On finite self-complementary metacirculants

Cai Heng Li · Guang Rao · Shu Jiao Song

Received: 25 October 2013 / Accepted: 17 March 2014 / Published online: 22 April 2014
© Springer Science+Business Media New York 2014

Abstract We prove that the automorphism group of a self-complementary metacirculant is either soluble or has A_5 as the only insoluble composition factor, extending a result of Li and Praeger which says the automorphism group of a self-complementary circulant is soluble. The proof involves a construction of self-complementary metacirculants which are Cayley graphs and have insoluble automorphism groups. To the best of our knowledge, these are the first examples of self-complementary graphs with this property.

Keywords Self-complementary · Vertex-transitive · Metacirculants

1 Introduction

A graph $\Gamma = (V, E)$ consists of a vertex set V and an edge set E which is a subset of the set $\{\{u, v\} \mid u, v \in V, u \neq v\}$. The *complement* $\overline{\Gamma}$ of a graph Γ is the graph with the same vertex set V such that $\{u, v\}$ is an edge of $\overline{\Gamma}$ if and only if $u \neq v$, and $\{u, v\}$ is not an edge of Γ . A graph is said to be *self-complementary* if its complement is isomorphic to itself.

A graph $\Gamma = (V, E)$ is called a *metacirculant* if $\text{Aut}\Gamma$ contains a metacyclic subgroup R which is transitive on V ; refer to [16] for the definition which is slightly

C. H. Li · G. Rao (✉) · S. J. Song
Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics,
The University of Western Australia, Crawley, WA 6009, Australia
e-mail: generalrao@hotmail.com

C. H. Li
e-mail: cai.heng.li@uwa.edu.au

S.J. Song
e-mail: shu-jiao.song@uwa.edu.au

more general than the original definition of Alspach and Parsons in [2]. (Recall that a group R is called *metacyclic* if R has a normal subgroup N such that both N and R/N are cyclic.) To emphasise the metacyclic group, we sometimes call Γ a *metacirculant of R* .

In this paper, we study metacirculants that are self-complementary, which will be called *self-complementary metacirculants*.

A special subclass of metacirculants consists of *circulants*, whose automorphism group contains a transitive cyclic subgroup. Self-complementary circulants have been studied for a long time and have been taken as good models for studying other combinatorial objects, such as Ramsey numbers and communication networks. The first family of self-complementary circulants was constructed by Sachs in 1962, and since then self-complementary circulants have been widely studied, see [6, 19, 22, 24, 25, 28] for the work till 1980s. In 1990s, the orders of self-complementary circulants were determined, see [1] for Alspach-Morris-Vilfred's proof and [7] for an alternative approach. Some special families of self-complementary circulants are constructed in [11, 18, 21]. More recently, it was proved that the automorphism groups of self-complementary circulants are soluble by Praeger and the first author in [14], and the structure of self-complementary circulants of prime-power order is characterised in [17].

On the other hand, self-complementary metacirculants form a subclass of self-complementary vertex-transitive graphs. One of the motivations of studying self-complementary vertex-transitive graphs is to investigate Ramsey numbers. For a positive integer m , we wish to construct a graph Γ with vertex set V such that neither Γ nor $\bar{\Gamma}$ contains a complete subgraph K_m and $|V|$ is as big as possible. By intuition, the distribution of pairs of elements of V between Γ and $\bar{\Gamma}$ should be “balanced” so that Γ and $\bar{\Gamma}$ should not be very different. In the extreme case, $\Gamma \cong \bar{\Gamma}$. Furthermore, since $K_m \not\leq \Gamma$ is a global property of Γ , the edges of Γ should be distributed “homogeneously”. In the ideal case, Γ is vertex-transitive. Because of this, self-complementary vertex-transitive graphs have been effectively used as models to find good lower bounds of Ramsey numbers (see [4, 5, 9, 23] for references).

The study of self-complementary vertex-transitive graphs has a long history, refer to the excellent survey of Beezer [3]. In particular, the orders of general self-complementary vertex-transitive graphs have been completely determined by Muzychuck [20]. Since 2000, the study of self-complementary vertex-transitive graphs has been advanced by the work in [10, 13]. The self-complementary vertex-transitive graphs whose orders are products of two primes are classified by the authors in [15].

Recall that for a group G and subgroups $K \triangleleft H \leq G$, the quotient group H/K is called a *section* of G . The main result of this paper gives a characterisation of the automorphism groups of self-complementary metacirculants.

Theorem 1.1 *Let Γ be a self-complementary metacirculant of R . Then either $\text{Aut}\Gamma$ is soluble or the following hold:*

- (i) *the only insoluble composition factor of $\text{Aut}\Gamma$ is the alternating group A_5 and*
- (ii) *$\text{Aut}\Gamma$ has a section that is of the form $\mathbb{Z}_p^2 : (\mathbb{Z}_\ell \circ \text{SL}(2, 5))$ such that $\mathbb{Z}_p^2$ is a section of R , where p is a prime number and $p \equiv 1$ or $9 \pmod{40}$, and ℓ divides $p - 1$ and is divisible by 4.*

Remark Although A_5 is the only insoluble composition factor of $\text{Aut}\Gamma$, this factor can appear multiple times, see Lemma 3.5.

In Sect. 3, we construct examples of self-complementary metacirculants that are Cayley graphs and have insoluble automorphism groups. Moreover, as a result of Lemma 3.3, we have the following.

Theorem 1.2 *There exist self-complementary metacirculants of which the automorphism groups are insoluble.*

Praeger and the first author in [14] proved that the automorphism group of a self-complementary circulant is soluble. A metacirculant Γ is called a *Sylow-circulant* if $\text{Aut}\Gamma$ has a transitive metacyclic subgroup R of which all Sylow subgroups are cyclic. In this case, R has a normal Hall subgroup N ; in particular, the orders of N and R/N are coprime. By Theorem 1.1 (ii), if $\text{Aut}\Gamma$ is insoluble, then there exists a prime p such that a Sylow p -subgroup of R is not cyclic. Therefore, a consequence of Theorem 1.1 is the following corollary, which extends the result for circulants.

Corollary 1.3 *If Γ is a self-complementary Sylow-circulant, then $\text{Aut}\Gamma$ is soluble. In particular, if Γ is a self-complementary metacirculant of square-free order, then $\text{Aut}\Gamma$ is soluble.*

We end this section with a conjecture.

Conjecture *Self-complementary metacirculants are Cayley graphs.*

We remark that a metacirculant is generally not necessarily a Cayley graph; for example, the Petersen graph is a metacirculant but not a Cayley graph.

2 Preliminaries

In this section, we collect some preliminary results that will be used.

Let $\Gamma = (V, E)$ be a self-complementary graph. Let $\sigma \in \text{Sym}(V)$ be an isomorphism from Γ to $\overline{\Gamma}$, called a *complementing isomorphism* of Γ . Then σ interchanges Γ and $\overline{\Gamma}$, and σ has order $o(\sigma) = 2^e b$ for $e \geq 1$, where b is an odd number. Observe that $\sigma = \sigma^b \cdot \sigma^{1-b}$, where σ^b is a complementing isomorphism of 2-power order and $\sigma^{1-b} \in \text{Aut}\Gamma$. We thus sometimes assume a complementing isomorphism σ to be of 2-power order. If $o(\sigma) = 2$, then σ interchanges some vertices u, v and fixes $\{u, v\}$, which is not possible. So 4 divides $o(\sigma)$.

Lemma 2.1 *A complementing isomorphism of a self-complementary graph has order divisible by 4.*

For a regular self-complementary graph of order n , a vertex has an equal number of neighbours and non-neighbours, and hence n must be an odd number. Furthermore, the graph has precisely half number of edges of the complete graph, and so $n(n-1)/4$ is an integer, and $n \equiv 1 \pmod{4}$.

Lemma 2.2 *The order of a regular self-complementary graph is congruent to 1 (modulo 4).*

We next introduce a classical method for constructing self-complementary graphs.

For a finite group R , let $R^\# := R \setminus \{1\}$, the set of all non-identity elements of R . For a subset $S \subseteq R^\#$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the associated *Cayley graph* $\text{Cay}(R, S)$ is the graph with vertex set $V = R$ and edge set $E = \{\{a, b\} \mid a, b \in R, ba^{-1} \in S\}$.

By the definition, the Cayley graph $\text{Cay}(R, R^\# \setminus S)$ is the complement of $\text{Cay}(R, S)$. Observe that any automorphism $\sigma \in \text{Aut}(R)$ induces an isomorphism from $\text{Cay}(R, S)$ to $\text{Cay}(R, S^\sigma)$. Thus, if there exist $S \subseteq R^\#$ and $\sigma \in \text{Aut}(R)$ such that

$$S^\sigma = R^\# \setminus S, \quad (*)$$

then $\text{Cay}(R, S) \cong \text{Cay}(R, S^\sigma) = \text{Cay}(R, R^\# \setminus S)$ is self-complementary. Since S and $R^\# \setminus S$ are disjoint, σ fixes no non-identity element of R , that is, σ is a *fixed-point-free* automorphism of R . Moreover, since $S = S^{-1}$, and σ cannot interchange any pair of vertices, we further have that σ^2 is fixed-point-free, and $g^{-1} \in g^{(\sigma^2)}$ for any $g \in R$.

Conversely, if a group R has an automorphism σ of order a power of 2 such that σ^2 is fixed-point-free, then the following construction indeed produces self-complementary graphs, see [12].

Construction 2.3 Let R be a group and let $\sigma \in \text{Aut}(R)$ of order a power of 2 such that σ^2 is fixed-point-free. Then every orbit of $\langle \sigma \rangle$ on $R^\#$ has length divisible by 4 and is divided into two parts by σ^2 . We define a subset S and a Cayley graph $\text{Cay}(R, S)$ as follows:

- (1) Let $\Delta_1, \Delta_2, \dots, \Delta_r$ be the $\langle \sigma \rangle$ -orbits on $R^\#$, and label the two orbits of $\langle \sigma^2 \rangle$ on Δ_i as Δ_i^+ and Δ_i^- , where $1 \leq i \leq r$.
- (2) Set $S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}$, where $\varepsilon_i = +$ or $-$. (We remark that there are 2^r different choices for such a subset S .)

Another construction method of self-complementary vertex-transitive graphs is based on the *lexicographic product* (sometimes called the *wreath product*). Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. The lexicographic product of Γ_2 by Γ_1 is defined as the graph with vertex set $V = V_1 \times V_2$ such that (u_1, v_1) is adjacent to (u_2, v_2) if and only if either u_1 is adjacent to u_2 in Γ_1 or $u_1 = u_2$ and v_1 is adjacent to v_2 . This graph is denoted by $\Gamma_1[\Gamma_2]$.

The lexicographic product has the following nice properties (see [3]):

Lemma 2.4 *If both Γ_1, Γ_2 are vertex-transitive or self-complementary, then so is $\Gamma_1[\Gamma_2]$.*

We will also need to know about the order of the maximal p -elements in $\text{GL}(d, p)$.

Lemma 2.5 *If $d \geq 2$, then the largest order p^e of p -elements of $\text{GL}(d, p)$ satisfies $p^e \geq d > p^{e-1}$.*

Proof Let U be the set of the elements of $\text{GL}(d, p)$ of the following form:

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1d} \\ 0 & 1 & a_{23} & & a_{2d} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{d-1,d} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then U forms a Sylow p -subgroup of $\text{GL}(d, p)$. Let $u \in U$ such that $o(u) = p^e$ is the largest order of elements of U , where e is an integer. Let $u = A + I$. Notice that for any $u \in U$, we have $u^{p^e} = I$ if and only if $(A + I)^{p^e} = I$, if and only if (using the binomial theorem) $A^{p^e} = 0$. By careful choice of u , we can ensure that $A^b \neq 0$ for $b < d$, so that $A^{p^e} = 0$ implies $p^e \geq d$ and $p^{e-1} < d$. $\square$

3 Examples with insoluble automorphism groups

We construct examples of self-complementary metacirculants which have insoluble automorphism groups.

Let $R = \mathbb{Z}_p^2$ with p prime. Then $\text{Aut}(R) = \text{GL}(2, p)$. Assume that $p \equiv \pm 1 \pmod{10}$. Then $\text{GL}(2, p)$ contains an irreducible subgroup $\text{SL}(2, 5)$, see [26, page 411]. Assume further that $p \equiv 1 \pmod{8}$. Then the centre $\mathbf{Z}(\text{GL}(2, p)) \cong \mathbb{Z}_{p-1}$ has order divisible by 8. Let σ be a 2-element of $\mathbf{Z}(\text{GL}(2, p))$ which has order at least 8. Since σ is a scalar matrix, its action on R is fixed-point-free. Then $\langle \sigma, \text{SL}(2, 5) \rangle = \langle \sigma \rangle \circ \text{SL}(2, 5)$ is a subgroup of $\text{GL}(2, p)$. We note that the condition that $p \equiv \pm 1 \pmod{10}$ and $p \equiv 1 \pmod{8}$ is equivalent to $p \equiv 1$ or $9 \pmod{40}$.

Lemma 3.1 *Let $M = \langle \sigma^2, \text{SL}(2, 5) \rangle$ and $H = \langle \sigma, \text{SL}(2, 5) \rangle$. Then M divides each H -orbit on $R^\#$ into two parts of equal size.*

Proof Let Δ be an H -orbit on $R^\#$, and let $x \in \Delta$. Then $x^\sigma \in \Delta$. Suppose that M is transitive on Δ . Then there exists an element $g \in M$ such that $x^g = x^\sigma$. Thus, $x^{\sigma g^{-1}} = x$. Since σ is in the centre of H , $\sigma g^{-1} = g^{-1}\sigma$. Hence $(\sigma g^{-1})^{o(g)} = \sigma^{o(g)}(g^{-1})^{o(g)} = \sigma^{o(g)}$. As no element of $\text{SL}(2, 5)$ has order divisible by 8, the power $\sigma^{o(g)}$ is not the identity. Since σg^{-1} fixes x , so does the power $(\sigma g^{-1})^{o(g)} = \sigma^{o(g)}$. This is not possible since the scalar $\sigma^{o(g)}$ is a fixed-point-free automorphism of R . Therefore, M is not transitive on Δ . Since M is a normal subgroup of H of index 2, M acts on Δ half-transitively and has exactly two equal size orbits. $\square$

This lemma enables us to have the following construction.

Construction 3.2 Using the notation above, we define a graph as below.

- (1) Let $\Delta_1, \dots, \Delta_r$ be the orbits of H acting on $R^\#$, and let Δ_i^+ and Δ_i^- be the two orbits of M on Δ_i , where $1 \leq i \leq r$.
- (2) Let $S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}$, where $\varepsilon_i = +$ or $-$, and let $\Gamma = \text{Cay}(R, S)$.

Lemma 3.3 *Let Γ be a Cayley graph constructed in Construction 3.2. Then Γ is a self-complementary metacirculant with insoluble automorphism group $\text{Aut}\Gamma \geq \mathbb{Z}_p^2 : (\langle \sigma^2 \rangle \circ \text{SL}(2, 5))$, and σ is a complementing isomorphism.*

Proof By definition, σ interchanges Δ_i^+ and Δ_i^- , where $1 \leq i \leq r$. Thus,

$$S^\sigma = \bigcup_{i=1}^r (\Delta_i^{\varepsilon_i})^\sigma = \bigcup_{i=1}^r \Delta_i^{-\varepsilon_i} = R^\# \setminus S.$$

Since σ is a scalar of order divisible by 8, the power $\sigma^{o(\sigma)/2}$ maps every element $x \in R$ into the inverse x^{-1} . Hence each Δ_i is self-inverse and so is S . So Γ is undirected and is self-complementary with σ being a complementing isomorphism.

Since R is metacyclic, the Cayley graph Γ of R is a metacirculant. Moreover, $\text{Aut}\Gamma \geq R:M = \mathbb{Z}_p^2 : \langle \sigma^2, \text{SL}(2, 5) \rangle = \mathbb{Z}_p^2 : (\langle \sigma^2 \rangle \circ \text{SL}(2, 5))$ is insoluble. $\square$

By iteratively taking the lexicographic products of graphs constructed via Construction 3.2, we can in fact construct self-complementary metacirculants with insoluble automorphism groups, which are Cayley graphs on $\mathbb{Z}_{n_1 n_2}^2$ for any coprime pair n_1, n_2 of square-free integers. Construction 6.1 and Lemma 6.2 of [8] show that the resulting graphs are in fact self-complementary metacirculants with insoluble automorphism groups in which the insoluble composition factor A_5 appears multiple times.

We next give a different construction of self-complementary metacirculants of which the automorphism groups contain a section $A_5 \times \cdots \times A_5$.

Let $p_1, \dots, p_t$ be distinct primes such that $p_i \equiv 1$ or $9 \pmod{40}$ for each $i \in \{1, \dots, t\}$. Let $R_i = \mathbb{Z}_{p_i}^2$, and let $L_i < \text{Aut}(R_i) = \text{GL}(2, p_i)$ be isomorphic to $\text{SL}(2, 5)$. Let $\sigma_i \in \mathbf{Z}(\text{Aut}(R_i))$ have order divisible by 8. Let $R = R_1 \times \cdots \times R_t$, $L = L_1 \times \cdots \times L_t$, and $\sigma = (\sigma_1, \dots, \sigma_t) \in \text{Aut}(R)$. Let $H = R : (\langle \sigma \rangle \circ L)$ and $M = R : (\langle \sigma^2 \rangle \circ L)$.

Lemma 3.4 *The subgroup M divides every H -orbit on $R^\#$ into two parts of equal size.*

Proof (The proof is similar to the proof of Lemma 3.1.) Let Δ be an H -orbit on $R^\#$, and let $(x_1, \dots, x_t) \in \Delta$. Suppose that there is an element $g \in M$ such that $(x_1, \dots, x_t)^g = (x_1, \dots, x_t)^\sigma$. Then $(x_1, \dots, x_t)^{\sigma g^{-1}} = (x_1, \dots, x_t)$. As $\sigma g^{-1} = g^{-1} \sigma$, we have $(\sigma g^{-1})^{o(g)} = \sigma^{o(g)} (g^{-1})^{o(g)} = \sigma^{o(g)}$. Noticing that no element of L has order divisible by 8, we know $\sigma^{o(g)} \neq 1$. Since σg^{-1} fixes $(x_1, \dots, x_t)$, so does the power $\sigma^{o(g)}$. Now some $x_i \neq 1$, and $x_i^{\sigma_i^{o(g)}} = x_i$, which contradicts that the scalar $\sigma_i^{o(g)}$ is a fixed-point-free automorphism of R_i . So M is intransitive on Δ and has exactly two orbits of equal size. $\square$

Thus, applying Construction 3.2 with H, M defined before Lemma 3.4 produces examples of self-complementary metacirculants of which the automorphism groups have a section A_5^t for any positive integer t .

Lemma 3.5 *Let $p_1, \dots, p_t$ be distinct primes such that $p_i \equiv 1$ or $9 \pmod{40}$ for every $i \in \{1, \dots, t\}$. Then there exist self-complementary metacirculants Γ of order*

$p_1^2 \dots p_t^2$ such that $\text{Aut} \Gamma \geq \mathbb{Z}_{p_1 \dots p_t}^2 : (\mathbb{Z}_\ell \circ \text{SL}(2, 5)^t)$, where $\ell \mid (p_i - 1)$ for each i and $8 \mid \ell$.

4 Proof of Theorem 1.1

Let $\Gamma = (V, E)$ be a self-complementary metacirculant of a metacyclic group R . By [14], we may assume that R is not cyclic. Let $G = \text{Aut} \Gamma$, and let σ be a complementing isomorphism of Γ . Let $X = \langle G, \sigma \rangle$. Then $\sigma^2 \in G$, and $X = G.\mathbb{Z}_2$. Let n be a positive integer, and let p be a prime factor of n . By n_p we denote the highest power of p that divides n , so that p and n/n_p are coprime.

4.1 The primitive case

We consider first the vertex-primitive case. Suppose that X is primitive on V . Then a result of [10] states that either X is affine or the socle $\text{soc}(X) = \text{PSL}(2, q^2)^\ell$ and $|V| = (\frac{1}{2}q^2(q^2 + 1))^\ell$, where q is odd and $\ell \geq 2$. We first exclude the latter case.

Lemma 4.1 *The group X is affine.*

Proof Suppose that X has socle $N = \text{PSL}(2, q^2)^\ell$, and $|V| = (\frac{1}{2}q^2(q^2 + 1))^\ell$. Since R is transitive on V , it follows that $q^{2\ell}$ divides $|R|$. Let $q = p^f$ with p prime and $f \geq 1$. Since R is metacyclic, R contains an element x of order $q^\ell = p^{f\ell}$. Let K be the largest subgroup of X which normalises each direct factor of N . Then $N \triangleleft K \leq \text{P}\Gamma\text{L}(2, q^2)^\ell$. Note that $\text{P}\Gamma\text{L}(2, q^2) = \text{PGL}(2, q^2) : \mathbb{Z}_{2f}$, and $\text{PGL}(2, q^2)$ has a Sylow p -subgroup $\mathbb{Z}_p^{2f}$. Hence a p -element of K has order at most pf_p . Thus,

$$X/K \geq \langle x \rangle K/K \cong \langle x \rangle / (\langle x \rangle \cap K),$$

and so $X/K \leq S_\ell$ contains an element of order $p^{f\ell-1}/f_p$. However, $p^{f\ell-1}/f_p$ does not divide $|S_\ell| = \ell!$, which is a contradiction. $\square$

We now show that only affine groups of dimension 2 can appear.

Lemma 4.2 *The primitive group X is affine of dimension 2.*

Proof By Lemma 4.1 the socle $N = \text{soc}(X) = \mathbb{Z}_p^d$, where p is a prime and $d \geq 1$. Then $X \leq \text{AGL}(d, p)$, and p^d divides $|R|$. Since R is metacyclic, there exists an element $g \in R$ of order divisible by $p^{\lceil \frac{d}{2} \rceil}$. Since $\text{AGL}(d, p) < \text{GL}(d+1, p)$ and they have isomorphic Sylow p -subgroups, by Lemma 2.5, the largest order p^e of the p -elements of $\text{AGL}(d, p)$ is such that $p^e \geq d+1 > p^{e-1}$. Thus, $p^{\lceil \frac{d}{2} \rceil - 1} \leq p^{e-1} < d+1$. Noticing that $p \geq 3$, this implies that $d \leq 4$.

Suppose that $d = 3$. Then $p^{\lceil \frac{3}{2} \rceil} = p^2 \leq p^e$, and $p \leq p^{e-1} < d+1 = 4$. Thus $p = 3$, and $|V| = 3^3 \not\equiv 1 \pmod{4}$, which is not possible by Lemma 2.2.

Suppose that $d = 4$. Then $\lceil \frac{4}{2} \rceil = 2$, and it follows that $p \leq p^{e-1} < d+1 = 5$. Thus $p = 3$, and $|V| = 3^4$. Without loss of generality, we may assume that R is a

3-group. Note that $\text{AGL}(4, 3) < \text{GL}(5, 3)$, by Lemma 2.5, the largest order of the 3-elements of $\text{GL}(5, 3)$ is 9. Thus, the metacyclic group $R = \mathbb{Z}_9.\mathbb{Z}_9$. However, checking the subgroups of $\text{GL}(5, 3)$ by GAP [27] leads to the conclusion that $\text{GL}(5, 3)$ does not have a metacyclic group $\mathbb{Z}_9.\mathbb{Z}_9$.

Finally, since we have assumed that R is not cyclic, we conclude that $d = 2$. $\square$

Next, we determine the insoluble case.

Lemma 4.3 *Assume that X is primitive and insoluble. Then we have*

$$X = \mathbb{Z}_p^2 : (\mathbb{Z}_\ell \circ \text{SL}(2, 5)), \text{ and } G = \mathbb{Z}_p^2 : (\mathbb{Z}_{\ell/2} \circ \text{SL}(2, 5)),$$

where $p \equiv 1$ or $9 \pmod{40}$, and ℓ divides $p - 1$ and is divisible by 8.

Proof Since X is primitive and insoluble, by Lemma 4.2, we have $|V| = p^2$, and $X \leq \text{AGL}(2, p)$, and the stabiliser X_v is an irreducible subgroup of $\text{GL}(2, p)$.

Notice that G has index 2 in X and is not 2-transitive on V , so G_v does not contain $\text{SL}(2, p)$. Inspecting the subgroups of $\text{SL}(2, p)$ listed in [26, Theorem 6.17], we conclude that X_v contains a subgroup isomorphic to $\text{SL}(2, 5)$ with $p = 5$ or $p \equiv \pm 1 \pmod{5}$. Noting that $\mathbb{Z}_{p-1} \circ \text{SL}(2, 5)$ is maximal in $\mathbb{Z}_{p-1} \circ \text{SL}(2, p)$ and there is no element of $\text{GL}(2, p) \setminus (\mathbb{Z}_{p-1} \circ \text{SL}(2, 5))$ which normalises $\text{SL}(2, 5)$, we have

$$X_v = \langle z \rangle \circ \text{SL}(2, 5) = \mathbb{Z}_\ell \circ \text{SL}(2, 5),$$

where z is a scalar of order ℓ dividing $p - 1$. Since G is transitive on V , the index $|X_v : G_v| = |X : G| = 2$. We may assume $\sigma \in X_v$. As $\text{SL}(2, 5)$ has no subgroup of index 2, $\sigma \notin \text{SL}(2, 5)$. Since σ normalises G_v , σ normalises $\text{SL}(2, 5)$, and we may choose $\sigma \in \langle z \rangle$. Then $\sigma \notin \langle z^2 \rangle$, and $G_v = \langle z^2 \rangle \circ \text{SL}(2, 5)$. By Lemma 2.1, the order $o(\sigma)$ is divisible by 4.

In particular, the minimal normal subgroup $\mathbb{Z}_p^2$ of X is regular on V and is a Sylow p -subgroup. So Γ is a metacirculant and a Cayley graph of R , say

$$\Gamma = \text{Cay}(R, S).$$

Suppose that ℓ is not divisible by 8. Then since $\sigma \in \mathbb{Z}_\ell$, σ is of order $4m$ with m odd, and

$$X_v = \langle \sigma^4 \rangle \times (\mathbb{Z}_4 \circ \text{SL}(2, 5)) = \mathbb{Z}_m \times (\mathbb{Z}_4 \circ \text{SL}(2, 5)),$$

and $G_v = \mathbb{Z}_m \times \text{SL}(2, 5)$. Let $\tau = \sigma^m \in \langle z \rangle$. Then τ is of order 4, and $X_v = \langle \tau \rangle \circ G_v$. For every element $x \in R$, we have $x^\tau = x^j$ for some integer j which has order 4 in the multiplicative group $\text{GF}(p)^*$ of $\text{GF}(p)$.

Let $y \in G_v$ be of order 4. Since 4 divides $p - 1$, the group $\langle y \rangle$ is reducible on $R = \text{soc}(X)$. Thus, y normalises a cyclic subgroup $\langle w \rangle \cong \mathbb{Z}_p$, where $w \in R$, and y induces an automorphism of $\langle w \rangle$ of order 4. So $w^y = w^k$, for some integer k which has order 4 in $\text{GF}(p)^*$. Since $\text{GF}(p)^*$ has a unique subgroup of order 4, $k = j$ or j^{-1} in $\text{GF}(p)^*$. Thus, $w^\tau = w^y$ or $w^{y^{-1}}$, respectively. Without loss of generality, assume

that $w \in S$. Since $y \in G$, both w^y and $w^{y^{-1}}$ belong to S . However, $w^\tau \in S^\tau = R^\# \setminus S$, which is a contradiction. Hence 8 divides $o(\sigma)$, and $p \equiv 1 \pmod{8}$.

Finally, since $p \equiv \pm 1 \pmod{5}$, it follows that $p \equiv 1$ or $9 \pmod{40}$. The proof is completed. $\square$

4.2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we assume that X is insoluble. We complete our proof by induction on the order $|V|$.

If X is primitive on V , then by Lemmas 4.2 and 4.3, the statement of Theorem 1.1 is true. We thus assume that X is imprimitive on V , and further assume that Theorem 1.1 holds for any self-complementary metacirculant of order properly dividing $|V|$.

Let $\mathcal{B}$ be an X -invariant partition of V , called a *block system* of X on V . Each element $B \in \mathcal{B}$ is called a *block*. By $[B]_\Gamma$ we denote the induced subgraph of Γ on B . We need the following result (see [13]).

Theorem 4.4 (Li-Praeger 2003) *Suppose that V has a block system $\mathcal{B}$ under the action of X . Then the following hold:*

- (i) *for each $B \in \mathcal{B}$, the induced subgraph $[B]_\Gamma$ is self-complementary, $G_B^B \leq \text{Aut}[B]_\Gamma$, and σ induces a complementing isomorphism of $[B]_\Gamma$;*
- (ii) *there is a self-complementary graph Σ with vertex set $\mathcal{B}$ such that $G^\mathcal{B} \leq \text{Aut}\Sigma$ and any element of $X^\mathcal{B} \setminus G^\mathcal{B}$ is a complementing isomorphism of Σ .*

Let $K = X_{(\mathcal{B})}$ be the kernel of X acting on $\mathcal{B}$. Then, as $|\mathcal{B}|$ properly divides $|V|$, by our assumption, Theorem 1.1 holds for $X^\mathcal{B}$, that is, if $X^\mathcal{B}$ is insoluble, then A_5 is the only insoluble composition factor of $X^\mathcal{B}$, and $X^\mathcal{B}$ has a section which has the form

$$\mathbb{Z}_q^2 : (\mathbb{Z}_m \circ \text{SL}(2, 5)),$$

where q is a prime, $8 \mid m$ and $m \mid (q-1)$ and $\mathbb{Z}_q^2$ is a section of $R^\mathcal{B}$.

If K is soluble, then $X = K.X^\mathcal{B}$ satisfies Theorem 1.1. We thus assume that K is insoluble. Then $K^\mathcal{B}$ is insoluble and so is $X_B^\mathcal{B}$.

To complete the proof, we may further assume that $\mathcal{B}$ is a minimal block system of X , that is, for a block $B \in \mathcal{B}$, the induced action $X_B^\mathcal{B}$ is primitive. By Lemmas 4.2 and 4.3, we have that $X_B^\mathcal{B}$ is affine of degree p or p^2 with p prime, and as $X_B^\mathcal{B}$ is insoluble,

$$X_B^\mathcal{B} = \mathbb{Z}_p^2 : (\mathbb{Z}_\ell \circ \text{SL}(2, 5)),$$

and $R_B^\mathcal{B} = \mathbb{Z}_p^2$, where $p \equiv 1$ or $9 \pmod{40}$ is a prime, as in part (ii) of Theorem 1.1.

Since $K \triangleleft X_B$ is insoluble, $K^\mathcal{B} = \mathbb{Z}_p^2 : (\mathbb{Z}_{\ell'} \circ \text{SL}(2, 5))$, where $\ell' \mid \ell$. Furthermore, since $K \leq K^{B_1} \times \cdots \times K^{B_t}$, where $\mathcal{B} = \{B_1, \dots, B_t\}$, we conclude that A_5 is the only insoluble composition factor of K . Finally, since $X = K.X^\mathcal{B}$ is an extension of K by $X^\mathcal{B}$, every composition factor of X is a composition factor of K or $X^\mathcal{B}$. Therefore, A_5 is the only insoluble composition factor of X . This completes the proof of our theorem. $\square$

Acknowledgments This work was partially supported by an ARC Discovery Project grant.

References

1. Alspach, B., Morris, J., Vilfred, V.: Self-complementary circulant graphs. *Ars Combin.* **53**, 187–191 (1999)
2. Alspach, B., Parsons, T.D.: A construction for vertex-transitive graphs. *Canad. J. Math.* **34**, 307–318 (1982)
3. Beezer, R.A.: Sylow subgraphs in self-complementary vertex transitive graphs. *Expo. Math.* **24**(2), 185–194 (2006)
4. Chvátal, V., Erdős, P., Hedrlín, Z.: Ramsey's theorem and self-complementary graphs. *Disc. Math.* **3**, 301–304 (1972)
5. Clapham, C.R.J.: A class of self-complementary graphs and lower bounds of some Ramsey numbers. *J. Graph Theory* **3**(3), 287–289 (1979)
6. Figueroa, R., Giudici, R.E.: Group generation of self-complementary graphs. In: *Combinatorics and Graph Theory* (Hefei, 1992), pp. 131–140. World Science Publisher, River Edge, NJ (1993)
7. Fronček, D., Rosa, A., Širáň, J.: The existence of self-complementary circulant graphs. *Eur. J. Combin.* **17**(7), 625–628 (1996)
8. Giudici, M., Li, C.H., Potoncik, P., Praeger, C.E.: Homogeneous factorisations of graph products. *Disc. Math.* **308**(16), 3652–3667 (2008)
9. Gulden, F., Tomasta, P.: New lower bounds of some diagonal Ramsey numbers. *J. Graph Theory* **7**(1), 149–151 (1983)
10. Guralnick, R., Li, C.H., Praeger, C.E., Saxl, J.: On orbital partitions and exceptionality of primitive permutation groups. *Trans. Am. Math. Soc.* **356**(12), 4857–4872 (2004)
11. Jajcay, R., Li, C.H.: Constructions of self-complementary circulants with no multiplicative isomorphisms. *Eur. J. Combin.* **22**(8), 1093–1100 (2001)
12. Li, C.H.: On finite graphs that are self-complementary and vertex-transitive. *Australas. J. Combin.* **18**, 147–155 (1998)
13. Li, C.H., Praeger, C.E.: On partitioning the orbitals of a transitive permutation group. *Trans. Am. Math. Soc.* **355**(2), 637–653 (2003)
14. Li, C.H., Praeger, C.E.: On finite permutation groups with a transitive cyclic subgroup. *J. Algebra* **349**, 117–127 (2012)
15. Li, C. H., and Rao, G.: Self-complementary vertex-transitive graphs of order a product of two primes. *Bull. Austral. Math. Soc.* **89**(2), 322–330 (2014)
16. Li, C.H., Song, S.J., Wang, D.J.: A characterization of metacirculants. *J. Combin. Theory Ser. A* **120**(1), 39–48 (2013)
17. Li, C.H., Sun, S.H., Xu, Jing: Self-complementary circulants of prime-power order. *Siam J. Disc. Math.* **28**(1), 8–17 (2014)
18. Liskovets, V., Poschel, R.: Non-Cayley-isomorphic self-complementary circulant graphs. *J. Graph Theory* **34**(2), 128–141 (2000)
19. Matheron, R.: On self-complementary strongly regular graphs. *Disc. Math.* **69**(3), 263–281 (1988)
20. Muzychuk, M.: On Sylow subgraphs of vertex-transitive self-complementary graphs. *Bull. Lond. Math. Soc.* **31**(5), 531–533 (1999)
21. Praeger, C.E., Li, C.H., Stringer, L.: Common circulant homogeneous factorisations of complete graphs. *Disc. Math.* **309**(10), 3006–3012 (2009)
22. Rao, S.B.: On regular and strongly regular self-complementary graphs. *Disc. Math.* **54**(1), 73–82 (1985)
23. Rödl, V., Šinajová, E.: Note on Ramsey numbers and self-complementary graphs. *Math. Slovaca* **45**(3), 243–249 (1995)
24. Sachs, H.: Über selbstcomplementäre graphen. *Publ. Math. Debrecen* **9**, 270–288 (1962)
25. Suprunenko, D.A.: Self-complementary graphs. *Cybernetics* **21**(5), 559–567 (1985)
26. Suzuki, M.: *Group Theory I*. Springer-Verlag, New York (1982)
27. The GAP Group, GAP - Groups, Algorithms, and Programming, Version 4.3 (The GAP Group, Aachen/St Andrews, 2002); <http://www-gap.dcs.st-and.ac.uk>
28. Zelinka, B.: Self-complementary vertex-transitive undirected graphs. *Math. Slovaca* **29**(1), 91–95 (1979)