Scattered linear sets generated by collineations between pencils of lines

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Received: 10 June 2013 / Accepted: 10 March 2014 / Published online: 25 March 2014
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Abstract Let A and B be two points of $\text{PG}(2, q^n)$, and let Φ be a collineation between the pencils of lines with vertices A and B . In this paper, we prove that the set of points of intersection of corresponding lines under Φ is either the union of a scattered $\text{GF}(q)$ -linear set of rank $n + 1$ with the line AB or the union of $q - 1$ scattered $\text{GF}(q)$ -linear sets of rank n with A and B . We also determine the intersection configurations of two scattered $\text{GF}(q)$ -linear sets of rank $n + 1$ of $\text{PG}(2, q^n)$ both meeting the line AB in a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B .

Keywords Linear sets · Collineations · Subgeometries

Mathematics Subject Classification 51E21 · 05B25

1 Introduction

In [3] and [4], a (degenerate or not) C_F -set in $\text{PG}(2, q^2)$ is defined as the set of points of intersection of corresponding lines under a collineation with accompanying automorphism $x \mapsto x^q$ between two pencils of lines with vertices two distinct points A and B mapping the line AB onto itself or not. Every C_F -set has $q^2 + 1$ points, and it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\text{PG}(2, q^2)$, and every $(q + 1)$ -secant line intersects a C_F -set in a Baer subline (see [3]). Moreover, every C_F -set is the union of A , and B with $q - 1$ Baer sublines and it is projectively equivalent to the set of $\text{GF}(q^2)$ -rational points of the algebraic curve with equation

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$$x_1x_2^q - x_3^{q+1} = 0.$$

A C_F -set is also one of the possible intersection configurations of two Hermitian curves of $\text{PG}(2, q^2)$ (see e.g. [6]).

In [4], it is proved that every degenerate C_F -set is the union of the line AB with a Baer subplane meeting the line AB in a Baer subline and vice versa every Baer subplane can be seen as a subset of a degenerate C_F -set. Moreover, every degenerate C_F -set is projectively equivalent to the set of $\text{GF}(q^2)$ -rational points of the algebraic curve with equation

$$x_3 \left(x_1x_3^{q-1} - x_2^q \right) = 0.$$

In this paper, we extend the definition of a (degenerate or not) C_F -set to that of a (degenerate or not) C_F^m -set in any projective plane $\text{PG}(2, q^n)$, with $n \geq 2$, by collineations with accompanying automorphism $x \mapsto x^{q^m}$ between two pencils of lines with vertices A and B . We will prove that every C_F^m -set of $\text{PG}(2, q^n)$ is the union of A and B with $q-1$ scattered $\text{GF}(q)$ -linear sets of rank n each of which is a projective geometry $\text{PG}(n-1, q)$ embedded in $\text{PG}(2, q^n)$, and that every degenerate C_F^m -set is the union of the line AB with a scattered $\text{GF}(q)$ -linear set of rank $n+1$ meeting the line AB in a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B and vice versa. Furthermore, a degenerate C_F^m -set is also the union of the line AB with an affine geometry $\text{AG}(n, q)$ embedded in $\text{PG}(2, q^n)$. We also calculate the full collineation group stabilizing a (degenerate or not) C_F^m -set of $\text{PG}(2, q^n)$.

In the last section, we determine the intersection configurations of two scattered $\text{GF}(q)$ -linear sets of rank $n+1$ of $\text{PG}(2, q^n)$ both meeting the line AB in a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B .

2 Preliminary results

Let $\Omega = \text{PG}(r-1, q^n) = \text{PG}(V, \text{GF}(q^n))$, $q = p^h$, p a prime. A set L is said a $\text{GF}(q)$ -linear set of Ω of rank t if it is defined by the non-zero vectors of a $\text{GF}(q)$ -vector subspace U of V of dimension t , that is

$$L = L_U = \{ \langle \underline{u} \rangle_{\text{GF}(q^n)} : \underline{u} \in U \setminus \{0\} \}.$$

Let $\Lambda = \text{PG}(W, \text{GF}(q^n))$ be a subspace of Ω of dimension s , we say that Λ has weight i with respect to L_U if $\dim_{\text{GF}(q)}(W \cap U) = i$. A $\text{GF}(q)$ -linear set L_U of Ω of rank t is said *scattered* if each point of L_U has weight 1 with respect to L_U . In [15], it is proved that L_U is a scattered $\text{GF}(q)$ -linear set of rank t if and only if $|L_U| = q^{t-1} + q^{t-2} + \dots + q + 1$, and in [1] it is shown that $t \leq rn/2$. If L is a scattered linear set of $\text{PG}(r-1, q^n)$ of rank $rn/2$, then it is called a *maximum* scattered linear set.

If $\dim_{\text{GF}(q)} U = \dim_{\text{GF}(q^n)} V = r$ and $\langle U \rangle_{\text{GF}(q^n)} = V$, then $L_U \cong \text{PG}(U, \text{GF}(q))$ is a *subgeometry* of Ω . In such a case each point has weight 1, and hence $|L_U| = q^{r-1} + q^{r-2} + \dots + q + 1$. Let $\Sigma = \text{PG}(t, q)$ be a subgeometry of $\Sigma^* = \text{PG}(t, q^n)$

and suppose that there exists a $(t - r)$ -dimensional subspace Ω^* of Σ^* disjoint from Σ . Let $\Omega = \text{PG}(r - 1, q^n)$ be an $(r - 1)$ -dimensional subspace of Σ^* disjoint from Ω^* , and let Γ be the projection of Σ from Ω^* to Ω i.e. $\Gamma = \{\langle \Omega^*, x \rangle \cap \Omega : x \in \Sigma\}$. Let $p_{\Omega^*, \Omega}$ be the map from $\Sigma^* \setminus \Omega^*$ to Ω defined by $x \mapsto \langle \Omega^*, x \rangle \cap \Omega$. We call Ω^* the *center* and Ω the *axis* of $p_{\Omega^*, \Omega}$. In [12], the following characterization of $\text{GF}(q)$ -linear sets is given:

Theorem 2.1 *If L is a projection of $\text{PG}(t, q)$ into $\Omega = \text{PG}(r - 1, q^n)$, then L is a $\text{GF}(q)$ -linear set of Ω of rank $t + 1$ and $\langle L \rangle = \Omega$. Conversely, if L is a $\text{GF}(q)$ -linear set of Ω of rank $t + 1$ and $\langle L \rangle = \Omega$, then either L is a subgeometry of Ω or there are a $(t - r)$ -dimensional subspace Ω^* of $\Sigma^* = \text{PG}(t, q^n)$ disjoint from Ω and a subgeometry Σ of Σ^* disjoint from Ω^* such that $L = p_{\Omega^*, \Omega}(\Sigma)$.*

In [13], the authors define a class of maximum scattered $\text{GF}(q)$ -linear sets of the projective line $\text{PG}(1, q^n)$. Let $P_1 = (y_1, y_2)$ and $P_2 = (z_1, z_2)$ be two distinct points of $\text{PG}(1, q^n)$, and let τ be an automorphism of $\text{GF}(q^n)$ such that $\text{Fix}(\tau) = \{x \in \text{GF}(q^n) : x^\tau = x\} = \text{GF}(q)$. For each $\rho \in \text{GF}(q^n)^* = \text{GF}(q^n) \setminus \{0\}$, the set

$$W_{\rho, \tau} = \{(\lambda y_1 + \rho \lambda^\tau z_1, \lambda y_2 + \rho \lambda^\tau z_2) : \lambda \in \text{GF}(q^n)\}$$

is a $\text{GF}(q)$ -vector subspace of $\text{GF}(q^n)^2$ of dimension n , and $L_{\rho, \tau} = L_{W_{\rho, \tau}}$ is a scattered $\text{GF}(q)$ -linear set of rank n called of *pseudoregulus type*. The points P_1 and P_2 are the *transversal points* of $L_{\rho, \tau}$.

For an element x of $\text{GF}(q^n)$, we denote by $N(x) = x^{\frac{q^n-1}{q-1}}$ the *norm* of x over $\text{GF}(q)$.

Remark 2.2 Let $\mathcal{T}$ be a scattered $\text{GF}(q)$ -linear set of pseudoregulus type of $\text{PG}(1, q^n)$ with transversal points $A' = (1, 0)$ and $B' = (0, 1)$. From [13], we have that $\mathcal{T} = \{(x, \rho x^{q^m}) : x \in \text{GF}(q^n)^*\}$ for some $\rho \in \text{GF}(q^n)^*$ and some m with $(m, n) = 1$. Hence, $\mathcal{T} = \{(1, b) : N(b) = N(\rho)\}$. If f is a projectivity of $\text{PG}(1, q^n)$ fixing both A' and B' and stabilizing the set $\mathcal{T}$, then $(x_1, x_2)^f = (a^{q^m-1}x_1, x_2)$ for some $a \in \text{GF}(q^n)^*$. It follows that the group $G = \{f : (x_1, x_2) \mapsto (cx_1, x_2), N(c) = 1\}$ has order $\frac{q^n-1}{q-1}$ and stabilizes the set $\mathcal{T}$. The orbit of a point $P = (1, x_2)$ of $\text{PG}(1, q^n) \setminus \{A', B'\}$, with $N(x_2) = N(\rho)$, under the action of G , is the set $\mathcal{T}$. The orbits of G on points of $\text{PG}(1, q^n) \setminus \{A', B'\}$ are the sets $\mathcal{O}_a = \{(1, b) : N(b) = a\}$ for $a \in \text{GF}(q)^*$. So, they are all the possible scattered $\text{GF}(q)$ -linear set of pseudoregulus type of $\text{PG}(1, q^n)$ with transversal points A' and B' . They are a partition of $\text{PG}(1, q^n) \setminus \{A', B'\}$. The projectivity

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with $N(t) = a$, maps the set $\mathcal{O}_1$ onto the set $\mathcal{O}_a$. Therefore, all the possible $\text{GF}(q)$ -linear sets of pseudoregulus type of $\text{PG}(1, q^n)$ with transversal points A' and B' are projectively equivalent, independently of m and ρ . Since the group $\text{PGL}(2, q^n)$ is 2-transitive on the points of $\text{PG}(1, q^n)$, it follows that all $\text{GF}(q)$ -linear sets of pseudoregulus type of $\text{PG}(1, q^n)$ are projectively equivalent.

A *blocking set* $\mathcal{B}$ in a projective plane $\text{PG}(2, q^n)$ is a set of points meeting every line of $\text{PG}(2, q^n)$; it is called *trivial* if it contains a line, and it is called *minimal* if no proper subset of it is a blocking set. A blocking set $\mathcal{B}$ is *small* if it has size less than $\frac{3(q^n+1)}{2}$ and it is of *Rédei type* if there exists a line ℓ such that $|\mathcal{B} \setminus \ell| = q^n$, the line ℓ is called a *Rédei line* of $\mathcal{B}$. If $\mathcal{B}$ is a $\text{GF}(q)$ -linear set of rank $n+1$, then from [16], it follows that $\mathcal{B}$ is a small minimal blocking set called a *$\text{GF}(q)$ -linear blocking set*. Such blocking sets were introduced by G. Lunardon (see [10], [11]).

A *linear space* is an incidence structure $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a nonempty set whose elements are called *points*, $\mathcal{L}$ is a set, with at least two elements, whose elements are called *lines*, satisfying the following properties:

- any two distinct points are incident with a unique line;
- every line is incident with at least two distinct points.

Lenz in [9] proved the following characterization theorem for affine spaces in terms of linear spaces:

Theorem 2.3 *Let $(\mathcal{P}, \mathcal{L})$ be a linear space with an equivalence relation (“parallelism”) among its lines. If the following properties are satisfied:*

- Through any point not on a line ℓ there is a unique line parallel to ℓ , and*
- Let ℓ and ℓ' be two distinct parallel lines, and let P, Q be two distinct points on ℓ and P' be a point of ℓ' . If R is a point on $PP' \setminus \{P\}$, then the lines RQ and ℓ' intersect,*

then $(\mathcal{P}, \mathcal{L})$ is an affine space.

3 C_F^m -sets in $\text{PG}(2, q^n)$

Let A and B be two distinct points of a projective plane $\text{PG}(2, q^n)$ over the Galois field $\text{GF}(q^n)$, $q = p^h$, p a prime number, and let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the pencils of lines with vertices A and B , respectively. Let α_F be the Frobenius automorphism of $\text{GF}(q^n)$ given by $\alpha_F : x \in \text{GF}(q^n) \rightarrow x^{q^m} \in \text{GF}(q^n)$, and let Φ be an α_F -collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$ which does not map the line AB onto itself.

A C_F^m -set of $\text{PG}(2, q^n)$ is the set of points of intersection of corresponding lines under Φ . The lines $(AB)^\Phi$ and $(AB)^{\Phi^{-1}}$ meet at a point C which do not lie on the line AB . We may, therefore, take A, B, C as triangle of reference, i.e., $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$.

The collineation Φ maps the line $ax_2 + bx_3 = 0$ through A onto the line $(\rho a^{q^m} + \lambda b^{q^m})x_1 + (\mu a^{q^m} + \theta b^{q^m})x_3 = 0$ through B , where $\begin{pmatrix} \rho & \lambda \\ \mu & \theta \end{pmatrix}$ is a non-singular matrix over $\text{GF}(q^n)$. Since Φ maps the line $x_2 = 0$ onto the line $x_3 = 0$ and the line $x_3 = 0$ onto the line $x_1 = 0$, ρ and θ are both zero and the set of points of intersection of corresponding lines under Φ has equation $\lambda x_1 x_2^{q^m} - \mu x_3^{q^m+1} = 0$, with $\lambda\mu \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda\mu^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

a C_F^m -set is projectively equivalent to the set of $\text{GF}(q^n)$ -rational points of the algebraic curve with equation $x_1x_2^{q^m} - x_3^{q^m+1} = 0$.

Every line through A (respectively through B) is a 2-secant line except the line AC (respectively the line BC) which is a tangent line, so every C_F^m -set has $q^n + 1$ points. The points A and B are called the *vertices*, and the point C , that is the point of intersection of the tangent lines at A and at B , is called the *center* of the C_F^m -set.

Without loss of generality, we may assume $(m, n) = 1$ and $m \leq n/2$. Indeed if $m > n/2$, then from $x_1x_2^{q^m} - x_3^{q^m+1} = 0$, it follows that $x_1^{q^{n-m}}x_2 - x_3^{q^{n-m}+1} = 0$. If $m = n/2$, then from $(m, n) = 1$ it follows $m = 1$ and $n = 2$. Since a C_F^1 set of $\text{PG}(2, q^2)$ has been already studied in [3] (where it was called a C_F -set) in the sequel, we may suppose that $m < n/2$.

Note that for $n = m = 1$, a C_F^1 -set of $\text{PG}(2, q)$ is a non degenerate conic of $\text{PG}(2, q)$ as consequence of Steiner's projective generation of conics (see [17]).

Proposition 3.1 *Every line of $\text{PG}(2, q^n)$ intersects a C_F^m -set in 0, 1, 2, or $q + 1$ points. In the last case, these points form a subline over $\text{GF}(q)$.*

Proof Let $\mathcal{C}$ be a C_F^m -set, and let ℓ be a line of $\text{PG}(2, q^n)$. If ℓ contains either A or B , then it is either a tangent line or a 2-secant line, so we may suppose that ℓ is neither through A nor through B . In that case, Φ induces an α_F -collineation ϕ_ℓ of the line ℓ into itself defined by

$$\phi_\ell : P \in \ell \longrightarrow (AP)^\Phi \cap \ell \in \ell.$$

The points of $\ell \cap \mathcal{C}$ are exactly the fixed points of ϕ_ℓ . The system of fixed points of ϕ_ℓ may be one of the following: the empty set, one point, two points, or a subline formed by all the points of ℓ coordinatized over the subfield $\text{Fix}(\alpha_F)$ (see e.g. [2]). This subfield is $\text{GF}(q)$ since $(m, n) = 1$. $\square$

Lemma 3.2 *Let $\ell = \text{PG}(1, F)$ be a projective line over a field F , and let σ be an automorphism of F . For every subline ℓ' of the line ℓ , coordinatized over the subfield $\text{Fix}(\sigma)$, there exists only one σ -collineation of the line ℓ into itself fixing pointwise the subline ℓ' .*

Proof Let f and g be two σ -collineations of the line ℓ fixing pointwise the subline ℓ' . Then, $f^{-1} \circ g$ is a projectivity of the line ℓ fixing pointwise ℓ' . Since $|\ell'| \geq 3$, from the fundamental theorem of projective geometry, we have that $f^{-1} \circ g$ is the identity, so $f = g$. Now, take three distinct points A' , B' , and C' on ℓ' . We may suppose that $A' = (1, 0)$, $B' = (0, 1)$, and $C' = (1, 1)$; hence, the σ -collineation induced by the σ -semilinear map

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^\sigma \\ x_2^\sigma \end{pmatrix}$$

fixes pointwise ℓ' . $\square$

Proposition 3.3 *Let ℓ_0 be a $\text{GF}(q)$ -subline of a line ℓ of $\text{PG}(2, q^n)$, and let A, B be two points not on ℓ such that the point $AB \cap \ell$ is not on ℓ_0 . Then, there exists a unique C_F^m -set with vertices A and B that meets ℓ in the subline ℓ_0 .*

Proof Observe that there is a bijective map Ψ between the set of the α_F -collineations of the line ℓ into itself and the set of the α_F -collineations between the pencils of lines $\mathcal{P}_A$ and $\mathcal{P}_B$ with vertices A and B . To every α_F -collineation f , Ψ corresponds the α_F -collineation Ψ_f that maps every line $r \in \mathcal{P}_A$ onto the line joining the points B and $(r \cap \ell)^f$. By the previous lemma, there exists only one α_F -collineation f_ℓ of the line ℓ into itself fixing pointwise the subline ℓ_0 . Since f_ℓ does not fix $AB \cap \ell$, Ψ_{f_ℓ} does not map the line AB onto itself. Moreover, Ψ_{f_ℓ} is the unique α_F -collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$ such that every point on ℓ_0 is a point of intersection of corresponding lines under Ψ_{f_ℓ} . Hence, the C_F^m -set defined by Ψ_{f_ℓ} is the unique C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B that meets ℓ exactly in the points of the subline ℓ_0 . $\square$

Proposition 3.4 *Every C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B is the union of $\{A, B\}$ with $q - 1$ pairwise disjoint subsets, each of which is a scattered $\text{GF}(q)$ -linear set of rank n .*

Proof Let $\mathcal{C}$ be the C_F^m -set of $\text{PG}(2, q^n)$ with equation $x_1 x_2^{q^m} = x_3^{q^m+1}$. It is $\mathcal{C} = \{(1, x^{q^m+1}, x^{q^m}) : x \in \text{GF}(q^n)\} \cup \{B\}$, and hence $\mathcal{C} \setminus \{A, B\}$ is the union of the sets $\mathcal{C}_a = \{(1, x^{q^m+1}, x^{q^m}) : x \in \text{GF}(q^n), N(x) = a\}$ for $a \in \text{GF}(q)^*$. Let a be an element of $\text{GF}(q)^*$. The projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{q^m+1} & 0 \\ 0 & 0 & t^{q^m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with $N(t) = a$, maps the set $\mathcal{C}_1$ onto the set $\mathcal{C}_a$. So, the sets $\mathcal{C}_a$ for $a \in \text{GF}(q)^*$ are pairwise projectively equivalent. Let N_1 be the set of all elements of $\text{GF}(q^n)$ with unitary norm. Consider the set $S_m = \{z^{q^{n-m}(q^n-m-1)} : z \in \text{GF}(q^n)^*\}$. Observe that S_m is a multiplicative subgroup of $\text{GF}(q^n)^*$, and since $(m, n) = 1$ it follows that $(q^{n-m} - 1, q^n - 1) = q - 1$. Therefore, $|S_m| = |N_1|$ and so $S_m = N_1$. Let $(1, x^{q^m+1}, x^{q^m})$ be an element of $\mathcal{C}_1$, there is an element $z \in \text{GF}(q^n)^*$ such that $x = z^{q^{n-m}(q^n-m-1)}$. Thus, $\mathcal{C}_1 = \{(1, z^{q^{n-m}(q^n-m-1)(q^m+1)}, z^{q^{n-m}(q^n-m-1)q^m}) : z \in \text{GF}(q^n)^*\}$. As $z^{q^{n-m}q^m} = z^{q^n} = z$, we have that $\mathcal{C}_1 = \{(z, z^{q^{2(n-m)}}, z^{q^{n-m}}) : z \in \text{GF}(q^n)^*\}$. Since $m < n/2$, it follows that $\mathcal{C}_1 = \{(z, z^{q^{n-2m}}, z^{q^{n-m}}) : z \in \text{GF}(q^n)^*\}$, so $\mathcal{C}_1$ is a $\text{GF}(q)$ -linear set of size $\frac{q^n-1}{q-1}$; hence, it is a rank n scattered $\text{GF}(q)$ -linear set. $\square$

Let $\mathcal{C}$ be a C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B and center C , generated by an α_F -collineation Φ between $\mathcal{P}_A$ and $\mathcal{P}_B$. Let $\mathcal{T}$ be a scattered linear set of pseudoregulus type of the line BC with transversal points B and C . Let $\mathcal{T}'$ be the set of lines joining the point A with a point P running in $\mathcal{T}$.

A component of $\mathcal{C}$ is the set of points of intersection of the lines of $\mathcal{T}'$ with corresponding lines under Φ . It is easy to see that for a C_F^m -set $\mathcal{C}$ of $\text{PG}(2, q^n)$ with equation

$x_1x_2^{q^m} - x_3^{q^{m+1}} = 0$, the $q - 1$ components of $\mathcal{C}$ are the sets $\mathcal{C}_a$ defined in the proof of the previous proposition. Since $\mathcal{C}$ is the union its $q - 1$ components with $\{A, B\}$, it follows easily that every $(q + 1)$ -secant line to $\mathcal{C}$ is a $(q + 1)$ -secant line to one of its components.

Remark 3.5 Consider the incidence structure $(\mathcal{C}_1, \mathcal{L})$, where $\mathcal{L}$ is the set of $(q + 1)$ -secant lines to $\mathcal{C}_1$. Embed $\Omega = \text{PG}(2, q^n)$ in $\Sigma^* = \text{PG}(n - 1, q^n)$ and let Ω^* be an $(n - 4)$ -dimensional subspace of Σ^* disjoint from Ω . Consider a subgeometry $\Sigma = \text{PG}(n - 1, q)$ of Σ^* disjoint from Ω^* . From Theorem 2.1, it follows that $\mathcal{C}_1$ is obtained via the projection $p_{\Omega^*, \Omega}$ from Ω^* onto Ω of the set Σ . As $\mathcal{C}_1$ is a $\text{GF}(q)$ -linear set contained in a set of type $(0, 1, 2, q + 1)$, we have that $\mathcal{C}_1$ is of type $(0, 1, q + 1)$. Via the projection $p_{\Omega^*, \Omega}$, it follows that Veblen-Young's axiom holds in $(\mathcal{C}_1, \mathcal{L})$ since it holds in Σ . Hence, $(\mathcal{C}_1, \mathcal{L})$ is a projective geometry $\text{PG}(n - 1, q)$ such that every three non-collinear points of $\mathcal{C}_1$ are also non-collinear points of $\text{PG}(2, q^n)$.

Proposition 3.6 *Let $\mathcal{C}$ be a C_F^m -set, and let $\mathcal{C}'$ be a $C_F^{m'}$ -set of $\text{PG}(2, q^n)$. The sets $\mathcal{C}$ and $\mathcal{C}'$ are $\text{P}\Gamma\text{L}$ -equivalent if and only if $m' = m$.*

Proof We can assume that $\mathcal{C} = \{(1, x^{q^m+1}, x^{q^m}) : x \in \text{GF}(q^n)\} \cup \{B\}$ and $\mathcal{C}' = \{(1, x^{q^{m'}+1}, x^{q^{m'}}) : x \in \text{GF}(q^n)\} \cup \{B\}$ and that $m \leq m'$. Let f be an element of $\text{P}\Gamma\text{L}(3, q^n)$ with accompanying automorphism $x \mapsto x^{p^i}$ that maps $\mathcal{C}$ to $\mathcal{C}'$. The map f fixes the set $\{A, B\}$, since A and B are the unique points of both $\mathcal{C}$ and $\mathcal{C}'$ through which all lines but one are 2-secant lines. Indeed, every other point P of $\mathcal{C}$ is contained in a component of $\mathcal{C}$, which is a projective geometry $\text{PG}(n - 1, q)$. Moreover, f fixes also the set $\{AC, BC\}$ as these are the unique tangents to both $\mathcal{C}$ and $\mathcal{C}'$ through A and B , so f fixes C .

First, suppose that f fixes both A and B . In this case, there exist two elements $a, b \in \text{GF}(q^n)^*$ such that $(x_1, x_2, x_3)^f = (ax_1^{p^i}, bx_2^{p^i}, x_3^{p^i})$. Hence, $(1, x^{q^m+1}, x^{q^m})^f = (a, bx^{(p^{hm}+1)p^i}, x^{p^{hm+i}})$ belongs to $\mathcal{C}'$ for every $x \in \text{GF}(q^n)$. It follows that $abq^{m'}x^{(p^{hm}+1)p^{hm'+i}} = x^{p^{hm+i}(p^{hm'}+1)}$; therefore, $(x^{p^{hm}-p^{hm'}})^{p^i} = abq^{m'}$ that is $(x^{1-p^{h(m'-m)}})^{p^{hm+i}} = abp^{hm'}$ for every x in $\text{GF}(q^n)^*$. If α denotes the inverse of the automorphism $x \mapsto x^{p^{hm+i}}$, then $x^{1-p^{h(m'-m)}} = (abp^{hm'})^\alpha$.

Hence, the polynomial $(abp^{hm'})^\alpha y^{q^{m'-m}-1} - 1 \in \text{GF}(q^n)[y]$ is a scalar multiple of the polynomial $y^{q^n-1} - 1$. Therefore, $(abp^{hm'})^\alpha = 1$ and either $m' = m$ or $m' = n - m$. The latter case cannot occur since $m, m' < n/2$. Next, assume that $A^f = B$. In this case, there exist two elements $a, b \in \text{GF}(q^n)^*$ such that $(x_1, x_2, x_3)^f = (ax_2^{p^i}, bx_1^{p^i}, x_3^{p^i})$. Therefore, $(1, x^{q^m+1}, x^{q^m})^f = (ax^{(p^{hm}+1)p^i}, b, x^{p^{hm+i}})$ belongs to $\mathcal{C}'$ for every $x \in \text{GF}(q^n)$. It follows that $abq^{m'}x^{p^i} - x^{p^{hm+hm'+i}} = 0$, and hence $abq^{m'} - x^{p^i(p^{hm+hm'}-1)} = 0$ that is $x^{p^{hm+hm'}-1} = (abq^{m'})^{p^{hn-i}}$ for every $x \in \text{GF}(q^n)^*$.

Hence, the polynomial $y^{p^{hm+hm'}-1} - (abq^{m'})^{p^{hn-i}} \in \text{GF}(q^n)[y]$ is a scalar multiple of the polynomial $y^{p^{hn}-1} - 1$. Therefore, $(abq^{m'})^{p^{hn-i}} = 1$ and $m + m' = n$, a contradiction since $m, m' < n/2$. $\square$

Remark 3.7 Let f be a collineation of $\text{PG}(2, q^n)$ with accompanying automorphism $x \mapsto x^{p^i}$ stabilizing the C_F^m -set $\mathcal{C}$ with equation $x_1 x_2^{q^m} - x_3^{q^m+1} = 0$. From the proof of previous proposition, we have that f fixes A, B , and C pointwise; moreover, there exist two elements a, b of $\text{GF}(q^n)^*$ such that $ab^{q^m} = 1$ and

$$(x_1, x_2, x_3)^f = (ax_1^{p^i}, bx_2^{p^i}, x_3^{p^i}).$$

Hence, the collineation group G stabilizing $\mathcal{C}$ is the semidirect product of the cyclic linear collineation group H of order $q^n - 1$ whose elements are the projectivities

$$(x_1, x_2, x_3) \mapsto (ax_1, bx_2, x_3),$$

where $ab^{q^m} = 1$, with the cyclic group K of order nh generated by the collineation

$$(x_1, x_2, x_3) \mapsto (x_1^p, x_2^p, x_3^p).$$

Remark 3.8 1. Let $\mathcal{C}$ be a C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B and center C , and let H be the full linear collineation group stabilizing $\mathcal{C}$. As previously shown, H is a cyclic group of order $q^n - 1$. The orbit of a point P , not on the edges of the triangle ABC , under the group H is the unique C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B and center C containing P . Let T be the unique subgroup of H of order $\frac{q^n-1}{q-1}$. The orbit of a point P , not on the edges of the triangle ABC , under the group T is a projective geometry $\text{PG}(n-1, q)$ embedded in $\text{PG}(2, q^n)$.

2. Kestenband in [7] studies in $\text{PG}(2, q^n)$ the set of $\text{GF}(q^n)$ -rational points of the algebraic curve with equation $x_1^{q+1} + x_1 x_2^q + \tau x_2 x_3^q + s x_3^{q+1} = 0$, where $\tau \neq 0$ and $-s$ is not a $(q+1)^{\text{th}}$ power. He proves that such a set is the union of two distinguished points with $q-1$ sets each of which is a projective geometry $\text{PG}(n-1, q)$ embedded in $\text{PG}(2, q^n)$. We observe that this set is a C_F^1 -set of $\text{PG}(2, q^n)$.

4 Degenerate C_F^m -sets in $\text{PG}(2, q^n)$

Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the pencils of lines with vertices two distinct points A and B of a projective plane $\text{PG}(2, q^n)$ over the Galois field $\text{GF}(q^n)$, $q = p^h$, p a prime number. Let Φ be an α_F -collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$ which maps the line AB onto itself. Without loss of generality, we may assume that $A = (1, 0, 0)$ and $B = (0, 1, 0)$. The collineation Φ maps the line $ax_2 + bx_3 = 0$ through A onto the line $(\rho a^{q^m} + \lambda b^{q^m})x_1 + (\mu a^{q^m} + \theta b^{q^m})x_3 = 0$ through B , where $\begin{pmatrix} \rho & \lambda \\ \mu & \theta \end{pmatrix}$ is a non-singular matrix over $\text{GF}(q^n)$. Since Φ maps the line $x_3 = 0$ onto itself, it follows that $\lambda = 0$. The set $\mathcal{C}$ of points of intersection of corresponding lines under Φ has equation

$$x_3 \left(\rho x_1 x_3^{q^m-1} + \mu x_3^{q^m} - \theta x_2^{q^m} \right) = 0,$$

where $\rho\theta \neq 0$, and it is called a *degenerate C_F^m -set* of $\text{PG}(2, q^n)$.

Assuming that the point $C = (0, 0, 1)$ belongs to $\mathcal{C}$, the previous equation assumes the following form:

$$x_3 \left(\rho \theta^{-1} x_1 x_3^{q^m-1} - x_2^{q^m} \right) = 0.$$

Moreover, every degenerate C_F^m -set of $\text{PG}(2, q^n)$ is projectively equivalent to the set of $\text{GF}(q^n)$ -rational points of the algebraic curve with equation

$$x_3 \left(x_1 x_3^{q^m-1} - x_2^{q^m} \right) = 0,$$

via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \rho \theta^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Every degenerate C_F^m -set of $\text{PG}(2, q^n)$ has $2q^n + 1$ points, $q^n + 1$ of them are the points of the line AB . The points A and B are called the *vertices* of the degenerate C_F^m -set.

We can assume in the sequel that $m \leq n/2$. Indeed if $m > n/2$, then from $x_1 x_3^{q^m} - x_3 x_2^{q^m} = 0$, we have that $x_1^{q^{n-m}} x_3 - x_3^{q^{n-m}} x_2 = 0$, and hence via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

we obtain $x_1' x_3'^{q^{n-m}} - x_3' x_2'^{q^{n-m}} = 0$.

If $m = n/2$, then from $(m, n) = 1$, it follows $m = 1$ and $n = 2$. Since a degenerate C_F^1 set of $\text{PG}(2, q^2)$ has been already studied in [4] (where it was called a degenerate C_F -set) in the sequel, we can assume that $m < n/2$.

Note that for $n = m = 1$, a degenerate C_F^1 -set of $\text{PG}(2, q)$ is a degenerate conic of $\text{PG}(2, q)$ as consequence of Steiner's projective generation of conics (see [17]).

Next proposition describes the behavior of a degenerate C_F^m -set with respect to lines of $\text{PG}(2, q^n)$ and can be proved similarly to Proposition 3.1.

Proposition 4.1 *Every line of $\text{PG}(2, q^n)$, different from the line AB , intersects a degenerate C_F^m -set either in 1, 2 or $q + 1$ points. In the last case, these points form a subline over $\text{GF}(q)$.*

Proposition 4.2 *Let ℓ_0 be a $\text{GF}(q)$ -subline of a line ℓ of $\text{PG}(2, q^n)$, and let A, B be two points not on ℓ such that the point $AB \cap \ell$ is on ℓ_0 . Then, there exists only one degenerate C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B that meets ℓ in the subline ℓ_0 .*

Proof Observe that there is a bijective map Ψ between the set of the α_F -collineations of the line ℓ into itself and the set of the α_F -collineations between the pencils of lines

$\mathcal{P}_A$ and $\mathcal{P}_B$ with vertices A and B . To every α_F -collineation f , Ψ corresponds the α_F -collineation Ψ_f defined by

$$\Psi_f : r \in \mathcal{P}_A \longrightarrow (r \cap \ell)^f B \in \mathcal{P}_B.$$

From Lemma 3.2, there exists only one α_F -collineation f_0 of the line ℓ into itself fixing pointwise the subline ℓ_0 . Since f_0 fixes $(AB) \cap \ell$, Ψ_{f_0} maps the line AB onto itself. Moreover, Ψ_{f_0} is the unique α_F -collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$ such that every point on ℓ_0 is a point of intersection of corresponding lines. Hence, the degenerate C_F^m -set defined by Ψ_{f_0} is the unique degenerate C_F^m -set of $\text{PG}(2, q^n)$, with vertices A and B , that meets ℓ exactly in the points of the subline ℓ_0 . $\square$

Proposition 4.3 *Every degenerate C_F^m -set $\mathcal{C}$ of $\text{PG}(2, q^n)$ with vertices A and B is the union of the line AB with a scattered $\text{GF}(q)$ -linear set $\mathcal{S}$ of rank $n + 1$, such that $\mathcal{S} \cap AB$ is a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B and vice versa.*

Proof Let $\mathcal{C}$ be a degenerate C_F^m -set of $\text{PG}(2, q^n)$ with vertices A and B with equation $x_3(x_1x_3^{q^m-1} - x_2^{q^m}) = 0$, and let $\mathcal{A} = \mathcal{C} \setminus AB$. We first prove that the union of the set $\mathcal{A}$ with the set of the directions of $\mathcal{A}$ on the line AB is a scattered $\text{GF}(q)$ -linear set of $\text{PG}(2, q^n)$ of rank $n + 1$. Let $X = (x^{q^m}, x, 1)$ and $Y = (y^{q^m}, y, 1)$ be any two distinct points of $\mathcal{A}$. The line XY meets the line AB in the point $((y-x)^{q^m}, y-x, 0)$, and hence the union of the set $\mathcal{A}$ with its directions on the line AB is given by $\mathcal{S} = \{(z^{q^m}, z, a) : z \in \text{GF}(q^n), a \in \text{GF}(q), (z, a) \neq (0, 0)\}$. It is clear that $\mathcal{S}$ is a $\text{GF}(q)$ -linear set. Since $\mathcal{S}$ has size $q^n + q^{n-1} + \dots + q + 1$, it follows that it is a scattered $\text{GF}(q)$ -linear set of $\text{PG}(2, q^n)$ of rank $n + 1$. Embed $\Omega = \text{PG}(2, q^n)$ in $\Sigma^* = \text{PG}(n, q^n)$, and let Ω^* be an $(n-3)$ -dimensional subspace of Σ^* disjoint from Ω . Consider a subgeometry $\Sigma = \text{PG}(n, q)$ of Σ^* disjoint from Ω^* . Let H be the hyperplane of Σ^* spanned by Ω^* and the line AB . From Theorem 2.1, it follows that $\mathcal{S}$ (resp. $\mathcal{A}$) is obtained via the projection $p_{\Omega^*, \Omega}$ from Ω^* onto Ω of the set Σ (resp. $\text{AG}(n, q) = \Sigma \setminus H$). As $\mathcal{C}$ is a set of type $(1, 2, q+1)$, we have that $\mathcal{A}$ is of type $(0, 1, q)$. Consider the incidence structure $(\mathcal{A}, \mathcal{L})$, where $\mathcal{L}$ is the set of q -secant lines to $\mathcal{A}$. Via the projection $p_{\Omega^*, \Omega}$, it follows that the axioms in Theorem 2.3 are satisfied in $(\mathcal{A}, \mathcal{L})$ since they hold in $\Sigma \setminus H$. So, $(\mathcal{A}, \mathcal{L})$ is an affine geometry $\text{AG}(n, q)$. Since $\mathcal{S} \cap AB = \{(z^{q^m}, z, 0) : z \in \text{GF}(q^n)^*\}$ from Remark 4.7 in [13] it follows that $\mathcal{S} \cap AB$ is a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B . Vice versa let $\mathcal{C}$ be the union of the line AB with a scattered $\text{GF}(q)$ -linear set $\mathcal{S}$ of rank $n + 1$, such that $\mathcal{S} \cap AB$ is a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B . We may assume that $\mathcal{S} \cap AB = \{(z^{q^m}, z, 0) : z \in \text{GF}(q^n)^*\}$ for some $m \in \{1, \dots, n-1\}$ with $(m, n) = 1$ (see [13]), and from the first part of this proof, the set $\mathcal{S}$ is the projection of Σ from Ω^* to Ω . Let Ψ be the collineation of Σ^* of order n with accompanying automorphism $x \mapsto x^q$ such that $\Sigma = \text{Fix}(\Psi)$. As in [13], we may assume that $\Omega^* = \langle B^{\Psi^i} : i \neq 0, m \rangle$, where B and $A = B^{\Psi^m}$ are the transversal points of $\mathcal{S} \cap AB$. Consider the following collineation with accompanying automorphism $x \mapsto x^{q^m}$:

$$\Phi : \ell \in \mathcal{P}_B \rightarrow \langle \ell, \Omega^* \rangle^{\Psi^m} \cap \Omega \in \mathcal{P}_A.$$

Since the collineation Ψ fixes the hyperplane $\langle \Omega^*, AB \rangle$, the collineation Φ maps the line AB into itself. For every line $\ell \in \mathcal{P}_B$ with ℓ different from AB , the hyperplane $\langle \ell, \Omega^* \rangle$ meets Σ in a unique point P that is also the unique point of Σ in $\langle \ell^\Phi, \Omega^* \rangle$, and hence $\ell \cap \ell^\Phi$ is the projection of P from Ω^* to Ω . It follows that $\mathcal{C}$ is a degenerate C_F^m -set of Ω . $\square$

Remark 4.4 1. From the proof of the previous proposition, it follows that every degenerate C_F^m -set of $\text{PG}(2, q^n)$ is the union of the line AB with a set $\mathcal{A}$ of q^n points isomorphic to an affine geometry $\text{AG}(n, q)$ such that every three non-collinear points of $\mathcal{A}$ are also non-collinear points of $\text{PG}(2, q^n)$.
2. The set $\mathcal{S}$ defined in the proof of Proposition 4.3 is a small minimal $\text{GF}(q)$ -linear blocking set of Rédei type.

Proposition 4.5 *Let $\mathcal{C}$ be a degenerate C_F^m -set, and let $\mathcal{C}'$ be a degenerate $C_F^{m'}$ set of $\text{PG}(2, q^n)$ ($m, m' < n/2$). The sets $\mathcal{C}$ and $\mathcal{C}'$ are $P\Gamma L$ -equivalent if and only if $m' = m$.*

Proof We can assume that $\mathcal{C} = \{(x^{q^m}, x, 1) : x \in \text{GF}(q^n)\} \cup AB$ and $\mathcal{C}' = \{(x^{q^{m'}}, x, 1) : x \in \text{GF}(q^n)\} \cup AB$ and that $m \leq m'$. Let f be an element of $P\Gamma L(3, q^n)$ with accompanying automorphism $x \mapsto x^{p^i}$ that maps $\mathcal{C}$ in $\mathcal{C}'$. From Proposition 4.3, we have that $\mathcal{C}$ (resp. $\mathcal{C}'$) is the union of the line AB with a scattered $\text{GF}(q)$ -linear set $\mathcal{S}$ (resp. $\mathcal{S}'$), such that $\mathcal{S} \cap AB$ ($\mathcal{S}' \cap AB$) is a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B . The map f sends $\mathcal{S} \cap AB$ to $\mathcal{S}' \cap AB$, and these $\text{GF}(q)$ -linear sets have the same transversal points. From Proposition 4.3 in [13], the transversal points of a scattered $\text{GF}(q)$ -linear set of pseudoregulus type are uniquely determined, so the map f necessarily stabilizes the set $\{A, B\}$.

First, suppose that f fixes both A and B . In this case, there exist four elements $a, b, c, d \in \text{GF}(q^n)$, $ac \neq 0$ such that $(x_1, x_2, x_3)^f = (ax_1^{p^i} + bx_3^{p^i}, cx_2^{p^i} + dx_3^{p^i}, x_3^{p^i})$. Hence, $(x^{q^m}, x, 1)^f = (ax^{p^{hm+i}} + b, cx^{p^i} + d, 1)$ belongs to $\mathcal{C}'$ for every $x \in \text{GF}(q^n)$. It follows that $c^{q^{m'}}x^{p^{hm'+i}} - ax^{p^{hm+i}} + d^{p^{hm'}} - b = 0$ for every $x \in \text{GF}(q^n)$. As 0 is a solution of the previous equation, we have that $d^{p^{hm'}} = b$; thus, $x^{p^{hm+i}}(c^{p^{hm'}}x^{p^{hm+i}(p^{h(m'-m)}-1)} - a) = 0$ for every $x \in \text{GF}(q^n)$. Hence, the polynomial $c^{p^{hm'}}y^{p^{hm+i}(p^{h(m'-m)}-1)} - a \in \text{GF}(q^n)[y]$ is a scalar multiple of the polynomial $y^{p^{hm}-1} - 1$. So $m' = m$ and $a = c^{p^{hm'}}$.

Next, assume that $A^f = B$. In this case, there exist four elements $a, b, c, d \in \text{GF}(q^n)$, $ac \neq 0$ such that $(x_1, x_2, x_3)^f = (ax_2^{p^i} + bx_3^{p^i}, cx_1^{p^i} + dx_3^{p^i}, x_3^{p^i})$. Hence, $(x^{q^m}, x, 1)^f = (ax^{p^i} + b, cx^{p^{hm+i}} + d, 1)$ belongs to $\mathcal{C}'$ for every $x \in \text{GF}(q^n)$. It follows that $c^{q^{m'}}x^{p^{hm+hm'+i}} - ax^{p^i} + d^{q^{m'}} - b = 0$ for every $x \in \text{GF}(q^n)$. As before $d^{q^{m'}} = b$ so $c^{q^{m'}}x^{p^i(p^{hm+hm'}-1)} - a = 0$ for every $x \in \text{GF}(q^n)^*$. Hence, the polynomial $c^{q^{m'}}(y^{p^i})^{p^{hm+hm'}-1} - a \in \text{GF}(q^n)[y]$ is a scalar multiple of the polynomial $y^{p^{hn}-1} - 1$. Therefore, $m + m' = n$, $i = 0$, $a = c^{q^{m'}}$, a contradiction since $m, m' < n/2$. $\square$

Remark 4.6 Let f be a collineation of $\text{PG}(2, q^n)$ with accompanying automorphism $x \mapsto x^{p^i}$ stabilizing the degenerate C_F^m -set with equation $x_3(x_1x_3^{q^{m-1}} - x_2^{q^m}) = 0$.

From the proof of previous proposition, we have that f fixes $\{A, B\}$ pointwise, and that there exist four elements a, b, c, d of $\text{GF}(q^n)$ with $ac \neq 0, a = c^{q^m}, b = d^{q^m}$ such that

$$(x_1, x_2, x_3)^f = (ax_1^{p^i} + bx_3^{p^i}, cx_2^{p^i} + dx_3^{p^i}, x_3^{p^i}).$$

Hence, the full collineation group stabilizing a degenerate C_F^m -set of $\text{PG}(2, q^n)$ is the semidirect product of the linear collineation group of order $q^n(q^n - 1)$ whose elements are the projectivities

$$(x_1, x_2, x_3) \mapsto (ax_1 + bx_3, cx_2 + dx_3, x_3),$$

where $ac \neq 0, a = c^{q^m}$, and $b = d^{q^m}$ with the cyclic group of order nh generated by the collineation

$$(x_1, x_2, x_3) \mapsto (x_1^p, x_2^p, x_3^p).$$

Remark 4.7 1. Let $\mathcal{C}$ be a degenerate C_F^m -set of $\text{PG}(2, q^n)$, and let T be the linear collineation group of order q^n , stabilizing $\mathcal{C}$, whose elements are the projectivities $(x_1, x_2, x_3) \mapsto (x_1 + a^{q^m}x_3, x_2 + ax_3, x_3)$ for any $a \in \text{GF}(q^n)$. The orbit of a point P , not on the line AB , under the action of T is a affine geometry $\text{AG}(n, q)$ embedded in $\text{PG}(2, q^n)$.

2. Let G be the full collineation group stabilizing a degenerate C_F^m -set $\mathcal{C}$ of $\text{PG}(2, q^n)$ calculated in Remark 4.6. Every element of G stabilizes the scattered $\text{GF}(q)$ -linear set $\mathcal{S}$, contained in $\mathcal{C}$, of rank $n + 1$ of $\text{PG}(2, q^n)$ such that $\mathcal{S} \cap AB$ is a $\text{GF}(q)$ -linear set of pseudoregulus type and vice versa. Thus, the full collineation group stabilizing $\mathcal{S}$ coincides with G .

5 Intersection of two scattered $\text{GF}(q)$ -linear sets of rank $n + 1$ in $\text{PG}(2, q^n)$

In this section, we study the possible intersection configurations of a degenerate C_F^m -set, say $\mathcal{C}$, with a degenerate $C_F^{m'}$ -set, say $\mathcal{C}'$, both of $\text{PG}(2, q^n)$ without the restriction $(n, m) = (n, m') = 1$ and with $m \geq m'$. We assume that both $\mathcal{C}$ and $\mathcal{C}'$ have the same vertices A and B . In the meantime, we also study the possible intersection configurations of two scattered $\text{GF}(q)$ -linear sets of rank $n + 1$ such that both meet AB in a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B .

Proposition 5.1 *Let $\mathcal{C}$ be a degenerate C_F^m -set of $\text{PG}(2, q^n)$ defined by the collineation Φ , and let $\mathcal{C}'$ be a degenerate $C_F^{m'}$ -set of $\text{PG}(2, q^n)$ defined by the collineation Φ' . If both $\mathcal{C}$ and $\mathcal{C}'$ have the same vertices A and B and $\mathcal{C} \neq \mathcal{C}'$, then $\mathcal{C} \cap \mathcal{C}' \setminus AB$ is one of the following:*

- i) the empty set;
- ii) one point;
- iii) a degenerate $C_F^{n-m+m'}$ -set of a subgeometry $\text{PG}(2, q^t)$ of $\text{PG}(2, q^n)$ minus the line AB , where $t = (n, n - m + m')$ ($m \neq m'$).

Proof If P is a point of $\mathcal{C} \cap \mathcal{C}' \setminus AB$, then $(AP)^\Phi = (AP)^{\Phi'}$; hence, the points of $\mathcal{C} \cap \mathcal{C}' \setminus AB$ are in one-to-one correspondence with the lines of $\mathcal{P}_A \setminus \{AB\}$ fixed by the collineation $\Phi^{-1} \circ \Phi'$ with accompanying automorphism $\tau : x \mapsto x^{q^{n-m+m'}}$. The set of fixed lines of $\Phi^{-1} \circ \Phi'$, different from AB , is one of the following: the empty set, a line, and a subpencil of $\mathcal{P}_A$ coordinatized over the subfield $\text{Fix}(\tau) = \text{GF}(q^t)$, where $t = (n, n - m + m')$. In this last case by considering Φ and Φ' restricted to the subpencil $\text{Fix}(\Phi^{-1} \circ \Phi')$, we have that $\mathcal{C} \cap \mathcal{C}'$ is a degenerate $C_F^{n-m+m'}$ -set of a subgeometry $\text{PG}(2, q^t)$ of $\text{PG}(2, q^n)$. Observe that if $m = m'$, then τ is the identity, $\Phi^{-1} \circ \Phi'$ is a projectivity and so case *iii*) cannot occur. $\square$

Next, we study the intersections of two scattered $\text{GF}(q)$ -linear sets. We note that the intersection problem for two $\text{GF}(q)$ -linear sets generalizes the intersection problem for two subgeometries. Very little is known on the intersection of two $\text{GF}(q)$ -linear sets (see [8],[14]), while the intersection problem for two subgeometries was solved in [5].

Proposition 5.2 *Let $\mathcal{S}$ and $\mathcal{S}'$ be two scattered $\text{GF}(q)$ -linear sets of rank $n + 1$ of $\text{PG}(2, q^n)$ such that both meet the line AB in a $\text{GF}(q)$ -linear set of pseudoregulus type with transversal points A and B . Then, $\mathcal{S} \cap \mathcal{S}'$ is one of the following:*

- *the empty set;*
- *one point;*
- *a degenerate $C_F^{n-m+m'}$ -set of a subgeometry $\text{PG}(2, q^t)$ of $\text{PG}(2, q^n)$ minus the line AB , where $t = (n, n - m + m')$ ($m \neq m'$);*
- *the $\text{GF}(q)$ -linear set of pseudoregulus type $\mathcal{S} \cap AB$ with transversal points A and B ;*
- *the union of a point with the set $\mathcal{S} \cap AB$;*
- *the union of a degenerate $C_F^{n-m+m'}$ -set of a subgeometry $\text{PG}(2, q^t)$ of $\text{PG}(2, q^n)$ minus the line AB , where $t = (n, n - m + m')$ ($m \neq m'$), with the set $\mathcal{S} \cap AB$.*

Proof From Proposition 4.3, the union of $\mathcal{S}$ with the line AB is a degenerate C_F^m -set $\mathcal{C}$ of $\text{PG}(2, q^n)$ for some m with $(m, n) = 1$, and the union of $\mathcal{S}'$ with the line AB is a degenerate $C_F^{m'}$ -set $\mathcal{C}'$ of $\text{PG}(2, q^n)$ for some m' with $(m', n) = 1$. The possible intersection configurations for $\mathcal{C} \cap \mathcal{C}' \setminus AB$ are listed in Proposition 5.1. Moreover, from Remark 2.2, the set $\mathcal{S} \cap \mathcal{S}' \cap AB$ is either the empty set or it coincides with $\mathcal{S} \cap AB$. $\square$

Acknowledgments The authors thank Olga Polverino for her helpful suggestions regarding Proposition 3.4.

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