\dots+r+1}\})$, where $r \in \mathbb{Z}_n^*$ is such that $r^{p-1} + \dots + r + 1 \equiv 0 \pmod{n}$.

Proof Let $X = \text{Cay}(D_{2n}, S)$, where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$ and $|S| = p$, where p is a prime, and let $H = \langle a \rangle$ be a normal subgroup of $\text{Aut}(X)$. Consider the quotient graph $Y = X_H$. Observe that Y is a multigraph with 2 vertices and p parallel edges (the so-called *dipole graph* $\mathcal{D}_p$), and X is a regular $\mathbb{Z}_n$ -cover of Y .

Let $V(Y) = \{0, 1\}$ and let $a_0, a_1, \dots, a_{p-1}$ be the p parallel arcs (directed from 0 to 1) in Y (see Fig. 3). Let $\alpha = (a_0 a_1 \dots a_{p-1})$ be an automorphism of Y cyclically permuting these arcs. Since X is arc-transitive, the automorphism α must lift. Choose a_0 to be the spanning tree of Y , and let $t_1, t_2, \dots, t_{p-1}$ be the voltages of the arcs $a_1, a_2, \dots, a_{p-1}$, respectively. The fundamental cycle containing the arc a_i is a 2-cycle consisting of a_i and a_0^- (that is, the opposite arc of a_0). The image of this cycle under α is the 2-cycle $[a_{i+1}, a_1^-]$ if $i = 1, \dots, p-2$, and $[a_0, a_1^-]$ if $i = p-1$. Therefore, α^* , which is an automorphism of $\mathbb{Z}_n^*$, maps the voltages $t_i \in \mathbb{Z}_n$ by the following rules:

$$\alpha^*(t_i) = t_{i+1} - t_1, \quad (i = 1, \dots, p-2) \quad (21)$$

$$\alpha^*(t_{p-1}) = -t_1. \quad (22)$$

Since $t_2 = t_1 + \alpha^*(t_1)$, we have $t_2 \in \langle t_1 \rangle$. Similarly, one can see that $t_i \in \langle t_1 \rangle$, and hence $\mathbb{Z}_n = \langle t_1 \rangle$. By Proposition 2.6, we can therefore assume that $t_1 = 1$. Let $r \in \mathbb{Z}_n^*$ be such that $\alpha^*(x) = r \cdot x$. Using (21) we obtain $t_i = r^{i-1} + \dots + r + 1$, and combining this with (22) we obtain $r^{p-1} + \dots + r + 1 \equiv 0 \pmod{n}$. Using the obtained voltages in Y , it is not difficult to see that $X \cong \text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, \dots, ba^{r^{p-2}+\dots+r+1}\})$. $\square$

Remark 3.8 For any primes n and p such that $n \equiv 1 \pmod{p}$, the order of the multiplicative group $\mathbb{Z}_n^*$ is divisible by p , and hence there exists an element $r \in \mathbb{Z}_n^*$ of order p in $\mathbb{Z}_n^*$. Consequently, $r^p - 1$ is divisible by n , and since $r - 1 < n$ we have $r^{p-1} + \dots + r + 1 \equiv 0 \pmod{n}$. Since for a fixed prime p there are infinitely many primes n such that $n \equiv 1 \pmod{p}$ we can conclude that there are infinitely many p -valent arc-transitive bipartite dihedrants described in the previous lemma.

The following result about pentavalent 2-arc-transitive dihedrants can be extracted from [22, Theorem 1].

Proposition 3.9 *Let X be a connected pentavalent 2-arc-transitive dihedrant on $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$. Then one of the following is true:*

- (i) X is isomorphic to the complete bipartite graph $K_{5,5}$, or the complete bipartite graph minus a matching $K_{6,6} - 6K_2$, or the incidence graph $B(PG(2, 4))$;
- (ii) $\text{core}_{\text{Aut}(X)}(\langle a \rangle) \neq 1$ and X is a cyclic regular cover of some graph mentioned in (i).

Lemma 3.10 *Let X be a connected pentavalent arc-transitive bipartite dihedrant. Then X is core-free if and only if X is isomorphic to the complete bipartite graph minus a matching $K_{6,6} - 6K_2$, or to the incidence graph $B(PG(2, 4))$.*

Proof Let X be a core-free connected pentavalent bipartite arc-transitive dihedrant $X = BC_n[\emptyset, M, \emptyset] (|M| = 5)$. If X is 2-arc-transitive then, by Proposition 3.9, either X is isomorphic to the complete bipartite graph minus a matching $K_{6,6} - 6K_2$ or to the incidence graph $B(PG(2, 4))$. Suppose now that X is not 2-arc-transitive, and let $m = |\text{Aut}(X)_{x_0}|$ be the order of the vertex stabilizer of $x_0 \in V(X)$ in $\text{Aut}(X)$. Then, by Proposition 2.1, $m \leq 20$. Since X is core-free, Proposition 2.2 implies that $n < 2m$, and thus $n < 40$. However, with the use of the computer algebra system MAGMA [2] one can see that no such graph exists. $\square$

Theorem 3.11 *Let X be a connected pentavalent bipartite arc-transitive dihedrant on $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$. Then X is isomorphic to one of the following graphs:*

- (i) $K_{6,6} - 6K_2$,
- (ii) $BC_{12}[\emptyset, \{0, 1, 2, 4, 9\}, \emptyset]$,
- (iii) $BC_{24}[\emptyset, \{0, 1, 3, 11, 20\}, \emptyset]$,
- (iv) $B(PG(2, 4))$,
- (v) $\text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$, where $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Proof Let X be a connected pentavalent bipartite arc-transitive dihedrant on $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$. Then a is a $(2, n)$ -semiregular automorphism of X . If X is core-free then, by Lemma 3.10, either $X \cong K_{6,6} - 6K_2$ or $X \cong B(PG(2, 4))$. Suppose now that X is not core-free. Then there exist a subgroup N of $\langle a \rangle$ which is normal in $\text{Aut}(X)$. If $N = \langle a \rangle$ then, by Lemma 3.7, $X \cong \text{Cay}(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$, where $r \in \mathbb{Z}_n^*$ is such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

If N is a non-trivial subgroup of $\langle \rho \rangle$ then, by Lemma 2.3, the quotient graph X_N is a connected pentavalent core-free arc-transitive bipartite dihedrant, and hence, by Lemma 3.10, either $X_N \cong K_{6,6} - 6K_2$ or $X_N \cong B(PG(2, 4))$. Suppose first that the latter holds. With the use of MAGMA [2] one can see that $B(PG(2, 4))$ is 4-transitive, and that there are no s -arc-transitive subgroups of $\text{Aut}(B(PG(2, 4)))$ for $s \leq 3$. This implies that X is 4-transitive as well. Since there exist a 6-cycle in X , by Godsfil and

Fig. 4 The spanning tree in $K_{6,6} - 6K_2$

Royle [7, Lemma 4.1.3], $\text{girth}(X) = 6$, and consequently, by Godsil and Royle [7, Lemma 4.1.4], X is bipartite with diameter 3. All these combined together imply that X has 42 vertices, and so $X \cong B(PG(2, 4))$, a contradiction. We may therefore assume that $X_N \cong K_{6,6} - 6K_2$.

Let the vertices of X_N be labeled with x_i and y_i , where $i \in \{0, 1, 2, 3, 4, 5\}$. Then we can define the edge set of X_N as $E(X_N) = \{\{x_i, y_j\} : i, j \in \{0, 1, 2, 3, 4, 5\}, i \neq j\}$. Let us choose the following automorphisms of X_N

$$\begin{aligned}\alpha &= (x_0)(x_1 \ x_2 \ x_4 \ x_5 \ x_3)(y_0)(y_1 \ y_2 \ y_4 \ y_5 \ y_3) \text{ and} \\ \beta &= (x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5)(y_0 \ y_1 \ y_2 \ y_3 \ y_4 \ y_5),\end{aligned}$$

and let $G = \langle \alpha, \beta \rangle$. It can be checked directly, using MAGMA, that every $(2, 6)$ -semiregular automorphism of $X_N \cong K_{6,6} - 6K_2$ giving the “dihedrant” structure is conjugate to β , and that every arc-transitive subgroup of $\text{Aut}(X_N)$ which contains a $(2, 6)$ -semiregular automorphism giving the “dihedrant” structure contains a subgroup conjugate to G . Fix the spanning tree T of X_N as the one consisting of the edges (see Fig. 4)

$$\begin{aligned}\{x_0, y_1\}, \{x_0, y_2\}, \{x_0, y_3\}, \{x_0, y_4\}, \{x_0, y_5\}, \{x_1, y_0\}, \{x_1, y_5\}, \\ \{x_2, y_5\}, \{x_3, y_5\}, \{x_4, y_5\}, \text{ and } \{x_5, y_4\}.\end{aligned}$$

There are 19 fundamental cycles in X_N , which are generated, respectively, by the following cotree arcs: (x_1, y_2) , (x_1, y_3) , (x_1, y_4) , (x_2, y_0) , (x_2, y_1) , (x_2, y_3) , (x_2, y_4) , (x_3, y_0) , (x_3, y_1) , (x_3, y_2) , (x_3, y_4) , (x_4, y_0) , (x_4, y_1) , (x_4, y_2) , (x_4, y_3) , (x_5, y_0) , (x_5, y_1) , (x_5, y_2) , and (x_5, y_3) .

Since X is connected, we have $\mathbb{Z}_m = \langle t_i \mid i \in \{1, \dots, 19\} \rangle$. In addition, α extends to an automorphism α^* of $\mathbb{Z}_m$, and by Proposition 2.7, β^* is the identity automorphism of $\mathbb{Z}_m$. This implies that for $i \in \{1, 2, \dots, 19\}$ we have $\zeta(C_i^\beta) = \zeta(C_i) = t_i$ (Rows 1–19 of Table 3) and $\zeta(C_i^\alpha) = \alpha^*(\zeta(C_i)) = \alpha^*(t_i)$ (Rows 20–39 of Table 1). We get from Table 3 (using Rows 1–19) that $\mathbb{Z}_m = \langle t_1, t_2, t_3, t_5 \rangle$, in particular, we have:

$$\begin{aligned}t_4 &= -t_3 \text{ (by Row 3 of Table 3)} \\ t_{19} &= -t_3 \text{ (by Row 19 of Table 3)} \\ t_{18} &= -t_2 \text{ (by Row 18 of Table 3)} \\ t_{17} &= -t_1 \text{ (by Row 17 of Table 3)}\end{aligned}$$

Table 3 Fundamental cycles and their images with corresponding voltages in $K_{6,6} - 6K_2$

	C	$\zeta(C)$	C^β	$\zeta(C^\beta)$	
1	C_1	$(x_1, y_2, x_0, y_5, x_1)$	t_1	$(x_2, y_3, x_1, y_0, x_2)$	$t_6 - t_2 - t_4$
2	C_2	$(x_1, y_3, x_0, y_5, x_1)$	t_2	$(x_2, y_4, x_1, y_0, x_2)$	$t_7 - t_3 - t_4$
3	C_3	$(x_1, y_4, x_0, y_5, x_1)$	t_3	$(x_2, y_5, x_1, y_0, x_2)$	$-t_4$
4	C_4	$(x_2, y_0, x_1, y_5, x_2)$	t_4	$(x_3, y_1, x_2, y_0, x_3)$	$t_9 - t_5 + t_4 - t_8$
5	C_5	$(x_2, y_1, x_0, y_5, x_2)$	t_5	$(x_3, y_2, x_1, y_0, x_3)$	$t_{10} - t_1 - t_8$
6	C_6	$(x_2, y_3, x_0, y_5, x_2)$	t_6	$(x_3, y_4, x_1, y_0, x_3)$	$t_{11} - t_3 - t_8$
7	C_7	$(x_2, y_4, x_0, y_5, x_2)$	t_7	$(x_3, y_5, x_1, y_0, x_3)$	$-t_8$
8	C_8	$(x_3, y_0, x_1, y_5, x_3)$	t_8	$(x_4, y_1, x_2, y_0, x_4)$	$t_{13} - t_5 + t_4 - t_{12}$
9	C_9	$(x_3, y_1, x_0, y_5, x_3)$	t_9	$(x_4, y_2, x_1, y_0, x_4)$	$t_{14} - t_1 - t_{12}$
10	C_{10}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}	$(x_4, y_3, x_1, y_0, x_4)$	$t_{15} - t_2 - t_{12}$
11	C_{11}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}	$(x_4, y_5, x_1, y_0, x_4)$	$-t_{12}$
12	C_{12}	$(x_4, y_0, x_1, y_5, x_4)$	t_{12}	$(x_5, y_1, x_2, y_0, x_5)$	$t_{17} - t_5 + t_4 - t_{16}$
13	C_{13}	$(x_4, y_1, x_0, y_5, x_4)$	t_{13}	$(x_5, y_2, x_1, y_0, x_5)$	$t_{18} - t_1 - t_{16}$
14	C_{14}	$(x_4, y_2, x_0, y_5, x_4)$	t_{14}	$(x_5, y_3, x_1, y_0, x_5)$	$t_{19} - t_2 - t_{16}$
15	C_{15}	$(x_4, y_3, x_0, y_5, x_4)$	t_{15}	$(x_5, y_4, x_1, y_0, x_5)$	$-t_3 - t_{16}$
16	C_{16}	$(x_5, y_0, x_1, y_5, x_0, y_4, x_5)$	t_{16}	$(x_0, y_1, x_2, y_0, x_1, y_5, x_0)$	$-t_5 + t_4$
17	C_{17}	$(x_5, y_1, x_0, y_4, x_5)$	t_{17}	$(x_0, y_2, x_1, y_5, x_0)$	$-t_1$
18	C_{18}	$(x_5, y_2, x_0, y_4, x_5)$	t_{18}	$(x_0, y_3, x_1, y_5, x_0)$	$-t_2$
19	C_{19}	$(x_5, y_3, x_0, y_4, x_5)$	t_{19}	$(x_0, y_4, x_1, y_5, x_0)$	$-t_3$
	C	$\zeta(C)$	C^α	$\zeta(C^\alpha)$	
20	C_1	$(x_1, y_2, x_0, y_5, x_1)$	t_1	$(x_2, y_4, x_0, y_3, x_2)$	$t_7 - t_6$
21	C_2	$(x_1, y_3, x_0, y_5, x_1)$	t_2	$(x_2, y_1, x_0, y_3, x_2)$	$t_5 - t_6$
22	C_3	$(x_1, y_4, x_0, y_5, x_1)$	t_3	$(x_2, y_5, x_0, y_3, x_2)$	$-t_6$
23	C_4	$(x_2, y_0, x_1, y_5, x_2)$	t_4	$(x_4, y_0, x_2, y_3, x_4)$	$t_{12} - t_4 + t_6 - t_{15}$
24	C_5	$(x_2, y_1, x_0, y_5, x_2)$	t_5	$(x_4, y_2, x_0, y_3, x_4)$	$t_{14} - t_{15}$
25	C_6	$(x_2, y_3, x_0, y_5, x_2)$	t_6	$(x_4, y_1, x_0, y_3, x_4)$	$t_{13} - t_{15}$
26	C_7	$(x_2, y_4, x_0, y_5, x_2)$	t_7	$(x_4, y_5, x_0, y_3, x_4)$	$-t_{15}$
27	C_8	$(x_3, y_0, x_1, y_5, x_3)$	t_8	$(x_1, y_0, x_2, y_3, x_1)$	$-t_4 + t_6 - t_2$
28	C_9	$(x_3, y_1, x_0, y_5, x_3)$	t_9	$(x_1, y_2, x_0, y_3, x_1)$	$t_1 - t_2$
29	C_{10}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}	$(x_1, y_4, x_0, y_3, x_1)$	$t_3 - t_2$
30	C_{11}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}	$(x_1, y_5, x_0, y_3, x_1)$	$-t_2$
31	C_{12}	$(x_4, y_0, x_1, y_5, x_4)$	t_{12}	$(x_5, y_0, x_2, y_3, x_5)$	$t_{16} - t_4 + t_6 - t_{19}$
32	C_{13}	$(x_4, y_1, x_0, y_5, x_4)$	t_{13}	$(x_5, y_2, x_0, y_3, x_5)$	$t_{18} - t_{19}$
33	C_{14}	$(x_4, y_2, x_0, y_5, x_4)$	t_{14}	$(x_5, y_4, x_0, y_3, x_5)$	$-t_{19}$
34	C_{15}	$(x_4, y_3, x_0, y_5, x_4)$	t_{15}	$(x_5, y_1, x_0, y_3, x_5)$	$t_{17} - t_{19}$
35	C_{16}	$(x_5, y_0, x_1, y_5, x_0, y_4, x_5)$	t_{16}	$(x_3, y_0, x_2, y_3, x_0, y_5, x_3)$	$t_8 - t_4 + t_6$
36	C_{17}	$(x_5, y_1, x_0, y_4, x_5)$	t_{17}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}
37	C_{18}	$(x_5, y_2, x_0, y_4, x_5)$	t_{18}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}
38	C_{19}	$(x_5, y_3, x_0, y_4, x_5)$	t_{19}	$(x_3, y_1, x_0, y_5, x_3)$	t_9

$$\begin{aligned}
t_{16} &= -t_5 - t_3 \text{ (combining Rows 3 and 16 of Table 3)} \\
t_{15} &= t_5 \text{ (by Row 16 of Table 3)} \\
t_{14} &= t_5 - t_2 \text{ (combining Rows 16 and 19 of Table 3)} \\
t_{13} &= -t_1 - t_2 + t_3 + t_5 \text{ (combining Rows 16 and 18 of Table 3)} \\
t_{12} &= -t_1 \text{ (combining Rows 3, 16, and 17 of Table 3)} \\
t_{11} &= t_1 \text{ (by Row 11 of Table 3)} \\
t_{10} &= t_1 - t_2 + t_5 \text{ (combining Rows 12 and 15 of Table 3)} \\
t_9 &= -t_2 + t_5 \text{ (combining Rows 12 and 14 of Table 3)} \\
t_8 &= -t_2 \text{ (combining Rows 3, 12, and 13 of Table 3)} \\
t_7 &= t_2 \text{ (by Row 8 of Table 3)} \\
t_6 &= t_1 + t_2 - t_3 \text{ (combining Rows 8 and 11 of Table 3).}
\end{aligned}$$

Using the results obtained above, we further have:

$$t_2 = -\alpha^*(t_1) \text{ (by Row 30 of Table 3)} \quad (23)$$

$$t_3 = \alpha^*(t_1) + t_1 \text{ (by Row 20 of Table 3)} \quad (24)$$

$$t_5 = -\alpha^*(t_2) = (\alpha^*)^2(t_1) \text{ (by Row 26 of Table 3).} \quad (25)$$

Therefore, $\mathbb{Z}_m = \langle t_1 \rangle$, and by Proposition 2.6, we may assume that $t_1 = 1$ and that $\alpha^*(x) = kx$ for some $k \in \mathbb{Z}_m^*$. We now have $t_2 = -k$, $t_3 = k + 1$, and $t_5 = k^2$. Row 22 of Table 3 implies that $k^2 = k$, and using Row 27 of Table 3 we can conclude that $k^2 = 1$, implying that $k = 1$. Further, Row 20 of Table 3 now implies that $4 = 0$, and thus either $m = 2$ or $m = 4$. If $m = 2$ then $X \cong BC_{12}[\emptyset, \{0, 1, 2, 4, 9\}, \emptyset]$ and if $m = 4$ then $X \cong BC_{24}[\emptyset, \{0, 1, 3, 11, 20\}, \emptyset]$. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.1 Suppose that X is a connected pentavalent arc-transitive bicirculant $BC_n[L, M, R]$, where $n \geq 5$. Then, by Theorem 3.2, $|M| \in \{2, 3, 5\}$, and therefore Theorems 3.3, 3.6, and 3.11 combined together give that X is isomorphic to one of the graphs given in the statement of the theorem, all of which are clearly arc-transitive. $\square$

Remark 3.12 By Theorem 3.11, there exists a $\mathbb{Z}_4$ -cover of $K_{6,6} - 6K_2$ which is a 2-arc-transitive bipartite dihedrant. This graph was missing in the classification of 2-arc-transitive dihedrants obtained in [4]. In the communication with the authors of [4], we concluded that there was a typing error in the statement of their main result, which should be stated as bellow.

Theorem 3.13 [4] *Let X be a dihedrant. Then X is 2-arc-transitive if and only if it is one of the graphs listed bellow:*

- (i) cycles C_{2n} , $n \geq 3$;
- (ii) complete graphs K_{2n} , $n \geq 3$;
- (iii) complete bipartite graphs $K_{n,n}$, $n \geq 3$;

- (iv) *complete bipartite graphs minus a matching* $K_{n,n} - nK_2$, $n \geq 3$;
- (v) *incidence and nonincidence graphs* $B(H_{11})$ and $B'(H_{11})$ *of the Hadamard design on 11 points*;
- (vi) *incidence and nonincidence graphs* $B(PG(d, q))$ and $B'(PG(d, q))$, *with* $d \geq 2$ *and* q *a prime power, of projective spaces*;
- (vii) *infinite family of regular* $\mathbb{Z}_d$ -*covers* K_{q+1}^{2d} *of* $K_{q+1, q+1} - (q+1)K_2$, *where* $q \geq 3$ *is an odd prime power and* d *is a divisor of* $q-1$, *obtained by identifying the vertex set of the base graph with two copies of the projective line* $PG(1, q)$, *where the missing matching consists of all pairs of the form* $[a, a']$, $a \in PG(1, q)$, *and the edge* $[a, b']$ *carries trivial voltage if* a *or* b *is infinity, and carries voltage* $h \in \mathbb{Z}_d$, *the residue class of* $h \in \mathbb{Z}$, *if and only if* $a - b = \theta^h$, *where* θ *generates the multiplicative group* $\mathbb{F}_q^*$ *of the Galois field* $\mathbb{F}_q$.

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