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A classification of semisymmetric graphs of order $2p^3$: unfaithful case

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Abstract A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive, but not vertex-transitive. Every semisymmetric graph is a bipartite graph with two biparts of equal size. It was proved by Folkman in (J Comb Theory Ser B 3:215–232, 1967) that there exist no semisymmetric graphs of order $2p$ and $2p^2$, where p is a prime. For any distinct primes p and q, the classification of semisymmetric graphs of order $2pq$ was given by Du and Xu in (Comm Algebra 28:2685–2715, 2000). Naturally, one of our long-term goals is to determine all the semisymmetric graphs of order $2p^3$, for any prime p. All these graphs Γ are divided into two subclasses: (I) the automorphism group Aut (Γ) acts unfaithfully on at least one bipart; and (II) Aut (Γ) acts faithfully on both biparts. In Wang and Du (Eur J Comb 36:393–405, 2014), a group theoretical characterization for Subclass (I) was given by the authors. Based on this characterization, this paper gives a complete classification for Subclass (I).

Keywords Permutation group · Vertex-transitive graph · Semisymmetric graph

1 Introduction

All graphs considered in this paper are finite, connected, simple, and undirected. For a permutation group G on Ω, a subset $\Delta \subset \Omega$ and a subgroup $N \leq G$ preserving Δ, by G_Δ and $G_{(\Delta)}$ we denote the stabilizer of G relative to Δ setwise and pointwise,

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respectively, and by Δ_N, the set of N-orbits on Δ. For a graph $\Gamma = (V, E)$ with the vertex set V and edge set E, by $\{u, v\}$ and (u, v), we denote an edge and an arc of Γ, respectively, and by Aut (Γ) its full automorphism group. A graph Γ is said to be *regular* if all the vertices have the same degree. Set $A = \text{Aut}\,(\Gamma)$. The graph Γ is said to be *vertex-transitive* and *edge-transitive* if A acts transitively on V and E, respectively. If Γ is bipartite with the bipartition $V = W \cup U$, then we let A^+ be a subgroup of A preserving both W and U. Since Γ is connected, we know that either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism which interchanges the two biparts. For $G \leq A^+$, Γ is said to be G-*semitransitive* if G acts transitively on both W and U, while an A^+-semitransitive graph is simply said to be *semitransitive*.

A graph is said to be *semisymmetric* if it is regular and edge-transitive, but not vertex-transitive. It is easy to see that every semisymmetric graph is a semitransitive bipartite graph with two biparts of equal size.

The first person who studied semisymmetric graphs was Folkman. In 1967, he constructed several infinite families of such graphs and proposed eight open problems, see [11]. Afterwards, much work have been done on semisymmetric graphs, see [2, 3,15–17,23]. They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, using group-theoretical methods, Iofinova and Ivanov [15] in 1985 classified cubic semisymmetric graphs whose automorphism group acts primitively on both biparts, this was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups and some methods from graph coverings, some new results of semisymmetric graphs have appeared, see [4,6–10,12,18–22,26] and so on.

In [11], Folkman proved that there are no semisymmetric graphs of order $2p$ and $2p^2$ where p is a prime. In [10], the first author and Xu classified semisymmetric graphs of order $2pq$ for two distinct primes p and q. Therefore, a natural question is to determine semisymmetric graphs of order $2p^3$, where p is a prime. Since the smallest semisymmetric graphs have order 20 (see [11]), we let $p \geq 3$. It was proved in [20] that the Gray graph of order 54 is the only cubic semisymmetric graph of order $2p^3$. The classification of all the semisymmetric graphs of order $2p^3$ is still one of attractive and difficult open problems. These graphs Γ are naturally divided into two subclasses:

Subclass (I): Aut (Γ) acts unfaithfully on at least one bipart;
Subclass (II): Aut (Γ) acts faithfully on both biparts.

In [24], a group theoretical characterization for Subclass (I) was given. Based on this characterization, this paper will give a complete classification for Subclass (I).

First, we introduce two definitions in the following two paragraphs.

Let $\Sigma = (V_1, E_1)$ be a connected edge-transitive graph with bipartition $V_1 = W_1 \cup U_1$, where $|W_1| = p^3$ and $|U_1| = p^2$ for an odd prime p. Now we define a bipartite graph $\Gamma = (V, E)$ with bipartition $V = W \cup U$, where

$$W = W_1, \quad U = U_1 \times Z_p = \{(u, i) \mid u \in U_1, i \in Z_p\},$$
$$E = \big\{\{w, (u, i)\} \mid \{w, u\} \in E_1, i \in Z_p\big\}.$$

Then we shall call that Γ is the *graph expanded from* Σ. By Lemma 2.12, Γ is edge-transitive and regular, and moreover, if any two vertices in W_1 have the different neighborhood in Σ, then Γ is semisymmetric.

Let $\mathcal{P}$ be a partition of the vertex set V. Then we let $\Gamma_{\mathcal{P}}$ be the *quotient graph* of Γ relative to $\mathcal{P}$, that is, the graph with the vertex set $\mathcal{P}$, where two subsets V_1 and V_2 in $\mathcal{P}$ are adjacent if there exist two vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that v_1 and v_2 are adjacent in Γ. In particular, when $\mathcal{P}$ is the set of orbits of a subgroup N of Aut (Γ), we denote $\Gamma_{\mathcal{P}}$ by Γ_N.

The starting point is the following two propositions, see [24].

Proposition 1.1 *Let p be an odd prime, and let $\Gamma = (V, E)$ be a semisymmetric graph of order $2p^3$ with the bipartition $V = W \cup U$ and $A = Aut\ (\Gamma)$. Suppose that A acts unfaithfully on W. Then A is faithful on U. Furthermore, let $K = A_{(W)}$ and set $\Sigma = \Gamma_K$. Then the following hold.*

(1) Every orbit of K on U has length p, $K \cong (S_p)^{p^2}$ and Γ is expanded from Σ.
(2) A/K acts faithfully on U_K and on $W_K \cup U_K$, and $A/K \cong Aut\ (\Sigma)$.
(3) Any two vertices in W have the different neighborhood in Γ.

By Proposition 1.1, we know that the graph Γ is uniquely determined by its quotient graph Σ, which is a bipartite edge-transitive graph whose biparts have length of p^3 and p^2, respectively, and whose automorphism group acts faithfully on both biparts. Therefore, we shall pay attention to such special family of graphs. A group theoretical characterization for Σ is given by the following proposition.

Proposition 1.2 *Let p be an odd prime, and let $\Sigma = (V, E)$ be a bipartite edge-transitive graph with biparts W and U where $|W| = p^3$ and $|U| = p^2$ and whose automorphism group $A = Aut\ (\Sigma)$ acts faithfully on both biparts. Then A acts imprimitively on U, with a system $\mathcal{U}$ of blocks. Set $\mathcal{W} = W_{A_{(\mathcal{U})}}$. Then, either*

(1) A acts primitively on W, $p = 3$, and $\Sigma \cong \Sigma_{11}$ or Σ_{12}, as defined in Example 2.1, or
(2) A acts imprimitively on W, and either
  (2.1) $|\mathcal{W}| = 1$, $A_{(\mathcal{U})}$ is solvable, and A is an affine group, or
  (2.2) $|\mathcal{W}| = p$ and either
    (2.2.1) the quotient graph $\Sigma_{\mathcal{W}\cup\mathcal{U}} \cong C_{2p}$, or
    (2.2.2) $\Sigma_{\mathcal{W}\cup\mathcal{U}}$ is of valency at least 3, $p \geq 5$, A contains a nonabelian normal p-subgroup acting regularly on W, $A_{(\mathcal{U})}$ is solvable, and $A/A_{(\mathcal{U})} \cong Z_p \rtimes Z_r$, where $r \mid (p-1)$ and $r \geq 3$.

Remark 1.3 Based on the above proposition, we shall finish the classification for Subclass (1), equivalently, we need to determine the graphs Σ contained in four cases: (1), (2.1), (2.2.1) and (2.2.2) in Proposition 1.2.

Theorem 1.4 (Classification Theorem) *Let p be an odd prime and let Γ be a semisymmetric graph of order $2p^3$, whose automorphism group acts unfaithfully on one bipart. Then Γ is expanded from one of the graphs Σ given in Table 1:*

Table 1 Graphs Σ

Graphs	p	Valency of $w \in W$	Aut (Σ)	Type in Prop 1.2	Defined in
Σ_{11}	3	3	$S_3 \wr S_3$	Case (1)	Exam 2.1
Σ_{12}	3	6	$S_3 \wr S_3$	Case (1)	Exam 2.1
$\Sigma_{21}(p)$	$p \geq 5$	p	$Z_p^3 \rtimes (Z_p \rtimes Z_{p-1}^2)$	Case (2.1)	Exam 2.2
$\Sigma_{22}(p,r)$	$p \geq 5$	$rp, r \geq 2$, $r \mid (p-1)$	$Z_p^3 \rtimes (Z_p \rtimes (Z_{p-1} \times Z_r))$	Case (2.1)	Exam 2.2
$\Sigma_{31}(p,r_2)$	$p \geq 5$	$r_2,\ r_2 \geq 3$, $r_2 \mid (p-1)$	$P \rtimes S, \lvert P\rvert = p^3$, P nonabelian, Exp $(P) = p$ $\lvert S\rvert = r_2(p-1)$, S abelian	Case (2.2.2)	Exam 2.3
$\Sigma_{32}(p,r_1,r_2,r_3)$	$p \geq 5$	$r_1r_2,\ r_2 \geq 3$, $r_1, r_2 \mid (p-1)$ $0 \leq r_3 \leq$ $(r_2, \frac{p-1}{r_1}) - 1$	$P \rtimes S, \lvert P\rvert = p^3$, P nonabelian, Exp $(P) = p$ $\lvert S\rvert = r_1r_2$, S abeliabn	Case (2.2.2)	Exam 2.3
$\Sigma_4(p,r)$	$p \geq 5$	$r,\ r \geq 3$, $r \mid (p-1)$	$P \rtimes Z_r$, $\lvert P\rvert = p^3$, P nonabelian, Exp $(P) = p^2$	Case (2.2.2)	Exam 2.4
$\Sigma_{51}(p)$	$p \geq 3$	2	$S_p \wr D_{2p}$	Case (2.2.1)	Exam 2.5
$\Sigma_{52}(p)$	$p \geq 3$	$2(p-1)$	$S_p \wr D_{22}$	Case (2.2.1)	Exam 2.5
Σ_{61}	11	10	PSL $(2, 11) \wr D_{22}$	Case (2.2.1)	Exam 2.6
Σ_{62}	11	12	PSL $(2, 11) \wr D_{22}$	Case (2.2.1)	Exam 2.6
Σ_{71}	$\frac{q^n-1}{q-1} \geq 5$	$2\frac{q^{n-1}-1}{q-1}$	$P\Gamma L(n,q) \wr D_{2p}$	Case (2.2.1)	Exam 2.7
Σ_{72}	$\frac{q^n-1}{q-1} \geq 5$	$2q^{n-1}$	$P\Gamma L(n,q) \wr D_{2p}$	Case (2.2.1)	Exam 2.7

Table 1 continued

Graphs	p	Valency of $w \in W$	Aut (Σ)	Type in Prop 1.2	Defined in
$\Sigma_{81}(p, r)$	$p \geq 5$	$2r, r \mid (p-1)$ $r \neq 1, p-1,$ $(p, r) \neq (7{,}3),$ $(11{,}5),$	$(Z_p \rtimes Z_r) \wr D_{2p}$	Case (2.2.1)	Exam 2.8
$\Sigma_{82}(p, r)$	$p \geq 5$	$2r, r \neq 2$ $2r \mid (p-1)$ $(p, r) \neq (7{,}3),$ $(11{,}5),$	$(Z_p \rtimes Z_r) \wr D_{2p}$	Case (2.2.1)	Exam 2.8

Remark 1.5 One may see from [24,25] and the proof of Theorems 3.1 and 4.1 that every graph Σ in Table 1 is edge-transitive and any two vertices in W have the different neighborhood, where $|W| = p^3$. By Lemma 2.12, Γ is semisymmetric.

Remark 1.6 Every graph in Examples 2.1–2.8 is described by the so-called bi-coset graph $\mathbf{B}(A; L, R, D)$, where A is exactly its full automorphism group and its edge-set is explicitly listed in the corresponding theorem in Sects. 3 and 4, [24,25]. Moreover, these graphs are uniquely determined by the given parameters.

In Sect. 2, eight examples will be defined and some preliminary results will be quoted. Cases (1) and (2.2.1) in Proposition 1.2 have been determined in [24] and [25], respectively, and here we just quote the results, that is Example 2.1 for Case (1), and Examples 2.5–2.8 for Case (2.2.1). The remaining two cases (2.1) and (2.2.2) will be discussed in Sects. 3 and 4, respectively.

2 Examples and preliminaries

For group-theoretic concepts and notation, the reader is refereed to [1,14]. Moreover, for a prime p, by $p^i \,\|\, n$ we mean that $p^i \mid n$ but $p^{i+1} \nmid n$. For a ring S, let S^* be the multiplicative group of all the units in S. For a group G and a subgroup H of G, by $[G : H]$ we denote the set of right cosets of H in G, where the action of G on $[G : H]$ is always assumed to be the right multiplication action. For any α in the n-dimensional row vector space $\mathbf{V} = \mathbf{V}(n, p)$ over $\mathrm{GF}(p)$, we denote by t_α the translation corresponding to α in the affine group $\mathrm{AGL}(n, p)$ and by N the translation subgroup. Then $\mathrm{AGL}(n, p) \cong N \rtimes \mathrm{GL}(n, p)$. We adopt matrix notation for $\mathrm{GL}(n, p)$ and so we have $g^{-1}t_\alpha g = (t_\alpha)^g = t_{\alpha g}$, for any $t_\alpha \in N \leq \mathrm{AGL}(n, p)$ and any $g \in \mathrm{GL}(n, p)$.

The following definition is basic for this paper.

Definition [10] Let G be a group, let L and R be subgroups of G and let $D = RdL$ be a double coset of R and L in G. Define a bipartite graph $\Gamma = \mathbf{B}(G, L, R; D)$ with the bipartition $V = [G : L] \cup [G : R]$ and edge set $E = \{(Lg, Rdg) \mid g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.

Now we are ready to define eight families of bi-coset graphs with biparts of size p^2 and p^3, respectively, which are all the graphs in Theorem 1.4.

Example 2.1 [24] Let $M = \langle a, b\rangle \cong S_3$ where $|a| = 3$ and $|b| = 2$. Let

$$\begin{aligned} A &= \big(M \times M \times M\big) \rtimes S_3 = M \wr S_3,\\ L &= \big(\langle b\rangle \times \langle b\rangle \times \langle b\rangle\big) \rtimes S_3, \;\; R = \big(\langle b\rangle \times M \times M\big) \rtimes \big\langle (23)\big\rangle,\\ D_1 &= RL, \quad D_2 = RaL, \end{aligned}$$

Define two bi-coset graphs

$$\Sigma_{11} = \mathbf{B}\big(A; L, R, D_1\big) \text{ and } \Sigma_{12} = \mathbf{B}\big(A; L, R, D_2\big).$$

The valency $d(L)$ of the vertex $L \in [G : L]$ is 3 for Σ_{11} and 6 for Σ_{12}.

Example 2.2 Let $p \geq 5$. Let Λ be the subgroup of order $r \geq 1$ of $\mathrm{GF}(p)^*$. Set

$$x = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = y(\lambda, \mu) = \begin{pmatrix} \mu^2\lambda^{-1} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix} \in \mathrm{GL}(3, p),$$
$$S = \big\langle y(\mu, \lambda) \bigm| \mu \in \mathrm{GF}(p)^*, \lambda \in \Lambda \big\rangle \leq N_{\mathrm{GL}(3,p)}\big(\langle x \rangle\big).$$

Let N be the translation subgroup of $\mathrm{AGL}(3, p)$ and $N_0 = \langle t_{(1,0,0)}, t_{(0,1,0)} \rangle \leq N$. Set

$$A = N \rtimes \big(\langle x \rangle \rtimes S\big), \quad L = \langle x \rangle \rtimes S, \quad R = N_0 \rtimes S$$

Define the graphs

$$\Sigma_{21}(p) = \mathbf{B}\big(A; L, R, D\big), \quad \text{where } \Lambda = \mathrm{GL}(p)^*, D = RL;$$
$$\Sigma_{22}(p, r) = \mathbf{B}\big(A; L, R, D\big), \quad \text{where } 2 \leq r,\ D = Rt_{(0,0,1)}L.$$

Moreover, $d(L) = p$ for $\Sigma_{21}(p)$ and $d(L) = rp$ for $\Sigma_{22}(p, r)$, see Sect. 3.

Example 2.3 For $p \geq 5$, let

$$P = \langle a, b, c \bigm| a^p = b^p = c^p = 1, [b, a] = c, [c, a] = [c, b] = 1 \rangle.$$

For any $s', t' \in \mathrm{GF}(p)^* = \langle \theta \rangle$, set $\phi(s', t') \in \mathrm{Aut}(P)$ such that $a \to a^{s'}, b \to b^{t'}$. Let r_1 and r_2 be two divisors of $p - 1$ where $r_2 \geq 3$. Define a abelian subgroup $S = \langle \phi(1, 2), \phi(s, t) \rangle$ of order $r_1 r_2$ such that

$$S \cap \big\langle \phi(1, \theta) \big\rangle = \big\langle \phi(1, e) \big\rangle \cong Z_{r_1}, \ S/\langle \phi(1, e) \rangle \cong Z_{r_2},$$

that is $|e| = r_1, |s| = r_2$ and $t^{r_2} \in \langle e \rangle$. Now, $T = \langle t \in \mathrm{GF}(p)^* \bigm| t^{r_2} \leq \langle e \rangle \rangle$ is a subgroup of $\mathrm{GF}(p)^*$ of order $r_1(r_2, \frac{p-1}{r_1})$ and so we choose

$$e = \theta^{\frac{p-1}{r_1}}, \ s = \theta^{\frac{p-1}{r_2}}, \ t = \big(\theta^{\frac{p-1}{r_1(r_2, \frac{p-1}{r_1})}}\big)^{r_3},$$

where $0 \leq r_3 \leq (r_2, \frac{p-1}{r_1}) - 1$, and if $r_1 = 1$ then we let $r_3 \notin \{1, r_2 - 1\}$.

Let

$$A = P \rtimes S, \quad L = S, \quad R = \langle b \rangle S.$$

Define the graphs

$$\Sigma_{31}(p, r_2) = \mathbf{B}\big(A; L, R, D\big), D = RaL, \ \text{where } r_1 = p - 1, r_3 = 0;$$
$$\Sigma_{32}(p, r_1, r_2, r_3) = \mathbf{B}\big(A; L, R, D\big), \ D = RcaL.$$

Moreover, $d(L) = r_2$ for $\Sigma_{31}(p, r_2)$ and $d(L) = r_1r_2$ for $\Sigma_{32}(p, r_1, r_2, r_3)$, see Sect. 4.1.

Example 2.4 Let $p \geq 5$ and let

$$P = \langle a, b \mid a^{p^2} = b^p = 1, [b, a] = a^p \rangle.$$

Let $Z^*_{p^2} = \langle \lambda \rangle$ and $r \geq 3$, a divisor of $p-1$. Let $S = \langle \phi(\lambda^{\frac{p(p-1)}{r}}) \rangle \leq \mathrm{Aut}\,(P)$ such that $\phi(a) = a^{\lambda^{\frac{p(p-1)}{r}}}$ and $\phi(b) = b$. Let

$$A = P \rtimes S, \quad L = S, \quad R = \langle b \rangle S, \quad D = RaL.$$

Define a graph $\Sigma_4(p, r) = \mathbf{B}(A; L, R, D)$. Moreover, $d(L) = r$, see Sect. 4.2.

The remaining four families of bi-coset graphs $\mathbf{B}(A; L, R, D)$ are quoted from [25]. For these graphs, A acts imprimitively on both $[A : L]$ and $[A : R]$. Let $p \geq 3$ and set

$$\sigma = (0, 1, \ldots, p-1), \quad \tau = (0)(1, -1), \ldots, \left(\frac{p-1}{2}, \frac{p+1}{2}\right) \in S_p.$$

Then $\langle \sigma, \tau \rangle \cong D_{2p}$. For a group M and a positive integer k, set $M^k = \overbrace{G \times \cdots \times G}^{k \text{ times}}$.

Example 2.5 For $p \geq 3$, let $M \cong S_p$, $H \leq M$ and $H \cong S_{p-1}$. Pick up an element $m \in M \setminus H$. Set

$$\begin{aligned}
A &= M \wr \langle \sigma, \tau \rangle = M^p \rtimes \langle \sigma, \tau \rangle, \\
L &= \left(M^{\frac{p-1}{2}-1} \times H \times H \times M^{\frac{p+1}{2}-1}\right) \rtimes \langle \tau \rangle, \\
R &= \left(H \times M^{p-1}\right) \rtimes \langle \tau \rangle, \\
D_1 &= R\sigma^{\frac{p-1}{2}} L, \;\; D_2 = R \left(m, \overbrace{1, \ldots, 1}^{p-1 \text{ times}} \right) \sigma^{\frac{p-1}{2}} L
\end{aligned}$$

Define

$$\Sigma_{51}(p) = \mathbf{B}\big(A; L, R, D_1\big), \;\; \Sigma_{52}(p) = \mathbf{B}\big(A; L, R, D_2\big).$$

Then $d(L) = 2$ for Σ_{51} and $2(p-1)$ for Σ_{52}.

Example 2.6 Let $M \cong PSL(2, 11)$ and let H and J be two subgroups of M which are isomorphic to A_5 and are not conjugate in M. It is well known that acting on $[M : H]$, the subgroup J has two orbits of length 5 and 6, respectively. No loss of any generality, set $J \cap H \cong A_4$ and $J \cap H^m \cong D_{10}$ for some $m \in M$ so that $|J : (J \cap H)| = 5$ and $|J : (J \cap H^m)| = 6$.

Set

$$\begin{aligned} A &= M \wr \langle \sigma, \tau \rangle = M^p \rtimes \langle \sigma, \tau \rangle, \\ L &= \left(M^{\frac{p-1}{2}-1} \times J \times J \times M^{\frac{p+1}{2}-1} \right) \rtimes \langle \tau \rangle, \\ R &= \left(H \times M^{p-1} \right) \rtimes \langle \tau \rangle. \\ D_1 &= R\sigma^{\frac{p-1}{2}} L, \; D_2 = R \left(m, \overbrace{1, \ldots, 1}^{p-1 \text{ times}} \right) \sigma^{\frac{p-1}{2}} L. \end{aligned}$$

Then define

$$\Sigma_{61} = \mathbf{B}\big(A; L, R, D_1\big) \text{ and } \Sigma_{62} = \mathbf{B}\big(A; L, R, D_2\big).$$

Then $d(L) = 10$ for Σ_{61} and 12 for Σ_{62}.

Example 2.7 Let $M \cong P\Gamma L(n, q)$, where $p = \frac{q^n-1}{q-1} \geq 5$. In the projective geometry PG (n, q), take a point v and two hyperplanes S_1 and S_2 such that $v \in S_1$ and $v \notin S_2$. Let H, J and J^m for some $m \in M$ be the stabilizers of v, S_1, and S_2, respectively. For the above M, J and m, let A, R, L, D_1, and D_2 have the same form as in Example 2.6. Then define

$$\Sigma_{71}(p) = \mathbf{B}\big(A; L, R, D_1\big) \text{ and } \Sigma_{72}(p) = \mathbf{B}\big(A; L, R, D_2\big).$$

Then $d(L) = 2\frac{q^{n-1}-1}{q-1}$ for Σ_{71} and $2q^{n-1}$ for Σ_{72}.

Example 2.8 For $p \geq 5$, let $S = \langle t \rangle \rtimes \langle c \rangle \cong Z_p \rtimes Z_{p-1}$. For $r \mid (p-1)$ and $r \neq 1, p-1$, let $M = \langle t \rangle \rtimes \langle c^{\frac{p-1}{r}} \rangle \cong Z_p \rtimes Z_r$ and $H = \langle c^{\frac{p-1}{r}} \rangle$. Set

$$\begin{aligned} A_1 &= M \wr \langle \sigma, \tau \rangle = M^p \rtimes \langle \sigma, \tau \rangle, \\ L_1 &= \left(M^{\frac{p-1}{2}-1} \times H \times H \times M^{\frac{p+1}{2}-1} \right) \rtimes \langle \tau \rangle, \\ R_1 &= \left(H \times M^{p-1} \right) \rtimes \langle \tau \rangle, \\ D_1 &= R\sigma^{\frac{p-1}{2}} L. \end{aligned}$$

Then define

$$\Sigma_{81}(p, r) = \mathbf{B}\big(A_1; L_1, R_1, D_1\big),$$

where $d(L) = 2r$.

Suppose that $2r \mid (p-1)$, where $r \geq 2$. Set $d = (c^{\frac{p-1}{2r}}, \cdots, c^{\frac{p-1}{2r}}) \in S^p$. Set

$$\begin{aligned} A_2 &= M \wr \langle \sigma, d\tau \rangle = M^p \rtimes \langle \sigma, d\tau \rangle, \\ L_2 &= \left(M^{\frac{p-1}{2}-1} \times H \times H \times M^{\frac{p+1}{2}-1}\right) \rtimes \langle d\tau \rangle, \\ R_2 &= \left(H \times M^{p-1}\right) \rtimes \langle d\tau \rangle, \\ D_2 &= R\left(t, \overbrace{1, \ldots, 1}^{p-1 \text{ times}}\right) \sigma^{\frac{p-1}{2}} L. \end{aligned}$$

Then define

$$\Sigma_{82}(p, r) = \mathbf{B}\big(A_2; L_2, R_2, D_2\big),$$

where $d(L) = 2r$.

Finally, we quote some results.

Proposition 2.9 [10] *The graph $\Gamma = \mathbf{B}(G, L, R; D)$ defined before Example 2.1 is a well-defined bipartite graph. Under the right multiplication action on V of G, the graph Γ is G-semitransitive and edge-transitive. The kernel of the action of G on V is $\mathrm{Core}_G(L) \cap \mathrm{Core}_G(R)$, the intersection of the cores of the subgroups L and R in G. Furthermore, we have*

(i) the degree of any vertex in $[G : L]$ (resp. $[G : R]$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so Γ is regular if and only if $|L| = |R|$;
(ii) Γ is connected if and only if G is generated by elements of $D^{-1}D$.
(ii) Suppose Γ' is a G-semitransitive and edge-transitive graph with bipartition $V = U \cup W$. Take $u \in U$ and $w \in W$. Set $D = \{g \in G \mid w^g \in \Gamma_1'(u)\}$. Then $D = G_w g G_u$ and $\Gamma' \cong \mathbf{B}(G, G_u, G_w; D)$.

Proposition 2.10 [5, Theorem 3.4, 3.5] *For an odd prime p, let H be a maximal subgroup of $G = GL(2, p)$ and $H \neq SL(2, p)$. Then up to conjugacy, H is isomorphic to one of the following subgroups:*

(i) $D \rtimes \langle b \rangle$; where D is the subgroup of diagonal matrices and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
(ii) $\langle a \rangle \rtimes \langle b \rangle$, where $b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\langle a \rangle$ is the Singer subgroup of G, defined by $a = \begin{pmatrix} \gamma & \delta\theta \\ \delta & \gamma \end{pmatrix} \in G$, where $GF(p)^* = \langle \theta \rangle$, $F_{p^2} = GF(p)(\mathbf{t})$ for $\mathbf{t}^2 = \theta$, and $F_{p^2}^* = \langle \gamma + \delta\mathbf{t} \rangle$;
(iii) $\langle a \rangle \rtimes D$, where $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

(iv) $H/\langle z\rangle$ *is isomorphic to* $A_4 \times Z_{\frac{p-1}{2}}$, *for* $p \equiv 3, 5(\,mod\, 8)$; $S_4 \times Z_{\frac{p-1}{2}}$ *for* $p \equiv 1, 7(\,mod\, 8)$; *or* $A_5 \times Z_{\frac{p-1}{2}}$ *for* $p \equiv \pm 1(\,mod\, 10)$, *where* $z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $Z_{\frac{p-1}{2}} = Z(G)/\langle z\rangle$;

(v) $H/\langle z\rangle = A_4 \rtimes \langle s\rangle$, $\langle s^2\rangle \leq Z(G)/\langle z\rangle$, *if* $p \equiv 1(\,mod\, 4)$.

Proposition 2.11 [24] *Let* G *be an imprimitive transitive group of degree* p^2 *with* $p \geq 3$ *and* $p^3 \bigm| |G|$. *Suppose that* G *has an imprimitive* p*-block system* $\mathcal{B}$ *with the kernel* K. *Let* P *be a Sylow* p*-subgroup of* G *and* $N = P \cap K$. *Then*

(1) $Exp\ (P) \leq p^2$, $|Z(P)| = p$ *and* $P = N\langle t\rangle$, *where* $t^p \in Z(P)$;

(2) K *is solvable,* $N char K$ *and so* $N \lhd G$, *provided either* $p = 3$; *or* $p \geq 5$ *and* $|N| \leq p^{p-1}$.

The following result has been used in several papers without any proof. Here we give a proof for the readers.

Lemma 2.12 *Let* $\Sigma = (V_1, E_1)$ *be a connected edge-transitive graph with bipartition* $V_1 = W_1 \cup U_1$, *where* $|W_1| = p^3$ *and* $|U_1| = p^2$ *for an odd prime* p, *as defined in Sect. 1. Suppose that* Γ *is expanded from the graph* Σ. *Then* Γ *is edge-transitive and regular. Moreover, if any two vertices in* W_1 *have the different neighborhood in* Σ, *then* Γ *is semisymmetric.*

Proof Recall that $\Gamma = (V, E)$ is defined as a bipartite graph with bipartition $V = W \cup U$, where

$$W = W_1, \quad U = U_1 \times Z_p = \{(u, i) \bigm| u \in U_1, i \in Z_p\},$$
$$E = \{\{w, (u, i)\} \bigm| \{w, u\} \in E_1, i \in Z_p\}.$$

To show Γ is edge-transitive, take any two edges $\{w_1, (u_1, i)\}$ and $\{w_2, (u_2, j)\}$, where $\{w_1, u_1\}, \{w_2, u_2\} \in E_1$ and $i, j \in Z_p$. Since Σ is edge-transitive with $|W_1| \neq |U_1|$, there exists an automorphism $\phi_1 \in \text{Aut}\ (\Sigma)$ such that $\phi_1\{w_1, u_1\} = \{w_2, u_2\}$. Pick a permutation σ on Z_p such that $\sigma(i) = j$. Define a map ϕ on $V(\Gamma)$ by

$$\phi\{w, (u, k)\} = \{\phi_1(w), (\phi_1(u), \sigma(k))\},$$

where $w \in W_1, u \in U_1, k \in Z_p$. Then one may check that ϕ is an automorphism of Γ mapping $\{w_1, (u_1, i)\}$ to $\{w_2, (u_2, j)\}$. Therefore, Γ is edge-transitive.

Take any $w \in W_1$ and $u \in U_1$ and let $d_1(w)$ and $d_1(u)$ be the respective valency of w and u in Σ. Since Σ is edge-transitive, we have $p^3 d_1(w) = p^2 d_1(u)$, that is, $p \cdot d_1(w) = d_1(u)$. For any $w \in W = W_1$ and $(u, i) \in U$, let $d(w)$ and $d(u, i)$ be the respective valency of w and u in Γ. By the definition of Γ, we know that $d(w) = p \cdot d_1(w) = d_1(u) = d(u, i)$. Therefore, Γ is regular.

By the definition, we see that for any $u \in U_1$, the p vertices $\{(u, i) \bigm| i \in Z_p\}$ in U have the same neighborhood in Γ. Therefore, if any two vertices in W_1 have the different neighborhood in Σ, then Γ cannot be vertex-transitive and so it is semisymmetric. □

3 Case (2.1) in Proposition 1.2

Throughout Sects. 3 and 4, the graph $\Sigma = (V, E)$ is assumed to be an edge-transitive graph with the bipartition $V = W \cup U$ such that $|W| = p^3, |U| = p^2$ and $A = \mathrm{Aut}\,(\Sigma)$ acts faithfully and imprimitively on both biparts. Let $\mathcal{U}$ be a block system of A on U with the kernel $A_{(\mathcal{U})}$. Let $\mathcal{W} = W_{A_{(\mathcal{U})}}$, the set of orbits of $A_{(\mathcal{U})}$ on W. Clearly, $\mathcal{W}$ is also a block system of A on W.

The aim of this section was to determine Case (2.1) in Proposition 1.2.

Theorem 3.1 *Suppose that $|\mathcal{W}| = 1$, $A_{(\mathcal{U})}$ is solvable, and A is an affine group on W. Then Σ is isomorphic to one of the graphs $\Sigma_{21}(p)$ and $\Sigma_{22}(p, r)$, as defined in Example 2.2.*

Proof Considering the imprimitive action of A on U, we find that $A \lesssim S_p \wr S_p = (S_p)^p \rtimes S_p$ and $A_{(\mathcal{U})} \lesssim (S_p)^p$. Let $P \in \mathrm{Syl}_p(A)$ and $N = P \bigcap A_{(\mathcal{U})}$. Then $P \lesssim (Z_p)^p \rtimes Z_p$ and $N \lesssim (Z_p)^p$. By the hypothesis of the theorem, $|\mathcal{W}| = 1$, that is $A_{(\mathcal{U})}$ is transitive on W, which implies that the abelian p-group N is regular on W, so that $|N| = p^3$. In fact, $N \lhd A$ by Proposition 2.11. Since P/N acts transitively on $\mathcal{U}$, where $|\mathcal{U}| = p$, we get $|P| = p^4$. Take $w \in W$ and $u \in U$. Since P acts transitively on W and U respectively, it follows that $P = N \rtimes P_w \cong Z_p^3 \rtimes Z_p$ and $P_u = N_u \cong Z_p^2$. By the hypothesis, A is an imprimitive affine group on W, that is $A = N \rtimes A_w$, where N is identified with the translation subgroup of $\mathrm{AGL}\,(3, p)$ and A_w is a reducible subgroup of $\mathrm{GL}\,(3, p)$. Let $\mathbf{V}$ be the 3-dimensional row vector space over $\mathrm{GF}\,(p)$.

The proof is divided into the following six steps:

Step (1): Determination of P, P_w and $Z(P)$.

By Proposition 2.11, $|Z(P)| = p$. Since P is nonabelian, $Z(P) \leq N$. From $P = N \rtimes P_w$, we get $Z(P) \leq C_P(P_w) \cap N \leq Z(P)$, which implies $C_P(P_w) \cap N = Z(P)$. Therefore, P_w centralizes exactly p elements in N, in other words, P_w fixes exactly p vectors in $\mathbf{V}$. Observe that P_w fixes setwise one 1-dimensional subspace if and only if it fixes pointwise this 1-dimensional subspace. Therefore, P_w fixes setwise exactly one 1-dimensional subspace. It is well known that (see [5]) $\mathrm{GL}\,(3, p)$ has two conjugacy classes of elements of order p with the respective representatives

$$x = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where x' fixes setwise more than one 1-dimensional subspaces and more than one 2-dimensional subspaces, but x fixes setwise only one 1-dimensional subspace, say $\langle \alpha \rangle$, and only one 2-dimensional subspace, say $\mathbf{V_1}$, where

$$\alpha = (0, 0, 1), \quad \mathbf{V_1} = \{(0, a_2, a_3) \mid a_2, a_3 \in \mathrm{GF}\,(p)\}.$$

Therefore, up to group isomorphism, we may set

$$P_w = \langle x \rangle \quad \text{and} \quad Z(P) = \langle t_\alpha \rangle,$$

recalling that t_α denotes the translation corresponding to α.

Step (2): Show that $\langle x\rangle \lhd A_w$.

Since A_w is a reducible subgroup and $x \in P_w \le A_w$ and since x fixes only one 1-dimensional subspace $\langle\alpha\rangle$ and only one 2-dimensional subspace $\mathbf{V_1}$, respectively, we know that A_w is contained in the subgroup of $\mathrm{GL}(3, p)$, fixing $\langle\alpha\rangle$ setwise, that is

$$G = \left\{ \begin{pmatrix} & & a_{13} \\ & X & \\ & & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \;\middle|\; X \in \mathrm{GL}(2, p), a_{13}, a_{23}, a_{33} \in \mathrm{GF}(p), a_{33} \neq 0 \right\}$$

or the subgroup fixing $\mathbf{V_1}$ setwise, that is

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & & \\ & X & \\ 0 & & \end{pmatrix} \;\middle|\; X \in \mathrm{GL}(2, p), a_{11}, a_{12}, a_{13} \in \mathrm{GF}(p), a_{11} \neq 0 \right\}.$$

Since the proofs for the two cases $A_w \le G$ and $A_w \le H$ use the similar arguments, we write the proof in detail only for the first case.

Let

$$\begin{aligned} G_1 &= \{g \in G \mid X = E, a_{33} = 1\} \cong Z_p^2, \\ G_2 &= \{g \in G \mid a_{13} = a_{23} = 0, a_{33} = 1\} \cong \mathrm{GL}(2, p), \\ G_3 &= \{g \in G \mid X = E, a_{13} = a_{23} = 0\} \cong Z_{p-1}. \end{aligned}$$

Then

$$G = G_1 \rtimes (G_2 \times G_3) \cong Z_p^2 \rtimes \left(\mathrm{GL}(2, p) \times Z_{p-1}\right).$$

We are now showing that $\langle x\rangle \lhd A_w$. For the contrary, suppose that $\langle x\rangle \ntrianglelefteq A_w$. Since $p \,\|\, |A_w|$, it follows that any subgroup of order p in A_w is not normal. Then from $A_w \cap G_1 \trianglelefteq A_w$, we get $A_w \cap G_1 = 1$. Therefore, in G/G_1 we have $\overline{A_w} \cong A_w$ and $\overline{\langle x\rangle} \cong \langle x\rangle$. Thus, $\overline{\langle x\rangle} \ntrianglelefteq \overline{A_w}$.

Now $A_w \lesssim G/G_1 \cong \mathrm{GL}(2, p) \times Z_{p-1}$. Clearly, every Sylow p-subgroup of $\mathrm{SL}(2, p)$ is also a Sylow p-subgroup of $\mathrm{GL}(2, p)$. As the Sylow p-subgroups of A_w are non-normal, inspecting the maximal subgroup of $\mathrm{GL}(2, p)$ (Proposition 2.10), we know that A_w contains a subgroup, say Q, such that $Q \cong \mathrm{SL}(2, p)$. Recalling that A_w is transitive on $\mathcal{U}$ and since $|\mathcal{U}| = p$, the group Q is also transitive on $\mathcal{U}$. For any $v \in U$, take $U_1 \le \mathcal{U}$ such that $v \in U_1$. Then $|Q_{U_1}| = |Q|/|\mathcal{U}| = p^2 - 1$. Note that $A_{U_1}/A_{(U_1)}$ is a transitive group of degree p and $NA_{(U_1)}/A_{(U_1)}$ is a normal subgroup of $A_{U_1}/A_{(U_1)}$. So, $A_{U_1}/A_{(U_1)} \lesssim Z_p \rtimes Z_{p-1}$. Clearly, $Q_{U_1}/Q_{(U_1)} \cong Q_{U_1}A_{(U_1)}/A_{(U_1)} \le A_{U_1}/A_{(U_1)}$. Since $|Q_{U_1}| = p^2 - 1$, if follows that $p + 1 \,\big|\, |Q_{(U_1)}|$, and so the unique involution, say e of Q ($\cong \mathrm{SL}(2, p)$) is in $Q_{(U_1)}$. Thus, e fixes v. By the arbitrariness of v, we know that $e \in A_{(U)}$, a contradiction.

Step (3): Determination of A_w and A_u.

By Step (2), $\langle x\rangle \lhd A_w$. Then $A_w \le N_{\mathrm{GL}(3,p)}(\langle x\rangle)$. One may check (see [5]) that $N_{\mathrm{GL}(3,p)}(\langle x\rangle) = \langle x, x'\rangle \rtimes \hat{S} \cong Z_p^2 \rtimes Z_{p-1}^2$, where x' is as defined in Step 1 and $\hat{S} = \langle y(\mu,\lambda) \mid \mu,\lambda \in \mathrm{GL}(p)^*\rangle \cong Z_{p-1}^2$ where,

$$y(\lambda,\mu) = \begin{pmatrix} \mu^2\lambda^{-1} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Since $\langle x\rangle \lhd A_w \le N_{\mathrm{GL}(3,p)}(\langle x\rangle)$ and $p \,\|\, |A_w|$, we can write

$$A_w = \langle x\rangle S \quad\text{and}\quad A = N \rtimes \big(\langle x\rangle S\big) = PS.$$

where $S \le N_{\mathrm{GL}(3,p)}(\langle x\rangle)$ and $|S| \mid (p-1)^2$. In particular, A is solvable.

Since $|A| = p^4|S|$ where $|S| \mid (p-1)^2$, it follows that $|A_u| = p^2|S|$ for some $u \in U$. Since A and A_u are solvable, we get that S is contained in a conjugacy of A_u and then no loss of any generality, let $S \le A_u$. Then S normalizes $N_u = N \cap A_u \cong Z_p^2$. Therefore, we get

$$A_u = N_u S.$$

Let $\mathbf{V}(S) = \langle \gamma \in \mathbf{V} \mid t_\gamma \in N_u\rangle$. Then $\mathbf{V}(S)$ is a 2-dimensional S-invariant subspace. Since $Z(P) = \langle t_\alpha\rangle \le N$, we get that t_α cannot fix any vertex in U, in particular, $\alpha \notin \mathbf{V}(S)$. Since x fixes setwise $\mathbf{V}_1 = \{(0, a_2, a_3) \mid a_2, a_3 \in \mathrm{GF}(p)\}$, we get $\mathbf{V}(S) \ne \mathbf{V_1}$.

To determine N_u, in other words, determine $\mathbf{V}(S)$, let Δ be the set of all 2-dimensional subspaces of $\mathbf{V}$ not containing α. Then it is easy to check that $|\Delta| = p^2$. Set

$$\mathbf{V}_0 = \big\{(a_1, a_2, 0) \mid a_1, a_2 \in \mathrm{GF}(p)\} \text{ and } N_0 = \{t_\gamma \mid \gamma \in \mathbf{V}_0\big\}.$$

Clearly, $\mathbf{V}_0 \in \Delta$. Let x, x' be defined as before. Then it is easy to check that $\langle x, x'\rangle$ acts regularly on Δ. Thus there exists $y \in \langle x, x'\rangle$ such that $\mathbf{V}(S)^y = \mathbf{V}_0$, equivalently, $N_u^y = N_0$.

By Proposition 2.9, our graph Σ is isomorphic to a bi-coset graph

$$\Sigma_1 = \mathbf{B}\big(A; \langle x\rangle S, N_u S, (N_u S)d(\langle x\rangle S)\big)$$

for some $d \in A$. Then

$$\Sigma_1 \cong \mathbf{B}\big(A^y; \langle x\rangle S^y, N_0 S^y, (N_0 S^y)d^y(\langle x\rangle S^y)\big).$$

Since S^y normalizes both $\langle x\rangle$ and N_0, we get

$$S^y \le N_{\mathrm{GL}(3,p)}\big(\langle x\rangle\big) \cap N_{\mathrm{AGL}(3,p)}(N_0) = \hat{S}.$$

Rechoosing S^y by S, d^y by d and A^y by A, respectively, we get that

$$\Sigma \cong \mathbf{B}\big(A; \langle x\rangle S, N_0 S, N_0 S d\langle x\rangle S\big).$$

In summary, we let $N_u = N_0$ and $S \le \hat{S}$ so that

$$A = N \rtimes \big(\langle x\rangle S\big) = PS, \quad A_w = \langle x\rangle S \quad \text{and} \quad A_u = N_0 S.$$

Clearly, in this case, A acts imprimitively on both W and U.

Step (4): Determination of the possible bi-coset graphs isomorphic to Σ.

For each given S in last step, define

$$\Lambda = \big\langle \lambda \in \text{ GF }(p)^* \;\big|\; y(\mu, \lambda) \in S\big\rangle \le \text{ GF }(p)^*$$

and assume $|\Lambda| = r$. Clearly, if $S = \hat{S}$ then $\Lambda = \text{ GF }(p)^*$. For convenience, set

$$a = t_{(1,0,0)}, \quad b = t_{(0,1,0)} \quad \text{and} \quad c = t_{(0,0,1)}.$$

Then for any $y(\mu, \lambda) \in R$, we have

$$a^x = ab^2c^2,\ b^x = bc^2,\ c^x = c;\ a^{y(\mu,\lambda)} = a^{\mu^2\lambda^{-1}},\ b^{y(\mu,\lambda)} = b^{\mu},\ c^{\phi(s,t)} = c^{\lambda}.$$

Set

$$\begin{aligned} \Sigma' &= \mathbf{B}\big(A; \langle x\rangle S, N_0 S, N_0 S d\langle x\rangle S\big),\\ W' &= [A : A_w] = \big\{A_w a^i b^j c^k \;\big|\; i, j, k \in Z_p\big\},\\ U' &= [A : A_u] = \big\{A_u c^h x^l \;\big|\; h, l \in Z_p\big\}. \end{aligned}$$

In the following, we shall determine the possible double cosets. Since there exist edges leading w to every block in U, we just need to consider the double cosets $D(m) := A_u c^m A_w$, where $m \in Z_p$.

Assume $m \ne 0$. Then under the conjugacy action, $y(m, m)$ fixes A_w and A_u, and maps $D(1)$ to $D(m)$. Therefore, $y(m, m)$ induces an isomorphism between two bi-coset graphs. So we just need to consider two cases for $D(m)$: $m = 0$ and $m = 1$.

For any $l \in Z_p$ and $y(\mu, \lambda) \in S$, we have

$$A_u c^m y(\mu, \lambda) x^l = A_u c^{m\lambda} x^l.$$

Then

$$D(m) = A_u c^m A_w = \big\{A_u c^{m\lambda} x^l \;\big|\; \lambda \in \Lambda, l \in Z_p\big\}.$$

Furthermore, for any $i, j, k \in \mathrm{GF}(p)$, by computing we have

$$\begin{aligned}\left(A_u c^{m\lambda} x^l\right) a^i b^j c^k &= A_u c^{m\lambda} \left(a^{x^{-l}}\right)^i \left(b^{x^{-l}}\right)^j \left(c^{x^{-l}}\right)^k x^l \\ &= A_u c^{m\lambda} a^i b^{-2lj} c^{2l^2 i - 2lj + k} x^l = A_u c^{m\lambda + 2l^2 i - 2lj + k} x^l,\end{aligned}$$

and so the neighborhood of $A_w a^i b^j c^k$ in the bi-coset graph is

$$\left\{A_u c^{m\lambda + 2l^2 i - 2lj + k} x^l \;\middle|\; l \in Z_p, \lambda \in \Lambda\right\}.$$

Therefore, the edge set is

$$E = \left\{ \left(A_w a^i b^j c^k, A_u c^{m\lambda + 2l^2 i - 2lj + k} x^l\right) \;\middle|\; i, j, k \in Z_p, l \in Z_p, \lambda \in \Lambda\right\}.$$

Define a bipartite graph Σ'' with the bipartition $W'' \cup U''$ where $W'' = Z_p^3$ and $U'' = Z_p^2$, and the edge-set

$$\left\{ ((i, j, k), (h, l)) \;\middle|\; i, j, k, h, l \in Z_p,\ h = m\lambda + 2l^2 i - 2lj + k\right\}.$$

Clearly, the map

$$\phi :\ A_w a^i b^j c^k \to (i, j, k), \quad A_u c^h x^l \to (h, l),$$

where $i, j, k, h, l \in Z_p$, gives an isomorphism from Σ' to Σ''.

Note that every element in S is of the form $y(\mu, \lambda)$ and that the above arguments are independent on the parameter μ. Therefore, all the graphs arising from different subgroups S corresponding to the same group Λ are all isomorphic to Σ''. Hence, given a group Λ, S can be taken as the biggest one, that is

$$S = \left\langle y(\mu, \lambda) \;\middle|\; \mu \in \mathrm{GF}(p)^*, \lambda \in \Lambda\right\rangle \cong Z_{p-1} \times Z_r.$$

Step (5): Determination of the isomorphism classes and automorphism groups for the above bi-coset graphs.

Let $\Sigma'(r, m)$ denote the corresponding coset graphs, where $m \in \{0, 1\}$ and $r = |\Lambda|$ is as defined in Step 4. Then for the graph $\Sigma'(r, 0)$, $d(A_w) = p$; and for the graph $\Sigma'(r, 1)$, $d(A_w) = rp$.

To determine the automorphism group of the graph $\Sigma'(r, m)$, we first recall Proposition 1.2. This proposition tells us that for the graph $\Sigma = (V, E)$, Aut (Σ) is imprimitive on U with a system $\mathcal{U}$ and the kernel $(\mathrm{Aut}\,(\Sigma))_{(\mathcal{U})}$. Then, either

- (i) Aut (Σ) acts primitively on W, $p = 3$ and $\Sigma \cong \Sigma_{11}, \Sigma_{12}$; or
- (ii) Aut (Σ) acts imprimitively on W, and either $(\mathrm{Aut}\,(\Sigma))_{(\mathcal{U})}$ is transitive on W, and it is a solvable and affine group on W; or $(\mathrm{Aut}\,(\Sigma))_{(\mathcal{U})}$ induces p blocks of length p^2 on W.

Now, come back to our graph $\Sigma' = \Sigma'(r, m)$. We know that $A \le \mathrm{Aut}\,(\Sigma'(r, m))$ and A acts edge-transitively on $\Sigma'(r, m)$. By Proposition 1.2, $\mathrm{Aut}\,(\Sigma'(r, m))$ acts imprimitively on U'.

(1) Suppose that $p = 3$. Then $r = 1$ or 2 so that $d(A_w) = 3$ or 6. These two graphs are clearly the graphs Σ_{11} or Σ_{12} as defined in Example 2.1. However, the automorphism group for two graphs is $S_3 \wr S_3$, acting primitively on U, which does not satisfy the hypothesis of this section.

(2) Suppose that $p \ge 5$. By Proposition 1.2, the graphs Σ_{11} and Σ_{12} in Example 2.1 where $p = 3$ are the only graphs whose automorphism group acts primitively on W'. Therefore, $\mathrm{Aut}\,(\Sigma'(r, m))$ is imprimitive on both biparts W' and U'. Since $N \le A_{(\mathcal{U})}$ acts transitively on W', $\mathrm{Aut}\,(\Sigma'(r, m))$ belongs to Case (2.1) in Proposition 1.2. Then by the proof of the above four steps, we get

$$A = N \rtimes \langle x\rangle S \le \mathrm{Aut}\,\big(\Sigma'(r, m)\big) \le N \rtimes \langle x\rangle \hat{S}.$$

In what follows, we deal with the two cases, separately.

(2.1) Graphs $\Sigma'(r, 0)$ and $\Sigma'(1, 1)$.

The respective edge set of the two graphs is

$$E = \big\{(A_w a^i b^j c^k, A_u c^{2l^2 i - 2lj + k} x^l) \bigm| i, j, k, l \in Z_p\big\},$$
$$E = \big\{(A_w a^i b^j c^k, A_u c^{1+2l^2 i - 2lj + k} x^l) \bigm| i, j, k, l \in Z_p\big\}.$$

The mapping ρ fixing W' pointwise and multiplying c to each vertex in U' from the right side gives an isomorphism between the two graphs. Thus, we may just consider the graphs $\Sigma'(r, 0)$.

Define a bipartite graph Σ''' with the bipartition $W''' \cup U'''$ where $W''' = Z_p^3$ and $U''' = Z_p^2$, and the edge-set

$$\big\{((i, j, k), (h, l)) \bigm| i, j, k, h, l \in Z_p,\ h = 2l^2 i - 2lj + k\big\}.$$

Clearly, the map

$$\phi :\ A_w a^i b^j c^k \to (i, j, k), \quad A_u c^h x^l \to (h, l),$$

where $i, j, k, h, l \in Z_p$, gives an isomorphism from Σ' to Σ'''.

Note that the edge-set of Σ''' is independent of the parameters λ and μ. Therefore, S may achieve at the biggest group $\hat{S}$, so that $\mathrm{Aut}\,(\Sigma(r, 0)) = N \rtimes \langle x\rangle \hat{S}$. Clearly, $\Sigma'(r, 0)$ is exactly the graph $\Sigma_{21}(p)$, as defined in Example 2.2.

(2.2) Graph $\Sigma'(r, 1)$ for $2 \le r \le p - 1$ and $r \bigm| (p - 1)$.

Since $d(A_w) = rp$, which is determined by $r = |\Lambda|$ and since S is the maximal subgroup of $\hat{S}$ such that $|\Lambda| = r$, one has $\mathrm{Aut}\,(\Sigma'(r, 1)) \cap \hat{S} = S$, noting that $2 \le r \le p - 1$. Therefore, $\mathrm{Aut}\,(\Sigma'(r, 1)) = N\langle x\rangle(\mathrm{Aut}\,(\Sigma'(r, 1))) \cap \hat{S}) = N\langle x\rangle S = A$. Therefore, $\Sigma'(r, 1) \cong \Sigma_{22}(p, r)$, where $2 \le r \le p - 1$, as defined in Example 2.2.

Step (6): Checking any two vertices in W'have the different neighborhood.

We already know that $\Sigma'(r,m)$ is edge-transitive. Suppose that there exist two vertices in W' which have the same neighborhood. Then by the vertex-transitivity of A on W', we know that A_w and $A_w a^i b^j c^k$ have the same neighborhood, for some $(i,j,k) \neq (0,0,0)$, that is

$$\left\{A_u c^{m\lambda} x^l \;\middle|\; \lambda \in \Lambda, l \in Z_p\right\} = \left\{A_u c^{m\lambda+2l^2 i-2lj+k} x^l \;\middle|\; \lambda \in \Lambda, l \in Z_p\right\}.$$

Then for any l, we have $2l^2 i - 2lj + k \in m\Lambda$, noting that $m \in \{0,1\}$, which is impossible. Therefore, any two vertices in W' have the different neighborhood. □

4 Case (2.2.2) in Proposition 1.2

The main result in this section is the following theorem.

Theorem 4.1 *Suppose that $\Sigma_{\mathcal{W}\cup\mathcal{U}}$ is of valency at least 3, $p \geq 5$, A contains a nonabelian normal p-subgroup acting regularly on W, $A_{(\mathcal{U})}$ is solvable, and $A/A_{(\mathcal{U})} \cong Z_p \rtimes Z_r$, where $r \mid (p-1)$ and $r \geq 3$. Then Σ is isomorphic to the graphs as defined either in Example 2.3 or Example 2.4.*

Since a nonabelian normal Sylow p-subgroup of A acts regularly on W, we know that $|P| = p^3$. Since there are two nonisomorphic nonabelian groups P of order p^3, that is, $\mathrm{Exp}\,(P) = p$ or p^2, we shall deal with them in Sects. 4.1 and 4.2, separately. Thus, the proof of Theorem 4.1 consists of Lemmas 4.4 and 4.5.

The following two group theoretical lemmas will be used in Sect. 4.1.

Lemma 4.2 *For any odd prime p, let*

$$P = \left\langle a,b,c \;\middle|\; a^p = b^p = c^p = 1, [b,a] = c, [c,a] = [c,b] = 1\right\rangle.$$

(1) $\mathrm{Aut}\,(P)$ *consists of the following automorphisms*

$$\pi : a \to a^i b^j c^k, \quad b \to a^m b^n c^l, \text{ where } in \neq jm.$$

Hence $|\mathrm{Aut}\,(P)| = p^3(p-1)^2(p+1)$.

(2) Let

$$\pi_1 : a \to ac, b \to b; \quad \pi_2 : a \to a, b \to bc.$$

Then $Q = \langle \pi_1, \pi_2 \rangle \cong Z_p^2$, $Q \lhd \mathrm{Aut}\,(P)$ *and* $\mathrm{Aut}\,(P)/Q \cong \mathrm{GL}(2,p)$.

(3) For any $s,t \in \mathrm{GF}\,(p)^*$, *let*

$$\phi(s,t) : a \to a^s, b \to b^t.$$

Then $\hat{S} := \langle \phi(s,t) \mid s,t \in \mathrm{GF}\,(p)*\rangle \cong Z_{p-1} \times Z_{p-1}$. *For every subgroup* $M \leq \mathrm{Aut}\,(P)$ *fixing a subgroup of P of order p^2 such that $|M| \mid (p-1)^2$, M is contained in a conjugacy of $\hat{S}$.*

(4) Let $\phi(s,t) \in \hat{S}$, where $(s,t) \neq (1,1)$. Then

$$C_{Aut\,(P)}(\phi(s,t)) = \begin{cases} \langle \pi : a \to a^i, b \to b^n c^l \mid i, n \in \mathrm{GF}\,(p)^*, l \in \mathrm{GF}\,(p)\rangle, & \text{if } s = 1, t \neq 1; \\ \langle \pi : a \to a^i c^k, b \to b^n \mid i, n \in \mathrm{GF}\,(p)^*, k \in \mathrm{GF}\,(p)\rangle, & \text{if } s \neq 1, t = 1; \\ \hat{S} & \text{if } s \neq t; s, t \neq 1; \\ \langle \pi : a \to a^i b^j c^{\frac{1}{2}ij}, b \to a^m b^n c^{\frac{1}{2}mn} \mid & \text{if } s = t \geq 2. \\ i, j, m, n \in GF\,(p), in \neq mj\rangle, & \end{cases}$$

(5) For $e, s \neq 1$ and $t^{|s|} \leq \langle e\rangle$, $C_{\mathrm{Aut}\,(P)}(\langle \phi(1,e), \phi(s,t)\rangle) = \hat{S}$.

Proof (1) Direct checking.

(2) Considering the induced action of Aut (P) on $P/\langle c\rangle \cong Z_p^2$, we know that the kernel is Q and Aut $(P)/Q \leq$ GL $(2,p)$. Since $|\mathrm{Aut}\,(P)/Q| = p(p-1)^2(p+1)$, it follows Aut $(P)/Q =$ GL $(2,p)$.

(3) Suppose that $M \leq$ Aut (P) is a subgroup fixing a subgroup of P of order p^2 such that $|M| \mid (p-1)^2$. Since $MQ/Q \leq$ GL $(2,p)$ and MQ/Q fixes a $1-$dimensional subspace in $V(2,p) := P/\langle c\rangle$, it follows from the structure of GL $(2,p)$, that $(MQ/Q)^g \leq \hat{S}Q/Q$ for some $g \in$ Aut (P), that is $M^g \leq \hat{S}Q$. Since M^g must be contained in a p-complement of Q in $Q\hat{S}$, we get that $M^{gg_1} \leq \hat{S}$ for some $g_1 \in \hat{S}Q$, as desired.

(4) Let $\pi \in C_{\mathrm{Aut}\,(P)}(\phi(s,t))$ and assume

$$a^\pi = a^i b^j c^k \quad \text{and} \quad b^\pi = a^m b^n c^l.$$

Then we have

$$\begin{aligned} a^{\phi(s,t)\pi} &= (a^t)^\pi = (a^i b^j c^k)^t = a^{it} b^{jt} c^{\frac{1}{2}ijt(t-1)+kt} \\ &= a^{\pi\phi(s,t)} = (a^i b^j c^k)^{\phi(s,t)} = a^{it} b^{js} c^{kst}, \\ b^{\phi(s,t)\pi} &= (b^s)^\pi = (a^m b^n c^l)^s = a^{ms} b^{ns} c^{\frac{1}{2}mns(s-1)+sl} \\ &= b^{\pi\phi(s,t)} = (a^m b^n c^l)^{\phi(s,t)} = a^{mt} b^{sn} c^{stl}. \end{aligned}$$

Thus we get

$$\begin{cases} j(t-s) = 0 \\ m(s-t) = 0 \\ k(s-1) = \frac{1}{2}ij(t-1) \\ l(t-1) = \frac{1}{2}mn(s-1). \end{cases}$$

If $t = 1$, then $k = j = m = 0$, that is, $a^\pi = a^i$, $b^\pi = b^n c^l$.
If $t \neq 1$ and $s = 1$, then $j = m = 0$ and $l = 0$, that is $a^\pi = a^i c^k$, $b^\pi = b^n$.
If $t \neq 1$, $s \neq 1$ and $s = t$, then $k = \frac{1}{2}ij$ and $l = \frac{1}{2}mn$, that is, $a^\pi = a^i b^j c^{\frac{1}{2}ij}$ and $b^\pi = a^m b^n c^{\frac{1}{2}mn}$.
If $t \neq 1$, $s \neq 1$ and $s \neq t$, then $j = m = k = l = 0$, that is $\langle \pi\rangle \leq \hat{S}$.

(5) By using (4), we get

$$C_{\mathrm{Aut}\,(P)}\big(\langle\phi(1,e),\phi(s,t)\rangle\big)=C_{\mathrm{Aut}\,(P)}(\phi(1,e))\cap C_{\mathrm{Aut}\,(P)}(\phi(s,t))=\hat{S}.$$

□

Let S be any subgroup of $\hat{S}=\langle\phi(1,\theta),\phi(\theta,1)\rangle$, as defined in Lemma 4.2. Suppose that

$$S\cap\big\langle\phi(1,\theta)\big\rangle=\big\langle\phi(1,e)\big\rangle\cong Z_{r_1},\quad S/\big\langle\phi(1,e)\big\rangle=\big\langle\overline{\phi(s,t)}\big\rangle\cong Z_{r_2},$$

where $e,s,t\in\ \mathrm{GF}\,(p)^*$ and r_1,r_2 are divisors of $p-1$. Then

$$S=\big\langle\phi(1,e),\phi(s,t)\big\rangle\quad\text{and}\quad\phi(s,t)^{r_2}\in\big\langle\phi(1,e)\big\rangle.$$

In other words, in $\mathrm{GF}\,(p)^*$,

$$|e|=r_1,\quad|s|=r_2,\quad t^{r_2}\in\langle e\rangle,$$

where if $e=1$, then $S=\langle\phi(s,t)\rangle\cong Z_{r_2}$ and $\langle t\rangle\le\langle s\rangle\cong Z_{r_2}$. Clearly, we may set

$$e=\theta^{\frac{p-1}{r_1}}\quad\text{and}\quad s=\theta^{\frac{p-1}{r_2}}.$$

Let $T=\{t\in\ \mathrm{GF}\,(p)^*\ \big|\ t^{r_2}\le\langle e\rangle\}$, which is a subgroup of order $r_1(r_2,\frac{p-1}{r_1})$. Therefore, we set

$$t=\left(\theta^{\frac{p-1}{r_1(r_2,\frac{p-1}{r_1})}}\right)^{r_3},$$

where $0\le r_3\le(r_2,\frac{p-1}{r_1})-1$. Clearly, $|S|=r_1r_2$.

Lemma 4.3 *For $l=1,2$, let $S_l=\langle\phi(1,e),\phi(s,t_l)\rangle$ be as above such that $S_1\ne S_2$ and $s\ne1$. Let $A_l=P\rtimes S_l$. Suppose that γ is an isomorphism from A_1 to A_2 such that $\gamma(S_1)=S_2$. Then γ satisfies*

$$\gamma(a)=b^{j_1},\ \gamma(b)=a^i,\ \gamma(\phi(1,e))=\phi(e,1),\ \gamma(\phi(s,t_1))=\phi(t_1,s),$$

for some $i,j_1\in\mathrm{GF}\,(p)^*$, and $S_2=\langle\phi(e,1),\phi(t_1,s)\rangle$.

Proof Suppose that γ is an isomorphism from A_1 to A_2 such that $\gamma(S_1)=S_2$. Since P is characteristic in A_l, we have $\gamma(P)=P$.

(1) Case $e\ne1$.

Assume that

$$\begin{aligned}\gamma(a)&=a^{i_1}b^{j_1}c^{k_1},\ \gamma(b)=a^ib^jc^k,\ \gamma(c)=c^l,\\ \gamma\big(\phi(1,e)\big)&=\phi(1,e)^{m_1}\phi(s,t_2)^{n_1},\ \gamma\big(\phi(s,t_1)\big)=\phi(1,e)^{m_2}\phi(s,t_2)^{n_2}.\end{aligned}$$

where $i_1 j - j_1 i \neq 0$. Since

$$\begin{aligned} \phi(1,e)^{-1} a \phi(1,e) &= a, \phi(1,e)^{-1} b \phi(1,e) = b^e, \phi(1,e)^{-1} c \phi(1,e) = c^e, \\ \phi(s,t_1)^{-1} a \phi(s,t_1) &= a^s, \phi(s,t_1)^{-1} b \phi(s,t_1) = b^{t_1}, \phi(s,t)^{-1} c \phi(s,t_1) = c^{t_1 s}, \end{aligned}$$

we get

$$\begin{cases} (\phi(1,e)^{m_1}\phi(s,t_2)^{n_1})^{-1} a^{i_1} b^{j_1} c^{k_1} \phi(1,e)^{m_1}\phi(s,t_2)^{n_1} = a^{i_1} b^{j_1} c^{k_1} \\ (\phi(1,e)^{m_1}\phi(s,t_2)^{n_1})^{-1} a^{i} b^{j} c^{k} \phi(1,e)^{m_1}\phi(s,t_2)^{n_1} = (a^{i} b^{j} c^{k})^e \\ (\phi(1,e)^{m_1}\phi(s,t_2)^{n_1})^{-1} c^l \phi(1,e)^{m_1}\phi(s,t_2)^{n_1} = c^{le} \\ (\phi(1,e)^{m_2}\phi(s,t_2)^{n_2})^{-1} a^{i_1} b^{j_1} c^{k_1} \phi(1,e)^{m_2}\phi(s,t_2)^{n_2} = (a^{i_1} b^{j_1} c^{k_1})^s \\ (\phi(1,e)^{m_2}\phi(s,t_2)^{n_2})^{-1} a^{i} b^{j} c^{k} \phi(1,e)^{m_2}\phi(s,t_2)^{n_2} = (a^{i} b^{j} c^{k})^{t_1} \\ (\phi(1,e)^{m_2}\phi(s,t_2)^{n_2})^{-1} c^l \phi(1,e)^{m_2}\phi(s,t_2)^{n_2} = c^{lt_1 s} \end{cases}.$$

Solving these equations, we get

$$\begin{cases} (i) \quad i_1(s^{n_1}-1)=0; \quad (ii) \quad j_1(e^{m_1}t_2^{n_1}-1)=0; \quad (iii) \quad k_1(e^{m_1}(t_2 s)^{n_1}-1)=0; \\ (iv) \quad i(s^{n_1}-e)=0; \quad (v) \quad j(e^{m_1}t_2^{n_1}-e)=0; \quad (vi) \quad k(e^{m_1}(t_2 s)^{n_1}-e) \\ \qquad = \frac{1}{2} ije(e-1); \\ (vii) \quad e^{m_1}(t_2 s)^{n_1}=e; \quad (viii) \quad i_1(s^{n_2}-s)=0; \quad (ix) \quad j_1(e^{m_2}t_2^{n_2}-s)=0; \\ (x) \quad k_1(e^{m_2}(t_2 s)^{n_2}-s) \quad (xi) \quad i(s^{n_2}-t_1)=0; \quad (xii) \quad j(e^{m_2}t_2^{n_2}-t_1)=0; \\ \qquad = \frac{1}{2} i_1 j_1 s(s-1); \\ (xiii) \quad k(e^{m_2}(t_2 s)^{n_2}-t_1) \quad (xiv) \quad e^{m_2}(t_2 s)^{n_2}=t_1 s. \\ \qquad = \frac{1}{2} ijt_1(t_1-1); \end{cases}$$

First suppose that $i_1 \neq 0$. By (i), we get $s^{n_1} = 1$ and by combing (iv) and $e \neq 1$, we get $i = 0$. Then $j \neq 0$. From $(viii)$, we get $s^{n_2} = s$ and then $n_2 \equiv 1(\bmod r_2)$, where $r_2 = |s|$. Thus, from (xii), one has $t_1 = e^{m_2}t_2^{n_2} = e^{m_2}t_2$, which forces that $S_1 = S_2$, a contradiction.

Suppose that $i_1 = 0$. Then $j_1, i \neq 0$. From (ii) and (iv), respectively, we get $e^{m_1}t_2^{n_1} = 1$ and $s_1^n = e$, which implies $\gamma(\phi(1,e)) = \phi(e,1)$. From (ix) and (xi), respectively, we get $e^{m_2}t_2^{n_2} = s$ and $s^{n_2} = t_1$, which implies $\gamma(\phi(s,t_1)) = \phi(t_1,s)$, as desired. Moreover, by (ii) and (v), we get $j = 0$; by (ii), (iii) and (iv), we get $k_1 = 0$; and finally by (ix), (xi) and $(xiii)$, we get $k = 0$. Therefore, $\gamma(a) = b^{j_1}$ and $\gamma(b) = a^i$.

(2) Case $e = 1$.

Assume that

$$\gamma(a) = a^{i_1} b^{j_1} c^{k_1}, \gamma(b) = a^i b^j c^k, \gamma(c) = c^l, \gamma(\phi(s,t_1)) = \phi(s,t_2)^n.$$

where $i_1 j - j_1 i \neq 0$ and by our notion, when $e = 1$, we have $|t_i| \mid r_2$. Then from

$$\begin{cases} (\phi(s,t_2)^n)^{-1} \gamma(a) \phi(s,t_2)^n = \gamma(a^s) \\ (\phi(s,t_2)^n)^{-1} \gamma(b) \phi(s,t_2)^n = \gamma(b^{t_1}) \\ (\phi(s,t_2)^n)^{-1} \gamma(c) \phi(s,t_2)^n = \gamma(c^{st_1}) \end{cases}$$

we can get

$$\begin{cases} (i)\; i_1(s^n-s)=0; \quad (ii)\; j_1(t_2^n-s)=0; \quad (iii)\; k_1((t_2s)^n-s)=\frac{1}{2}i_1j_1s(s-1); \\ (iv)\; i(s^n-t_1)=0; \; (v): j(t_2^n-t_1)=0; \; (vi)\; k((t_2s)^n-t_1)=\frac{1}{2}ijt_1(t_1-1); \\ (vii): (t_2s)^n=t_1s. \end{cases}$$

First suppose that $i_1\neq 0$. Then by (i) we get $n\equiv 1(\bmod r_2)$ and by combing (vii), we get $t_2s=t_2^ns^n=t_1s$ and then $t_1=t_2$, a contradiction.

Second suppose that $i_1=0$. Then $j_1, i\neq 0$. By (ii) and (iv) we get $t_2^n=s$ and $t_1=s^n$, which implies $\gamma(\phi(s,t_1))=\phi(t_1,s)$.

By (iii), we get $k_1=0$; by (v), we get $j=0$ and then by (vi), we get $k=0$. Therefore, $\gamma(a)=b^{j_1}$ and $\gamma(b)=a^i$. □

4.1 Exp $(P)=p$

Throughout this subsection, we shall use the notation of Lemmas 4.2 and 4.3, in particular, for $\phi(s,t)$, $\hat{S}$, r_1, r_2 and r_3. We shall not explain them again.

Lemma 4.4 *Suppose that Exp $(P)=p$. Then Σ is isomorphic to the graphs defined in Example 2.3.*

Proof We divide the proof into five steps.

Step (1): Determination of the structure of $A=$ Aut (Σ), A_w and A_u.

Suppose that Exp $(P)=p$. Then let

$$P=\langle a,b,c \mid a^p=b^p=c^p=1, [b,a]=c, [c,a]=[c,b]=1\rangle.$$

Set $S=A_w$ for some $w\in W$. Then $A=P\rtimes S$, where S is core-free. Now S can be viewed as a subgroup of Aut (P). Note that all the subgroups of order $|S|$ in A are conjugate. Considering the action of A on U, where $|U|=p^2$, we know that $S\leq A_u$ for some u. Then S normalizes $P_u=A_u\cap P$ of order p and also $\langle c\rangle$. By Lemma 4.2.(3), S is contained in a conjugacy of $\hat{S}$ in Aut (P). Therefore, up to graph isomorphism, we may assume that

$$S=\langle \phi(1,e),\phi(s,t)\rangle\leq \hat{S}$$

as stated before Lemma 4.3. Clearly, $\mathcal{U}$ is exactly the set of $\langle c\rangle$-orbits on U. Since $\Sigma_{\mathcal{W}\cup\mathcal{U}}$ is of valency at least 3, we know that $|S|\geq 3$.

Now we determine P_u, noting that $u\in U$ and $S\leq A_u$. Suppose that $P_u=\langle g\rangle$, where $g=a^ib^jc^k$. Then S normalizes $\langle g\rangle$. Let $\mu=\phi(s_1,t_1)$ be any element in S, that is $\mu: a\to a^{s_1}, b\to b^{t_1}$. Since S normalizes P_u, we set $g^\mu=g^l$, for some integer l. Then

$$a^{is_1}b^{jt_1}c^{ks_1t_1}=a^{il}b^{jl}c^{kl+\frac{1}{2}ijl(l-1)}.$$

So we have

$$i(s_1 - l) = j(t_1 - l) = k(s_1 t_1 - l) - \frac{1}{2} ijl(l-1) = 0.$$

Noting the arbitrariness of μ and $|S| \geq 3$, from the above equations, we conclude all possibilities for $P_u = \langle g \rangle$:

(i) $g = b, S \leq \hat{S}$;
(ii) $g = a, S \leq \hat{S}$;
(iii) $g = ac^k, k \neq 0, t_1 = 1$;
(iv) $g = b^j c^k, k \neq 0, s_1 = 1$;
(v) $g = a^i bc^{\frac{i}{2}}, i \neq 0, s_1 = t_1 \geq 3$.

In case (iii), for any $\phi(s_1, t_1) \in S$, we have $t_1 = 1$ and so $A_w = S = \langle \phi(s_1, 1) \rangle$ and then $[b, S] = 1$. Thus, S fixes every block in $u^{\langle a,c \rangle b^h}$ setwise, where $h \in$ GF (p), that is $S \leq A_{(\mathcal{U})}$, a contradiction. The same arguments can be applied for case (iv).

Let $\tau_1 \in$ Aut (P) such that $\tau_1 : a \to b$ and $b \to a$. Then $\hat{S}^{\tau_1} = \hat{S}$ and so $\langle a \rangle S \to \langle b \rangle S^{\tau_1}$. By Proposition 2.9, every bi-coset graph $\mathbf{B}(PS; S, \langle a \rangle S, \langle a \rangle S d S)$ is isomorphic to a coset graph $\mathbf{B}(PS^{\tau_1}; S^{\tau_1}, \langle b \rangle S^{\tau_1}, \langle b \rangle S^{\tau_1} d^{\tau_1} S^{\tau_1})$, noting that $S^{\tau_1} \leq \hat{S}$. Therefore, for cases (i) and (ii), we may just consider one of them, say $g = \langle b \rangle$.

As for the case (v), $S = \langle \phi(s_1, s_1) \rangle$. Let $\tau_2 \in$ Aut (P) such that $\tau_2 : a \to a^i bc^{\frac{i}{2}}$ and $b \to b$. Then by Lemma 4.2.(4), $S^{\tau_2} = S$, and $\langle a \rangle S \to \langle a^i bc^{\frac{i}{2}} \rangle S$. Again, using the same arguments as in the last paragraph, one may see that cases (i) and (v) may give the isomorphic bi-coset graphs.

In summary, we conclude $\langle g \rangle = \langle b \rangle$, which is the case (i). Then, we have

$$A = P \rtimes S, \ A_w = S, \ A_u = \langle b \rangle S,$$

where $S = \langle \phi(1, e), \phi(s, t) \rangle$. Moreover, $\phi(1, e)$ fixes each block $u^{\langle c, a^i \rangle}$ setwise, where $i \in$ GF (p), and $\phi(s, t)$ acts on $\mathcal{U}$ as a permutation of order r_2 with the kernel $\phi(1, t^{r_2})$. Since $\Sigma_{\mathcal{W} \cup \mathcal{U}}$ is of valency at least 3, we know that $r_2 \geq 3$. In what follows, take into account that for any s' and t', we have

$$a^{\phi(s',t')} = a^{s'}, \ b^{\phi(s',t')} = b^{t'}, \ c^{\phi(s',t')} = c^{s't'}.$$

Step (2): Determination of the possible bi-coset graphs isomorphic to Σ.

Our graph Σ is isomorphic to a bi-coset graph $\Sigma' = \mathbf{B}(A; S, \langle b \rangle S, D)$, where $D = \langle b \rangle S d S$ for some $d \in A$. Set

$$W' = [A : S] = \{Sg \mid g \in A\} = \{Sa^i b^j c^k \mid i, j, k \in Z_p\},$$
$$U' = [A : \langle b \rangle S] = \{\langle b \rangle Sg \mid g \in A\} = \{\langle b \rangle Sc^h a^l \mid h, l \in Z_p\}$$

and let $\mathcal{W}'$ and $\mathcal{U}'$ to correspond to $\mathcal{W}$ and $\mathcal{U}$, respectively.

First we determine all possible double cosets. Every coset is of the form $D(h,l) := (\langle b\rangle S)c^h a^l S$, for any $h, l \in Z_p$.

If $l = 0$, then $\langle D(h,0)^{-1}D(h,0)\rangle \neq A$. By Proposition 2.9, Σ' is disconnected. Therefore, $l \neq 0$.

For $h \neq 1$, let $\mu = \phi(l, l^{-1}h)$. Then in $P \rtimes \mathrm{Aut}\,(P)$, under the conjugacy action, μ fixes $\langle b\rangle$, S and A, and maps $D(1,1)$ to $D(h,l)$. Then by Proposition 2.9, σ induces an isomorphism between two bi-coset graphs. So we just need to consider two cases for $D(h,1)$, where $h = 0$ and 1.

Recall $S = \langle \phi(1,e), \phi(s,t)\rangle$ is abelian. For any $m_1 \in Z_{r_1}$ and $m_2 \in Z_{r_2}$, we have

$$\begin{aligned}\langle b\rangle Sc^h a\phi(1,e)^{m_1}\phi(s,t)^{m_2} &= \langle b\rangle S(c^h)^{\phi(1,e)^{m_1}} a^{\phi(1,e)^{m_1}}\phi(s,t)^{m_2}\\ &= \langle b\rangle Sc^{he^{m_1}} a\phi(s,t)^{m_2} = \langle b\rangle S(c^{he^{m_1}})^{\phi(s,t)^{m_2}} a^{\phi(s,t)^{m_2}}\\ &= \langle b\rangle Sc^{he^{m_1}(st)^{m_2}} a^{s^{m_2}}.\end{aligned}$$

Then

$$D(h,1) = \langle b\rangle Sc^h aS = \left\{\langle b\rangle Sc^{he^{m_1}(st)^{m_2}} a^{s^{m_2}} \;\middle|\; m_1 \in Z_{r_1}, m_2 \in Z_{r_2}\right\}.$$

Furthermore, for any $i, j, k \in Z_p$, we have

$$\begin{aligned}\langle b\rangle Sc^h a\phi(1,e)^{m_1}\phi(s,t)^{m_2} b^i c^j a^k &= \langle b\rangle Sc^{he^{m_1}(st)^{m_2}} a^{s^{m_2}} b^i c^j a^k\\ &= \langle b\rangle Sc^{he^{m_1}(st)^{m_2}} (a^{s^{m_2}})^{b^i} c^j a^k\\ &= \langle b\rangle Sc^{he^{m_1}(st)^{m_2}-is^{m_2}+j} a^{k+s^{m_2}}.\end{aligned}$$

Then for any point $Sb^i c^j a^k$ in W', its neighborhood is

$$\left\{\langle b\rangle Sc^{he^{m_1}(st)^{m_2}-is^{m_2}+j} a^{k+s^{m_2}} \;\middle|\; i, j, k \in Z_p, m_1 \in Z_{r_1}, m_2 \in Z_{r_2}\right\}.$$

Since Σ' is now determined by r_1, r_2, r_3, and h, we denote it by $\Sigma' = \Sigma'(r_1, r_2, r_3, h)$.

Step (3): Determination of the automorphism group for $\Sigma'(r_1, r_2, r_3, h)$.

Let $\tilde{A} = \mathrm{Aut}\,(\Sigma')$. Similar to Step (5) in the proof of Theorem 3.1, we shall determine $\tilde{A}$ with a help of Proposition 1.2.

By Proposition 1.2, the graphs Σ_{11} and Σ_{12} in Example 2.1 in which $p = 3$ are the only graphs whose automorphism group acts primitively on W'. Since $p \geq 5$ for our graph Σ', it follows that $\mathrm{Aut}\,(\Sigma'(r,m))$ is imprimitive on both biparts W' and U', and so Σ' belongs to Case (2) of Proposition 1.2.

Take $w \in W'$. Then $d(w) = r_2$ for $h = 0$; and $r_1 r_2$ for $h = 1$, where r_2 and r_3 are divisor of $p-1$. However, $d(w) = p$ for $\Sigma_{21}(p)$; and rp for $\Sigma_{22}(p,r)$. Therefore, $\Sigma' \not\cong \Sigma_{21}(p), \Sigma_{22}(p,r)$, and then it belongs to Case (2.2) of Proposition 1.2. Furthermore, since the degree of $\Sigma'_{\mathcal{W}'\cup\mathcal{U}'}$ is at least 3, Σ' belongs to Case (2.2.2) of Proposition 1.2. Thus, $\tilde{A} = P \rtimes \tilde{S}$ for some subgroup $\tilde{S}$, where $\mathrm{Exp}\,(P) = p$ and $S \leq \tilde{S} = \tilde{A}_w \leq \tilde{A}_u$. Then $\tilde{S}$ is contained in a conjugacy of $\hat{S}$ in $P\mathrm{Aut}\,(P)$. In particular, $\tilde{S}$ is abelian. Then we prove the following facts:

(1) *For* $\Sigma' = \Sigma'(r_1, r_2, r_3, 1)$, *where either* $e \neq 1$ *or* $e = 1$ *but* $t \neq 1, s, s^{-1}$, *we have* $\tilde{A} = A$.

Suppose that either $e \neq 1$ or $e = 1$ but $t \neq 1, s, s^{-1}$. For any $z \in \tilde{S}$, set $z = g\mu_2$, where $g = a^i b^j c^k \in P$ and $\mu_2 \in \mathrm{Aut}\,(P)$. Since $\tilde{S}$ is abelian, we get

$$[g\mu_2, \phi(1, e)] = [g\mu_2, \phi(s, t)] = 1,$$

which implies

$$[g, \phi(1, e)] = 1, [g, \phi(s, t)] = 1, [\mu_2, \phi(1, e)] = 1, [\mu_2, \phi(s, t)] = 1.$$

Noting $|s| = r_2 \geq 3$, the first two equations give

$$j(e - 1) = 0, k(e - 1) = 0, i = 0, j(t - 1) = 0, k(st - 1) = 0.$$

Since either $e \neq 1$ or $e = 1$ and $t \neq 1, s^{-1}$, it follows that $i = j = k = 0$, that is $g = 1$. In other words, $\mu_2 \leq C_{\mathrm{Aut}\,(P)}(S)$. Moreover, by Lemma 4.2.(5) for $e \neq 1$ (noting $s \neq 1$) and by Lemma 4.2.(4) for $e = 1$ and $t \neq 1, s$, we get $\mu_2 \in \hat{S}$. In summary, we have $S \leq \tilde{S} \leq \hat{S}$. Observing that $|\tilde{S}| = d(w) = |S|$ for $w \in W'$, we get $\tilde{S} = S$ and then $\tilde{A} = A$.

(2) *For* $\Sigma' = \Sigma'(r_1, r_2, r_3, 0)$ *or* $\Sigma'(1, r_2, r_3, 1)$ *but* $e = 1$, $t = 1, s, s^{-1}$, *we have that* $\Sigma'(r_1, r_2, r_3, 0) \cong \Sigma'(1, r_2, r_3, 1)$ *and* $\tilde{A} \cong P \rtimes \langle \phi(1, \theta), \phi(s, 1)\rangle$.

For these graphs, $d(w) = r_2$ and $|\Sigma'_1(w) \cap U_i| = 0$ or 1 for any $w \in W'$ and block U_i in $\mathcal{U}'$.

For the graph $\Sigma'(r_1, r_2, r_3, 0)$, the edge set is

$$\left\{(Sa^i b^j c^k, \langle b\rangle Sc^{-is^{m_2}+j} a^{k+s^{m_2}}) \;\middle|\; i, j, k \in Z_p, m_2 \in Z_{r_2}\right\}.$$

Clearly, this graph is independent on r_1 and with the similar arguments as in case (2.1) of Step (5) of Theorem 3.1, we take $r_1 = p - 1$ so that $r_3 = 1$ and $\tilde{A} \cong P \rtimes \langle \phi(1, \theta), \phi(s, 1)\rangle$.

For the graph $\Sigma'(1, r_2, r_3, 1)$, where $t = 1$ or s, the edge set is $E = (S, \langle b\rangle SacS)^P$. Take an automorphism τ of P such that for $t = 1$, $\tau : a \to ac^{-1}, b \to b$; for $t = s$, $\tau : a \to abc^{\frac{1}{2}}, b \to b^{\frac{1}{2}}$. By Lemma 4.2.(4), we get τ fixes S and $\langle b\rangle S$ and maps $\langle b\rangle SacS$ to $\langle b\rangle SaS$. By Proposition 2.9, τ gives an isomorphism from $\Sigma'(1, r_2, r_3, 1)$ to $\Sigma'(r_1, r_2, r_3, 0)$.

For the graph $\Sigma'(1, r_2, r_3, 1)$, where $t = s^{-1}$, the edge set is $E = \{(S, \langle b\rangle Sca^{t^{m_2}}) \mid m_2 \in Z_{r_2}\}^P$. Fixing W' pointwise and right multiple c^{-1} to U' pointwise, we may get an isomorphism from this graph to $\Sigma'(r_1, r_2, r_3, 0)$.

Step (4): Determination of the isomorphism classes for $\Sigma'(r_1, r_2, r_3, h)$.

(1) From Step 3, we know that $\Sigma'(r_1, r_2, r_3, 0) \cong \Sigma'(1, r_2, r_3, 1)$, where $e = 1$, $t = 1, s, s^{-1}$. In fact, $\Sigma'(r_1, r_2, r_3, 0)$ is the graph $\Sigma_{31}(p, r_2)$ in Example 2.3. Clearly, this graph is uniquely determined by r_2, which is the valency of $w \in W'$.

(2) Suppose that $\Sigma' = \Sigma'(r_1, r_2, r_3, 1)$, where either $e \neq 1$ or $e = 1$ but $t \neq 1, s, s^{-1}$. We shall show that this graph is uniquely determined by the three parameters: r_1, r_2 and r_3.

First note that r_2 determines the valency of $\Sigma'_{\mathcal{W}'\cup\mathcal{U}'}$ and $d(w) = r_1r_2$ for $w \in W'$.

By Step (3), Aut $(\Sigma') = A$. In viewing of the edge set of Σ', each group S gives just one graph Σ'.

For $i = 1, 2$, let $S_i = \langle \phi(1, e), \phi(s, t_i)\rangle$ but $S_1 \neq S_2$. Let $A_1 = P \rtimes S_1 \cong A_2 = P \rtimes S_2$ and let Σ'_i be the graph corresponding to A_i. Then we show that $\Sigma'_1 \ncong \Sigma'_2$ below.

For the contrary, suppose that γ is an isomorphism from Σ'_1 and Σ'_2. For convenience, label the vertex set of both graphs by $W' = P$ and $U' = [P : \langle b\rangle]$. Then A_1, A_2, and $\langle\gamma\rangle$ are permutation groups on both W' and U', in particular, P acts on them by right multiplication and S_i by conjugacy. Since $\gamma^{-1}A_1\gamma$ preserves E_2, we get $\gamma^{-1}A_1\gamma \leq A_2$ and then $\gamma^{-1}A_1\gamma = A_2$ by considering $|A_1| = |A_2|$. Since P is characteristic in A_i and all the subgroups of order r_1r_2 in A_i are conjugate, there exists an isomorphism from Σ'_1 to Σ'_2, denoted by γ again, such that $\gamma^{-1}P\gamma = P$ and $\gamma^{-1}S_1\gamma = S_2$. Now γ induces an isomorphism from A_1 and A_2 and an automorphism of P as well. By Lemma 4.3, such γ is defined by $a^\gamma = b^{j_1}$ and $b^\gamma = a^i$, for some $i, j_1 \in$ GF $(p)^*$. Now considering $\langle\gamma\rangle$ and S_i as permutation subgroups on P, we know that $S_1^\gamma = S_2$ and $[\gamma^2, S_i] = 1$. Since $P \rtimes \langle S_1, S_2, \gamma\rangle$ has a representation of degree p^2 on U', $\langle S_1, S_2, \gamma\rangle$ must fix a subgroup of P of order p. However, S_i just fixes $\langle a\rangle$ and $\langle b\rangle$ but γ interchanges $\langle a\rangle$ and $\langle b\rangle$, a contradiction. Therefore, $\Sigma'_1 \ncong \Sigma'_2$.

Finally, note that $\Sigma'(r_1, r_2, r_3, 1) \cong \Sigma_{32}(p, r_1, r_2, r_3)$ in Example 2.3.

Step (5): Checking any two vertices in W' have the different neighborhood.

Similar to the proof of Theorem 3.1. □

4.2 Exp $(P) = p^2$

Lemma 4.5 *Suppose that Exp $(P) = p^2$. Then Σ is isomorphic to the graphs as defined in Example 2.4.*

Proof We divide the proof into four steps.

Step (1): Determination of the structure of A, A_u, and A_w.

Suppose that Exp $(P) = p^2$ and let

$$P = \left\langle a, b \;\middle|\; a^{p^2} = b^p = 1, [b, a] = a^p \right\rangle,$$

where set $c = a^p$. Assume $S = A_w$ for some $w \in W$. Then $A = P \rtimes S$, where S is corefree subgroup of order r, where r is defined in Proposition 1.2.

A routine checking shows that every automorphism of P has the following form:

$$\pi : a \to a^i b^j, b \to a^{pk} b,$$

where $i \in Z^*_{p^2}$, $j, k \in Z_p$. Let $Z^*_{p^2} = \langle\lambda\rangle$. Let

$$\phi(\lambda) : a \to a^\lambda, b \to b, \quad \tau : a \to ab, b \to b.$$

Then

$$\text{Aut}\,(P)/\,\text{Inn}\,(P) \cong \langle\tau\rangle \rtimes \langle\phi(\lambda)\rangle \cong Z_p \rtimes Z_{p-1}.$$

In particular, Aut (P) is solvable. Since $|\mathrm{Aut}\,(P)| = p^3(p-1)$, Aut (P) has only one conjugacy class of subgroup of order $p-1$ with a representative S, and so we let $S = \langle \phi(\lambda^{\frac{p(p-1)}{r}})\rangle$ and $A = P \rtimes S$, recalling $|S| = r$. Set $\hat{S} = \langle \phi(\lambda)\rangle$ and $t = \lambda^{\frac{p(p-1)}{r}}$.

Now we determine P_u, where $u \in U$ and $S \le A_u$. Suppose that $P_u = \langle g\rangle$, where $g = a^{pk}b^j$. Since S normalizes $\langle g\rangle$, we have

$$g^{\phi(t)} = (a^{pk}b^j)^{\phi(t)} = a^{pkt}b^j = g^l = a^{pk}b^{jl},$$

for some l. Then we conclude the following possibilities for $\langle g\rangle$:

$$(i)\, g = b;\ (ii)\, g = a^p;\ (iii)\, g = a^{pk}b, t = l = 1.$$

In case (iii), $t = 1$ and so $S = 1$, contradiction. In case (ii), $g \in Z(P)$, which implies that g fixes U pointwise, a contradiction. Therefore, $g = b$ and $A_u = \langle b\rangle S$.

Step (2): Determine the possible bi-coset graphs isomorphic to Σ.

Now our graph Σ is isomorphic to the bi-coset graph $\Sigma' = \mathbf{B}(A; S, \langle b\rangle S, D)$, where $D = \langle b\rangle SdS$. Let $W' = [A : S]$ and $U' = [A : \langle b\rangle S]$.

Every double coset has the form $D(l) := \langle b\rangle Sa^l S$, for any $l \in Z_{p^2}$. If $p \mid l$, then $\langle D^{-1}D\rangle \ne A$ and the graph Σ' is disconnected and so $(p, l) = 1$. Define $\sigma : A \to A$ by $a \to a^l, b \to b, \phi(t) \to \phi(t)$. Then $\sigma \in$ Aut (A). Clearly, σ fixes S and $\langle b\rangle S$, and maps $D(1)$ to $D(l)$. So we just consider the following case

$$D(1) = \langle b\rangle SaS = \left\{\langle b\rangle Sa\phi(t)^m \mid m \in Z_r\right\} = \left\{\langle b\rangle Sa^{t^m} \mid m \in Z_r\right\}.$$

Then for any point Sb^ia^l in W', its neighborhood is

$$\left\{\langle b\rangle Sa^{t^m}b^ia^l \mid m \in Z_r\right\} = \left\{\langle b\rangle Sa^{l+t^m-it^m p} \mid m \in Z_r\right\}.$$

Set $l \equiv jp + k$, where $i, j, k \in Z_p$ so that $jp + k + t^m - it^m p \equiv yp + z \pmod{p^2}$, for some $y, z \in Z_p$.

Step (3): Determination of Aut (Σ').

Suppose that w is adjacent to another vertex, say $\langle b\rangle Sa^{t^m}$ in the block U_i containing $\langle b\rangle Sa$. Then $a^{t^m} = a^{\lambda^{(p-1)k}}$ for some $0 \ne m \in Z_r$ and $k \in Z_p$. Then $\lambda^{m\frac{p(p-1)}{r}} = \lambda^{(p-1)k}$, that is, $mp\frac{p-1}{r} = (p-1)k$. Then $r \mid m$ and so $a^{t^m} = a$, that is $m \equiv 0 \pmod{r}$, a contradiction. Therefore, for any $u' \in U_i$ such that $wu' \in E$, u' is the only vertex in U_i which is adjacent to w.

Let $\tilde{A} = $ Aut (Σ'). With the similar arguments as in Step (3) in the proof of Lemma 4.4, Σ' belongs to case (2.2.2) of Proposition 1.2. In particular, Exp $(P) = p^2$ and $A = P \rtimes S$. By the proof of Step (1), $S \le \hat{S}$. Since $d(w) = r$ for $w \in W'$, we get $S \cong Z_r$ so that $\tilde{A} = A$ and then different r give nonisomorphic graphs. Note that Σ' is exactly the graph $\Sigma_4(p, r)$ as defined in Example 2.4.

Step (4): Checking any two vertices in W' have the different neighborhoods.

Similar to the proof of Theorem 3.1. □

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