Every 3-equivalenced association scheme is Frobenius

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Abstract For a positive integer k we say that an association scheme with more than one point is k -*equivalenced* if each non-diagonal relation has valency k . In this paper we prove that every 3-equivalenced association scheme is Frobenius, that is, the set of relations coincides with the set of orbitals of a Frobenius group.

Keywords Association schemes · 3-Equivalenced · Frobenius

1 Introduction

Let G be a group acting transitively on a finite set Ω . Then G acts on $\Omega \times \Omega$ by

$$(\alpha, \beta)^g = (\alpha^g, \beta^g),$$

where $\alpha, \beta \in \Omega$ and $g \in G$. An orbit of the action of G on $\Omega \times \Omega$ is called an *orbital* of G , and we denote the set of orbitals of G by S_G . Then, each orbital $s \in S_G$ defines a digraph (Ω, s) whose out-degree is constant. We shall write the constant out-degree of (Ω, s) as n_s . Then, we have the following characterization:

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- (i) G is 2-transitive on Ω if and only if $|S_G| = 2$;
- (ii) G is 3/2-transitive on Ω if and only if n_s is constant on $s \in S_G \setminus \{1_\Omega\}$, where $1_\Omega = \{(\alpha, \alpha) \mid \alpha \in \Omega\}$;
- (iii) G is regular on Ω if and only if $n_s = 1$ for each $s \in S_G$.

An association scheme is a configuration consisting of a point set and a set of binary relations on the points with certain properties (see Sect. 2 for the details). For short we call it just a scheme for the remainder of this section. It is known (see [1, Ex. 2.1]) that (Ω, S_G) is a scheme. We say that a scheme (Ω, S) is *schurian* if $S = S_G$ for some permutation group G on Ω . In [1] we can see that schurian schemes play important roles in the theory of schemes or distance-regular graphs. But, a scheme is a purely combinatorial object just generalizing the orbitals of transitive permutation groups, and quite many non-schurian schemes are listed in [4]. For a positive integer k we say that a scheme (Ω, S) with more than one point is *k-equivalenced* if $n_s = k$ for each $s \in S \setminus \{1_\Omega\}$, where n_s is the valency of s , namely, the constant out-degree of (Ω, s) . This is a combinatorial property generalizing 3/2-transitivity. We say that (Ω, S) is *regular* (resp. *Frobenius*) if $S = S_G$ for some regular (resp. Frobenius) permutation group on Ω .

It is an easy observation that the adjacency matrices of a 1-equivalenced scheme form a group. This implies that every 1-equivalenced scheme is regular (see [7, Theorem A]). However, the situation for k -equivalenced schemes with $k > 1$ is completely different. In order to find non-trivial automorphisms from the property being k -equivalenced, we need to know some intersection numbers $\{c_{ss^*}^t \mid s, t \in S\}$ to determine the local structure of a non-diagonal relation (see Sect. 2 for the details). One may notice that the difficulty in determining the local structures from $\{c_{ss^*}^t \mid s, t \in S\}$ satisfying Lemma 2.1 (iv) with $u = s$ and $v = s^*$ is getting higher as k increases. In the case of $k = 2$, however, it works well and it is proved that every 2-equivalenced scheme is Frobenius (see [6]). In order to avoid repeating similar statements, we define $\mathcal{F}$ to be the set of positive integers k such that every k -equivalenced scheme is Frobenius. According to the classification of schemes of small orders (see [4]), we find non-schurian k -equivalenced schemes with exactly three relations for $k = 7, 9, 11, 12, 13, 14, 15$. Therefore,

$$7, 9, 11, 12, 13, 14, 15 \notin \mathcal{F}.$$

On the other hand, it is still open whether $4, 5, 6, 8, 10 \in \mathcal{F}$. However, in [5, Theorem 6.2], Muzychuk and Ponomarenko generalized the concept of pseudocyclic schemes for non-commutative ones and showed the following:

- (i) A scheme (Ω, S) is pseudocyclic of valency k if and only if (Ω, S) is k -equivalenced and, for each $t \in S \setminus \{1_\Omega\}$,

$$\sum_{s \in S} c_{ss^*}^t = k - 1;$$

- (ii) Every Frobenius scheme is pseudocyclic;
- (iii) Every pseudocyclic scheme (Ω, S) of valency k is Frobenius if $|S| \geq 4(k-1)k^3$.

The last one would be an immortal result in the theory of schemes to give a criterion for pseudocyclic schemes to be Frobenius though $|S| \geq 4(k-1)k^3$ is a rough bound, the first one shows that being k -equivalenced is quite weaker than pseudocyclicity, and the results suggest indirectly the following questions:

- (i) Whether the bound $|S| \geq 4(k-1)k^3$ can be improved for small k ;
- (ii) Whether every k -equivalenced scheme (Ω, S) is pseudocyclic for small k ;
- (iii) Whether $k \in \mathcal{F}$ for small k .

The following gives the answer for the questions when $k = 3$:

Theorem 1.1 *Every 3-equivalenced scheme is Frobenius.*

In the case of $k = 3$ we have two kinds of the local structure of a digraph (Ω, s) (see Lemma 3.2). It means that we need a slightly complicated argument in order to prove $3 \in \mathcal{F}$ in comparison with the case of $k = 2$.

Obviously, if a k -equivalenced scheme of order m exists, then $m \equiv 1 \pmod{k}$, and the converse is true if $k = 1$ (resp. 2), since a cyclic group of order m (resp. a dihedral group of degree m) exists for each positive integer m . On the other hand, it is known (see [4]) that there is no 3-equivalenced scheme of order 10 or 22. Thus, it is natural to ask about the size $|\Omega|$ such that a 3-equivalenced scheme (Ω, S) exists, or equivalently, the degree m such that a Frobenius group of order $3m$ exists. This question can be solved by the following statement given in [2, Theorem 1]: a Frobenius group of degree m and order mk exists if and only if each maximal prime power which divides m is congruent to one modulo k .

In Sect. 2 we prepare basic notation to make this paper as self-contained as possible. In Sect. 3 we give a proof of Theorem 1.1.

2 Preliminaries

We use the same notation on association schemes as in [5]. Let Ω be a finite set and S a partition of $\Omega \times \Omega$. For $s \in S$ we denote $\{(\beta, \alpha) \mid (\alpha, \beta) \in s\}$ by s^* . We denote the set of (α, α) with $\alpha \in \Omega$ by 1_Ω . For $\alpha \in \Omega$ and $s \in S$ we set

$$\alpha s = \{\beta \in \Omega \mid (\alpha, \beta) \in s\}.$$

We say that the pair (Ω, S) is an *association scheme* if it satisfies the following conditions (see [1], [7], and [8] for a background of the theory of association schemes):

- (i) $1_\Omega \in S$;
- (ii) For each $s \in S$ we have $s^* \in S$;
- (iii) For all $u, v, w \in S$ the number $|\alpha u \cap \beta v^*|$ is constant whenever $(\alpha, \beta) \in w$. We denote the constant by c_{uv}^w .

For the remainder of this paper we assume that (Ω, S) is an association scheme, and we denote $S \setminus \{1_\Omega\}$ by $S^\#$.

For $\alpha, \beta \in \Omega$ we denote the unique element of S which contains (α, β) by $r(\alpha, \beta)$. For $s \in S$ we denote $c_{ss^*}^{1_\Omega}$ by n_s , and we call it the *valency* of s .

Lemma 2.1 ([7]) *For all $u, v, w \in S$ we have the following:*

- (i) $c_{u^*v}^{1\Omega} = \delta_{u,v}n_u$, where δ is Kronecker's delta;
- (ii) $c_{uv}^w = c_{v^*u^*}^{w^*}$;
- (iii) $c_{uv}^{w^*}n_w = c_{vw}^{u^*}n_u = c_{wu}^{v^*}n_v$;
- (iv) $n_un_v = \sum_{s \in S} c_{uv}^s n_s$.

Applying Lemma 2.1 for a k -equivalenced association scheme (Ω, S) we obtain the following:

$$k = \delta_{v,u^*} + \sum_{w \in S^\#} c_{uv}^w \quad \text{for all } u, v \in S^\#. \quad (1)$$

In [7] the *complex product* UV of two subsets $U, V \subseteq S$ is defined to be

$$UV = \left\{ w \in S \mid \sum_{u \in U, v \in V} c_{uv}^w > 0 \right\}.$$

We shall write uV, UV instead of UV if U, V are singletons $\{u\}, \{v\}$, respectively.

Lemma 2.2 *For $u, v, w \in S$ with $w \in uv^*$ and $\alpha, \beta \in \Omega$ with $\beta \in \alpha u$ we have the following:*

- (i) $u^*v = \{r(\beta, \delta) \mid \delta \in \alpha v\}$;
- (ii) β has exactly $c_{wv^*}^{u^*}$ out-neighbors in αv with respect to w ;
- (iii) If $u = v$ and w is symmetric but not diagonal, then $n_u c_{wu}^{u^*}$ is even.

Proof (i) Let $s \in u^*v$. Then $c_{u^*v}^s > 0$. By Lemma 2.1 (iii), $c_{vs^*}^u = c_{u^*v}^s n_s / n_u > 0$. Thus, there exists $\delta \in \alpha v \cap \beta s$, implying $s = r(\beta, \delta)$ with $\delta \in \alpha v$. Conversely, for $\delta \in \alpha v$, $r(\beta, \delta) \in r(\beta, \alpha)r(\alpha, \delta) = u^*v$.

(ii) follows from the definition of the constants c_{uv}^w .

(iii) follows since $|w \cap (\alpha u^* \times \alpha u^*)| = n_u c_{wu}^{u^*} = n_u c_{wu}^u$ by Lemma 2.1 (iii), and it is twice of the number of edges in the graph $(\alpha u, (\alpha u \times \alpha u) \cap w)$. $\square$

For $s \in S$ we define a matrix A_s called the *adjacency matrix* of s as follows:

$$(A_s)_{\alpha\beta} := \begin{cases} 1 & \text{if } (\alpha, \beta) \in s \\ 0 & \text{if } (\alpha, \beta) \notin s \end{cases}$$

where the rows and columns of A_s are indexed by the elements of Ω . It follows from the definition of association schemes that

$$A_u A_v = \sum_{w \in S} c_{uv}^w A_w. \quad (2)$$

Recall that the Frobenius inner product on $\text{Mat}_\Omega(\mathbb{C})$ is defined such that, for all $A, B \in \text{Mat}_\Omega(\mathbb{C})$,

$$\langle A, B \rangle = \frac{1}{|\Omega|} \text{Tr}(AB^*),$$

where B^* is the Hermitian conjugate of B . As far as there is no confusion in the context we shall write just u instead of A_u , and $u \cdot v$ instead of $A_u A_v$ in order to distinguish it from the complex product uv for all $u, v \in S$. For example, we shall write (2) as

$$u \cdot v = \sum_{w \in S} c_{uv}^w w.$$

The following lemma is frequently used in this paper. We shall use it with no additional mention.

Lemma 2.3 *For all $u, v, w \in S$ we have the following:*

- (i) $\langle u, v \rangle = n_u \delta_{u,v}$;
- (ii) $\langle u \cdot v, w \rangle = \langle u, w \cdot v^* \rangle = \langle v, u^* \cdot w \rangle = c_{uv}^w n_w$;
- (iii) $\langle u \cdot v, u \cdot v \rangle = \langle u^* \cdot u, v \cdot v^* \rangle$.

Proof (i) follows from Lemma 2.1 (i), while (ii) and (iii) follow from the definition of $\langle *, * \rangle$ and (i). $\square$

Applying Lemma 2.3 for a k -equivalenced association scheme (Ω, S) we obtain from (1) that, for all $u, v \in S^\#$ with $v \neq u^*$,

$$\begin{aligned} \sum_{w \in S^\#} (c_{uv}^w)^2 k &= \langle u \cdot v, u \cdot v \rangle = \langle u^* \cdot u, v \cdot v^* \rangle \\ &= k^2 + \sum_{w \in S^\#} c_{u^*u}^w c_{vv^*}^w k \\ &\leq k^2 + k \sum_{w \in S^\#} c_{u^*u}^w \sum_{w \in S^\#} c_{vv^*}^w = k^2 + k(k-1)^2. \end{aligned}$$

Since $c_{uu^*}^w < k$ for every $w \in S^\#$ by (1), this implies that, for every k -equivalenced association scheme (Ω, S) with $k > 1$ and all $u, v, w \in S$,

$$c_{uv}^w = k \Leftrightarrow v = u^* \neq 1_\Omega \text{ and } w = 1_\Omega. \quad (3)$$

We use the same notation on group actions as in [3]. We denote the symmetric (resp. alternating) group on Ω by $\text{Sym}(\Omega)$ (resp. $\text{Alt}(\Omega)$). We define the *automorphism group* of (Ω, S) to be the set of $\sigma \in \text{Sym}(\Omega)$ such that, for all $\alpha, \beta \in \Omega$,

$$r(\alpha, \beta) = r(\alpha^\sigma, \beta^\sigma).$$

We denote it by $\text{Aut}(\Omega, S)$. Suppose that G is a transitive permutation group on Ω . Then G is contained in $\text{Aut}(\Omega, S_G)$. Thus, in order to prove Theorem 1.1, it suffices to find a Frobenius group G in the automorphism group of a 3-equivalenced association scheme (Ω, S) such that $S = S_G$.

3 Proof of the main result

In this section we assume that (Ω, S) is a 3-equivalenced association scheme.

Lemma 3.1 *For all $u, v \in S^\#$ with $u \neq v^*$ one of the following holds:*

- (i) $u \cdot v = w_1 + w_2 + w_3$ for some distinct $w_1, w_2, w_3 \in S^\#$;
- (ii) $u \cdot v = w_1 + 2w_2$ for some distinct $w_1, w_2 \in S^\#$.

Proof Applying (1) with $k = 3$ we have

$$\sum_{w \in S^\#} c_{uv}^w = 3.$$

Since $c_{uv}^w \leq n_w = 3$, it follows from (3) that $c_{uv}^w \leq 2$. This completes the proof. $\square$

Lemma 3.2 *For every $u \in S^\#$ one of the following holds:*

- (i) $u \cdot u^* = 3 \cdot 1_\Omega + w + w^*$ for some $w \in S$ with $w \neq w^*$;
- (ii) $u \cdot u^* = 3 \cdot 1_\Omega + 2u$ and $u = u^*$.

Proof Applying (1) with $k = 3$ we have

$$\sum_{w \in S^\#} c_{uu^*}^w = 2.$$

Thus, we have either $c_{uu^*}^w = 1$ or $c_{uu^*}^w = 2$ for some $w \in S$. In the first case we have $w \neq w^*$ by Lemma 2.2 (iii), and $c_{uu^*}^w = c_{uu^*}^{w^*}$ by Lemma 2.1 (ii). Thus, (i) holds.

In the second case,

$$u \cdot u^* = 3 \cdot 1_\Omega + 2u.$$

Since $c_{wu}^u = c_{uu^*}^w = 2$ by Lemma 2.1 (iii), it follows from Lemma 2.2 (ii) with $k = 3$ that the digraph $(\alpha u^*, w \cap (\alpha u^* \times \alpha u^*))$ forms a triangle, especially, w is symmetric. Thus, $w \in ww$ and $w = w^*$. It follows from Lemma 2.2 (iii) that c_{ww}^w is even. Therefore,

$$w \cdot w = 3 \cdot 1_\Omega + 2w.$$

Note that

$$\langle w \cdot u, w \cdot u \rangle = \langle u \cdot u^*, w^* \cdot w \rangle = 21.$$

If $u \neq w$, then the left-hand side is equal to 9 or 15 by Lemma 3.1, a contradiction. Therefore, (ii) holds. $\square$

Lemma 3.3 *For all $u, v \in S^\#$, if $u = u^*$ and $u \in vv^*$, then $u = v$ and $1_\Omega \cup u$ is an equivalence relation on Ω .*

Proof The assertion $u = v$ follows from Lemma 3.2. It is obvious that $1_\Omega \cup u$ is reflexive and symmetric. The equation as in Lemma 3.2 (ii) implies that $1_\Omega \cup u$ is transitive. $\square$

Lemma 3.4 *For all $u, v, w \in S^\#$ with $v \neq w$ we have $\langle u \cdot v, u \cdot w \rangle \leq 9$, and equality holds if and only if $v \cdot w^* = 2s + s^*$, where $u^* \cdot u = 3 \cdot 1_\Omega + s + s^*$ and $s \neq s^*$.*

Proof Applying Lemma 3.1 and Lemma 3.2 to $v \cdot w^*$ and $u^* \cdot u$, respectively, we obtain from $vw^* \subseteq S^\#$ that

$$\langle u \cdot v, u \cdot w \rangle = \langle v \cdot w^*, u^* \cdot u \rangle \in \{0, 3, 6, 9, 12\}.$$

If $\langle u \cdot v, u \cdot w \rangle = 12$, then $c_{u^*u}^u = c_{vw^*}^u = 2$, implying that $u = u^*$, $u \cdot v = v + 2w$ and $u \cdot w = 2v + w$. Thus, $u^* = u \in vv^* \cap ww^*$. By Lemma 3.2, $u = v = w$, a contradiction. Therefore, $\langle u \cdot v, u \cdot w \rangle \leq 9$, and equality holds if and only if $u^* \cdot u = 3 \cdot 1_\Omega + s + s^*$ and $v \cdot w^* = 2s + s^*$, where $u^* \cdot u = 3 \cdot 1_\Omega + s + s^*$ and $s \neq s^*$. $\square$

Lemma 3.5 *For all $u, v_1, v_2, v_3 \in S^\#$, if $u \cdot v_1 = u \cdot v_2 = u \cdot v_3$, then $|\{v_1, v_2, v_3\}| < 3$.*

Proof Without loss of generality we may assume that $v_1 \neq v_2$ and $v_1 \neq v_3$. Since $u \cdot v_1 = u \cdot v_2$, it follows from Lemma 3.4 that $v_1 \cdot v_2^* = 2s + s^*$, where $u^* \cdot u = 3 \cdot 1_\Omega + s + s^*$ and $s \neq s^*$. By the same argument, we have $v_1 \cdot v_3^* \in \{2s + s^*, s + 2s^*\}$. Thus, $\langle v_1 \cdot v_2^*, v_1 \cdot v_3^* \rangle = 15$ or 12 . By Lemma 3.4, this forces $v_2 = v_3$. $\square$

Fix $\alpha \in \Omega$. Then Ω is partitioned into $\{\alpha u \mid u \in S\}$. We define Γ_α to be the set of $\sigma \in \text{Sym}(\Omega)$ such that σ fixes only α and $(\alpha u)^\sigma = \alpha u$ for each $u \in S$. Remark that $|\Gamma_\alpha| = 2^{|S|-1}$ since there are exactly two fixed-point-free permutations on each αu with $u \in S^\#$. For $\sigma \in \Gamma_\alpha$ and $u, v \in S^\#$ we shall write $u \sim_\sigma v$ if $r(\beta, \gamma) = r(\beta^\sigma, \gamma^\sigma)$ for each $(\beta, \gamma) \in \alpha u \times \alpha v$. It is clear that σ is an automorphism of (Ω, S) if and only if $\sim_\sigma$ is the trivial relation on $S^\#$, that is, $u \sim_\sigma v$ for all $u, v \in S^\#$.

Lemma 3.6 *For all $u, v \in S^\#$ and $\sigma \in \Gamma_\alpha$, $\{(\alpha u \times \alpha v) \cap w \mid w \in u^*v\}$ is a partition of $\alpha u \times \alpha v$ which is invariant under exactly one of the following actions of σ on $\alpha u \times \alpha v$:*

- (i) $(\beta, \gamma)^\sigma = (\beta^\sigma, \gamma^\sigma)$ for each $(\beta, \gamma) \in \alpha u \times \alpha v$;
- (ii) $(\beta, \gamma)^\sigma = (\beta^\sigma, \gamma^{\sigma^{-1}})$ for each $(\beta, \gamma) \in \alpha u \times \alpha v$.

Moreover, $u \not\sim_\sigma v$ if and only if $r(\beta, \gamma) = r(\beta^\sigma, \gamma^{\sigma^{-1}})$ for each $(\beta, \gamma) \in \alpha u \times \alpha v$.

Proof By Lemma 2.2 (i), $\{(\alpha u \times \alpha v) \cap w \mid w \in u^*v\}$ is a partition of $\alpha u \times \alpha v$. By Lemma 2.2 (i), (ii), $\sum_{w \in u^*v} c_{wv^*}^{u^*} = 3$. Since $|u^*v| \in \{2, 3\}$ by Lemma 3.1 and Lemma 3.2, there exists $w \in u^*v$ such that $c_{wv^*}^{u^*} = 1$. Let $(\beta, \gamma) \in w$. Then $\beta^\sigma w \cap \alpha v = \{\gamma^{\sigma^i}\}$ for a unique $i = 1, 2$ since $c_{w^*u^*}^{v^*} = c_{wv^*}^{u^*} = 1$ by Lemma 2.1 (ii), (iii). Thus, we have

$$(\alpha u \times \alpha v) \cap w = \left\{ (\beta, \gamma), (\beta^\sigma, \gamma^{\sigma^i}), (\beta^{\sigma^2}, \gamma^{\sigma^{2i}}) \right\}.$$

If $|u^*v| = 2$, then we are done. If $|u^*v| = 3$, then $c_{zv^*}^{u^*} = 1$ for each $z \in u^*v$, and hence, by Lemma 2.2 (ii),

$$(\alpha u \times \alpha v) \cap r(\beta, \gamma^\sigma) = \{(\beta, \gamma^\sigma), (\beta^\sigma, \gamma^{\sigma^{i+1}}), (\beta^{\sigma^2}, \gamma^{\sigma^{2i+1}})\}, \text{ and} \\ (\alpha u \times \alpha v) \cap r(\beta, \gamma^{\sigma^2}) = \{(\beta, \gamma^{\sigma^2}), (\beta^\sigma, \gamma^{\sigma^{i+2}}), (\beta^{\sigma^2}, \gamma^{\sigma^{2i+2}})\}.$$

This completes the proof of the first assertion since $\sigma^2 = \sigma^{-1}$.

If $u \sim_\sigma v$, then, by the definition of $\sim_\sigma$, $\{(\alpha u \times \alpha v) \cap w \mid w \in u^*v\}$ is invariant under the first action. By the first assertion, the contrapositive of “if” part of the second assertion holds. If $u \not\sim_\sigma v$, then $r(\beta_0, \gamma_0) \neq r(\beta_0^\sigma, \gamma_0^\sigma)$ for some $(\beta_0, \gamma_0) \in \alpha u \times \alpha v$. By the first assertion, $\{(\alpha u \times \alpha v) \cap w \mid w \in u^*v\}$ is invariant under the second action. This completes the proof of the “only if” part of the second assertion. $\square$

It is obvious that $\sim_\sigma$ is symmetric on $S^\#$, and $\sim_\sigma$ is reflexive by the following:

Lemma 3.7 For all $u \in S^\#$ and $\sigma \in \Gamma_\alpha$ we have $u \sim_\sigma u$.

Proof Since $1_\Omega \in u^*u$ by Lemma 2.1 (i),

$$1_\Omega = r(\beta, \beta) \neq r(\beta^\sigma, \beta^{\sigma^{-1}}) \text{ for } \beta \in \alpha u.$$

By the contrapositive of the third statement of Lemma 3.6, we have $u \sim_\sigma u$. $\square$

Lemma 3.8 Let $\sigma \in \Gamma_\alpha$, $u \in S^\#$ and $\beta \in \alpha u$. We shall denote the permutation $(\beta, \beta^\sigma)\sigma(\beta, \beta^\sigma)$ of Ω by σ^u . Then we have the following:

- (i) σ^u does not depend on the choice of $\beta \in \alpha u$;
- (ii) $\sigma^u \in \Gamma_\alpha$;
- (iii) For all $v, w \in S^\# \setminus \{u\}$, $v \sim_\sigma w \Leftrightarrow v \sim_{\sigma^u} w$ and $v \sim_\sigma u \Leftrightarrow v \not\sim_{\sigma^u} u$.

Proof It is a routine work to check that the action of σ^u on $\Omega \setminus \alpha u$ coincides with that of σ , and the action of σ^u on αu coincides with that of σ^2 . This completes the proof of (i), (ii), and the former part of (iii).

Let $v \in S^\# \setminus \{u\}$. By Lemma 3.6, $v \sim_\sigma u$ if and only if $v \not\sim_{\sigma^u} u$. This completes the proof. $\square$

Lemma 3.9 Let $u \in S^\#$. Then there exists $\sigma \in \Gamma_\alpha$ such that $u \sim_\sigma v$ for all $v \in S^\#$.

Proof Choose $\sigma \in \Gamma_\alpha$ in such a way that $\{w \in S^\# \mid u \sim_\sigma w\}$ is maximal. Suppose that $\{w \in S^\# \mid u \sim_\sigma w\} \neq S^\#$. Then $u \not\sim_\sigma v$ for some $v \in S^\#$. By Lemma 3.7, $u \neq v$. Thus, by Lemma 3.8 (iii),

$$\{w \in S^\# \mid u \sim_{\sigma^u} w\} = \{w \in S^\# \mid u \sim_\sigma w\} \cup \{v\},$$

which contradicts the choice of σ . $\square$

Lemma 3.10 For each $\sigma \in \Gamma_\alpha$, the graph $(S^\#, \sim_\sigma)$ has no triangle.

Proof Suppose to the contrary that $u \not\sim_\sigma v$, $v \not\sim_\sigma w$ and $w \not\sim_\sigma u$ for some distinct $u, v, w \in S^\#$ and $\sigma \in \Gamma_\alpha$. Since $|st^*| = |ts^*| \in \{2, 3\}$ for all distinct $s, t \in S^\#$ by Lemma 3.1, it suffices to show a contradiction for each of the following cases: (i) $|u^*v| = |v^*w| = 2$; (ii) $|u^*v| = |v^*w| = 3$.

(i) Suppose $|u^*v| = |v^*w| = 2$. For $i = 1, 2$ there exists a unique $b_i \in u^*v$ ($c_i \in v^*w$) such that $c_{u^*v}^{b_i} = i$ (resp. $c_{v^*w}^{c_i} = i$) by Lemma 3.1. Thus, $u^*v = \{b_1, b_2\}$ and $v^*w = \{c_1, c_2\}$. Let $(\beta, \gamma, \delta) \in \alpha u \times \alpha v \times \alpha w$ such that $(\beta, \gamma) \in b_1$ and $(\gamma, \delta) \in c_1$. By Lemma 2.1 (i), (ii), $c_{b_2v^*}^{u^*} = c_{b_2v^*}^{b_2} = 2$ and $c_{c_2v^*}^{w^*} = c_{c_2v^*}^{c_2} = 2$. Thus, by Lemma 2.2 (ii), β has exactly two out-neighbors in αv with respect to b_2 , and δ also has exactly two out-neighbors in αv with respect to c_2^* . Since

$$\alpha v = \{\gamma, \gamma^\sigma, \gamma^{\sigma^2}\} \text{ and } \gamma \in \beta b_1 \cap \delta c_1^*,$$

it follows that

$$\gamma^\sigma, \gamma^{\sigma^2} \in \beta b_2 \cap \delta c_2^*.$$

Since $u \not\sim_\sigma v$ and $v \not\sim_\sigma w$, it follows from Lemma 3.6 that, for $i = 0, 1, 2$,

$$\{\gamma^\sigma, \gamma^{\sigma^2}\}^{\sigma^{-i}} \subseteq \beta^{\sigma^i} b_2 \cap \delta^{\sigma^i} c_2^*.$$

By the uniqueness of d with $c_{b_2c_2^*}^d = 2$, $d = r(\beta^{\sigma^i}, \delta^{\sigma^i})$ for $i = 0, 1, 2$. By Lemma 3.6, $u \sim_\sigma w$, which contradicts $u \not\sim_\sigma w$.

(ii) Suppose $|u^*v| = |v^*w| = 3$. By Lemma 3.1, we can take $d \in w^*u$ such that $c_{w^*u}^d = 1$.

We claim that $db = w^*v$ for each $b \in u^*v$. Let $(\beta, \gamma, \delta) \in \alpha u \times \alpha v \times \alpha w$ such that $(\delta, \beta) \in d$ and $(\beta, \gamma) \in b$. Applying Lemma 3.6 for $v \not\sim_\sigma w$ and $w \not\sim_\sigma u$ we obtain that, for $i = 0, 1, 2$,

$$d = r(\delta, \beta) = r(\delta^{\sigma^i}, \beta^{\sigma^{-i}}) \text{ and } b = r(\beta, \gamma) = r(\beta^{\sigma^{-i}}, \gamma^{\sigma^i}).$$

Thus, $r(\delta^{\sigma^i}, \gamma^{\sigma^i}) \in r(\delta^{\sigma^i}, \beta^{\sigma^{-i}})r(\beta^{\sigma^{-i}}, \gamma^{\sigma^i}) = db$. Since $w \not\sim_\sigma v$, it follows from Lemma 3.6 with $|w^*v| = |v^*w| = 3$ that $r(\delta^{\sigma^i}, \gamma^{\sigma^i})$ with $i = 0, 1, 2$ are distinct. Since $\{r(\delta^{\sigma^i}, \gamma^{\sigma^i}) \mid i = 0, 1, 2\} \subseteq w^*v$ by Lemma 2.2 (i), they are equal, and hence, $w^*v \subseteq db$. Since $|w^*v| = 3$ and $|db| \leq 3$, the claim holds.

By the claim with $|w^*v| = 3$, $d \cdot b = d \cdot b'$ for all $b, b' \in u^*v$. Since $|u^*v| = 3$, it contradicts Lemma 3.5. $\square$

Proposition 3.11 For each $\sigma \in \Gamma_\alpha$, $\sim_\sigma$ is an equivalence relation on $S^\#$.

Proof From the definition of $\sim_\sigma$, it is clear that $\sim_\sigma$ is symmetric on $S^\#$. By Lemma 3.7, $\sim_\sigma$ is reflexive on $S^\#$. It remains to show that $\sim_\sigma$ is transitive on $S^\#$. Let $u, v, w \in S^\#$ such that $u \sim_\sigma v$ and $v \sim_\sigma w$. If $u \not\sim_\sigma w$, then u, v , and w are distinct since $\sim_\sigma$ is reflexive. By Lemma 3.8 (iii), $u \not\sim_{\sigma^v} v$, $v \not\sim_{\sigma^v} w$, and $w \not\sim_{\sigma^v} u$, which contradicts Lemma 3.10. Therefore, $u \sim_\sigma w$. $\square$

Lemma 3.12 *If $\sigma \in \text{Aut}(\Omega, S)$ fixes two distinct points in Ω , then $|S| = 2$ or σ is the identity.*

Proof Suppose $\alpha, \beta \in \Omega$ are distinct and fixed by σ . We set $u = r(\alpha, \beta)$ for short. Let $\alpha u = \{\beta_0, \beta_1, \beta_2\}$, where $\beta = \beta_0$. Suppose $|S| > 2$. Then, there exists $v \in S^\#$ with $u \neq v$. By Lemma 3.1, there exists $w \in S^\#$ such that $c_{u^*v}^w = 1$. Since $c_{vw^*}^u = c_{u^*v}^w = 1$ by Lemma 2.1 (iii), for $i = 0, 1, 2$, there exists a unique $\gamma_i \in \alpha v \cap \beta_i w$. By $c_{uw}^v = c_{u^*v}^w = 1$, γ_0, γ_1 , and γ_2 are distinct. Since α and β are fixed by σ , we have

$$\{\gamma_i^\sigma\} = \alpha v \cap \beta_i^\sigma w \text{ for } i = 0, 1, 2, \text{ in particular, } \gamma_0^\sigma = \gamma_0. \quad (4)$$

We claim that $\beta_i^\sigma = \beta_i$ for $i = 1, 2$. Suppose to the contrary that $\beta_1^\sigma = \beta_2$ and $\beta_2^\sigma = \beta_1$. Then

$$r(\beta_1, \beta_2) = r(\beta_1^\sigma, \beta_2^\sigma) = r(\beta_2, \beta_1) = r(\beta_1, \beta_2)^*.$$

By (4),

$$r(\beta_1, \gamma_0) = r(\beta_1^\sigma, \gamma_0^\sigma) = r(\beta_2, \gamma_0).$$

Thus,

$$r(\beta_1, \beta_2) \in r(\beta_1, \gamma_0)r(\gamma_0, \beta_2) = r(\beta_1, \gamma_0)r(\beta_1, \gamma_0)^*.$$

Applying Lemma 3.3 for $r(\beta_1, \beta_2) \in u^*u \cap r(\beta_1, \gamma_0)r(\beta_1, \gamma_0)^*$, we obtain that $r(\beta_1, \beta_2) = u = r(\beta_1, \gamma_0)$, and $u = r(\alpha, \gamma_0)$ since $1_\Omega \cup u$ is transitive, which contradicts $\gamma_0 \in \alpha v$ since $u \neq v$.

By the above claim, σ fixes each element of αu . Moreover, by (4), σ fixes each element of αv . Since v is arbitrarily taken from $S^\# \setminus \{u\}$, it follows that σ fixes each element of Ω . $\square$

Proof of Theorem 1.1 First, we claim that, for each $\alpha \in \Omega$,

$$\Gamma_\alpha \cap \text{Aut}(\Omega, S) \neq \emptyset.$$

Applying Lemma 3.9 for an arbitrarily taken $u \in S^\#$ we obtain that there exists $\sigma \in \Gamma_\alpha$ such that $u \sim_\sigma v$ for all $v \in S^\#$. Applying Proposition 3.11 for $\sim_\sigma$ we obtain that $\sim_\sigma$ is the trivial relation on $S^\#$. This implies that $\sigma \in \text{Aut}(\Omega, S)$.

We shall denote by G the subgroup of $\text{Sym}(\Omega)$ generated by

$$\bigcup_{\alpha \in \Omega} \Gamma_\alpha \cap \text{Aut}(\Omega, S).$$

By the first claim, $G_\alpha \cap \Gamma_\alpha \neq \emptyset$. This implies that G_α acts transitively on αs for each $s \in S$.

We claim that G is transitive on Ω . Suppose $\alpha \in \Omega \setminus \beta^G$ for $\beta \in \Omega$. Then, for each $\gamma \in \beta^G$, $\alpha r(\alpha, \gamma)$ is contained in β^G since it is an orbit of G_α . Since $\{\alpha u \cap \beta^G \mid u \in S^\# , \alpha u \cap \beta^G \neq \emptyset\}$ is a partition of β^G , it follows that $|\beta^G| \equiv 0 \pmod{3}$. Therefore, we conclude that the size of each orbit of G on Ω is divisible by 3, which contradicts $|\Omega| = \sum_{s \in S} n_s \equiv 1 \pmod{3}$. Therefore, G is transitive on Ω .

By the second claim, S is the set of orbitals of G since G_α acts transitively on αs for each $s \in S$.

It remains to show that G is Frobenius. If $|S| = 2$, then $|\Omega| = 4$ and $G = \text{Alt}(\Omega)$, which is a Frobenius group. If $|S| > 2$, then G is Frobenius by Lemma 3.12. $\square$

Corollary 3.13 *Let (Ω, S) be a 3-equivalenced scheme with $|S| > 2$. Then $\text{Aut}(\Omega, S)$ is a Frobenius group of order $3|\Omega|$.*

Proof This is an immediate consequence of Theorem 1.1 and Lemma 3.12. $\square$

From now on we consider separability of 3-equivalenced schemes (see [5] for definitions of separability, coherent configurations, one-point fission (Ω, S_α) of (Ω, S) , fibers, semiregularity, and so on).

Lemma 3.14 *Let (Ω, S) be a 3-equivalenced scheme with $|S| > 2$ and $\alpha \in \Omega$. Then the coherent configuration (Ω, S_α) is semiregular on $\Omega \setminus \{\alpha\}$.*

Proof Let S_0 denote the set of orbitals of $\text{Aut}(\Omega, S)_\alpha$. Since $\text{Aut}(\Omega, S)_\alpha$ is semiregular on $\Omega \setminus \{\alpha\}$ by Corollary 3.13, the coherent configuration obtained from (Ω, S_0) by restricting to $\Omega \setminus \{\alpha\}$ is semiregular, i.e., (Ω, S_0) is semiregular on $\Omega \setminus \{\alpha\}$. Moreover, S_α is a fusion of S_0 , i.e., each element of S_α is a union of elements of S_0 . For $u, v \in S^\#$ we shall write the set of elements of S_α (resp. S_0) contained in $\alpha u \times \alpha v$ as $S_\alpha(u, v)$ (resp. $S_0(u, v)$). We define a binary relation $\sim$ on $S^\#$ such that $u \sim v$ if and only if $S_0(u, v) = S_\alpha(u, v)$.

First, we claim that $S_\alpha(u, v) \cap S_0(u, v) \neq \emptyset$ for all $u, v \in S^\#$ and $u \sim v$ whenever $|u^*v| = 3$. Since $|u^*v| \in \{2, 3\}$ by Lemma 3.1 and 3.2 and S_α is a fusion of S_0 , it follows from Lemma 2.2 that $\alpha u \times \alpha v$ contains at least one element (resp. three elements) of $S_\alpha \cap S_0$ as a subset when $|u^*v| = 2$ (resp. $|u^*v| = 3$). Since $|S_0(u, v)| = 3$, we have $S_\alpha(u, v) = S_0(u, v)$ whenever $|u^*v| = 3$.

Second, we claim that, if $u \sim v$ for some $v \in S^\#$, then $u \sim w$ for all $w \in S^\#$. By the first claim, there exists $s \in S_\alpha(v, w) \cap S_0(v, w)$. Since (Ω, S_0) is semiregular on $\Omega \setminus \{\alpha\}$ and $S_\alpha(u, v) = S_0(u, v)$, the products ts with $t \in S_\alpha(u, v)$ are distinct singletons in $S_\alpha(u, w) \cap S_0(u, w)$, where the product ts on a coherent configuration is defined by the similar way to the complex product on a scheme (see the last sentence of [5, Sect. 2.1]). Therefore, $S_\alpha(u, w) = S_0(u, w)$.

Third, we claim that $\sim$ is not empty. Let $u, v \in S^\#$ with $u \neq v$. If $u \neq u^*$, then $|u^*u| = 3$ by Lemma 3.2, which implies $u \sim u$ by the first claim. If $u = u^*$, then by Lemma 3.1,

$$\langle u \cdot v, u \cdot v \rangle = \langle u^* \cdot u, v \cdot v^* \rangle = 9 + 2 \cdot 3 \cdot c_{vv^*}^u,$$

which is equal to 9 since $c_{vv^*}^u = 0$ by Lemma 3.1. By Lemma 3.2, $|u^*v| = |uv| = 3$, which implies $u \sim v$ by the first claim.

Since $\sim$ is symmetric by the definition of coherent configurations and $\sim$ is transitive by the second claim, it follows from the second and third claims that $\sim$ is the trivial relation. Therefore, $S_0 = S_\alpha$. $\square$

Corollary 3.15 *Every 3-equivalenced scheme is separable.*

Proof The assertion is obvious if $|S| = 2$. If $|S| > 2$, Lemma 3.14 shows that the assumption of [5, Theorem 7.2] is satisfied. Therefore, the assertion follows from [5, Theorem 7.2]. $\square$

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