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Unimodality via Kronecker products

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Abstract We present new proofs and generalizations of unimodality of the q-binomial coefficients $\binom{n}{k}_q$ as polynomials in q. We use an algebraic approach by interpreting the differences between numbers of certain partitions as Kronecker coefficients of representations of S_n. Other applications of this approach include strict unimodality of the diagonal q-binomial coefficients and unimodality of certain partition statistics.

Keywords Unimodal sequences · Gaussian binomial coefficients · Kronecker product · Strict unimodality · Integer partitions

1 Introduction

A sequence $(a_1, a_2, \ldots, a_n)$ is called *unimodal*, if for some k we have

$$a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

The study of unimodality of combinatorial sequences is a classical subject going back to Newton, and has intensified in recent decades. There is a remarkable diversity of applicable tools, ranging from analytic to topological and from representation theory to probabilistic analysis. The results have a number of applications, but are also important in their own right. We refer to [6,7,30] for a broad overview of the subject.

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In this paper, we present two extensions of the following classical unimodality result. The *q-binomial* (Gaussian) *coefficients* are defined as

$$\binom{m+\ell}{m}_q = \frac{(q^{m+1}-1)\cdots(q^{m+\ell}-1)}{(q-1)\cdots(q^{\ell}-1)} = \sum_{n=0}^{\ell m} p_n(\ell,m)\, q^n.$$

The unimodality of a sequence

$$p_0(\ell,m),\ p_1(\ell,m),\ \ldots,\ p_{\ell m}(\ell,m)$$

is a celebrated result first conjectured by Cayley in 1856, and proved by Sylvester in 1878 [34] (see also [28]). Historically, it has been a starting point of many investigations and various generalizations, both of combinatorial and algebraic nature, and the problem remains very difficult. We refer to Sect. 7 for discussion of various proofs, connections with the Sperner property, historical remarks and references.

Recall that $p_n(\ell,m) = \#\mathcal{P}_n(\ell,m)$, where $\mathcal{P}_n(\ell,m)$ is the set of partitions $\lambda \vdash n$, such that $\lambda_1 \le m$ and $\lambda_1' \le \ell$. Denote by $\nu(\lambda)$, the number of distinct part sizes in the partition λ. The sequence $(a_1,\ldots,a_n)$ is called *symmetric* if $a_i = a_{n+1-i}$, for all $i \le i \le n$.

Theorem 1.1 *Let*

$$p_n(\ell,m,r) = \sum_{\lambda\in\mathcal{P}_n(\ell,m)} \binom{\nu(\lambda)}{r}.$$

Then the sequence

$$p_r(\ell,m,r),\ p_{r+1}(\ell,m,r),\ldots,\ p_{\ell m}(\ell,m,r)$$

is symmetric and unimodal.

Note that $p_n(\ell,m,r) = 0$ for $n < \binom{r+1}{2}$ or $n > \ell m - \binom{r}{2}$, and that $\nu(\lambda)$ can be viewed as the *number of corners* of the corresponding Young diagram $[\lambda]$. Moreover, $p_n(\ell,m,0) = p_n(\ell,m)$ and therefore, for $r = 0$, Theorem 1.1 gives the unimodality of q-binomial coefficients. Our next theorem is a different extension of this result in the diagonal case.

Theorem 1.2 *Let $a_n = p_n(m,m)$. Then, for all $m \ge 7$, we have*

$$a_1 < a_2 < \cdots < a_{\lfloor m^2/2\rfloor} = a_{\lceil m^2/2\rceil} > \cdots > a_{m^2-2} > a_{m^2-1}.$$

Of course, the new contributions of this theorem are the strict inequalities (see also 7.10). The idea behind the proof of Theorem 1.1 is to consider tensor products $S^\lambda \otimes S^\nu$ of irreducible representations of S_n, where $\nu = (n-k,k)$ is a two-row partition. We study the *Kronecker coefficients* $g(\lambda, \mu, \nu)$, defined as the multiplicity of S^ν in the tensor product representation $S^\lambda \otimes S^\nu$, namely

$$S^\lambda \otimes S^\mu = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu)\, S^\nu \tag{1}$$

and interpret these coefficients combinatorially, as the difference in the number of certain *Littlewood–Richardson* (LR) *tableaux*. We then prove that these tableaux are in bijection with the desired partitions. The inequality $g(\lambda, \mu, \nu) \geq 0$ then implies unimodality.

The proof of Theorem 1.2 is more intricate and uses further ingredients. We employ the main lemma in [22] to show that $g(\lambda, \mu, \nu) > 0$ and thereby to reduce strict positivity of Kronecker coefficients to *strict unimodality* of sufficiently large coefficients of a polynomial

$$\mathcal{A}_m(q) = \prod_{i=1}^{m} \left(1 + q^{2i-1}\right), \qquad \text{for all } m \geq 27.$$

To prove this result (Theorem 5.2), we strengthen Almkvist's proof of (non-strict) unimodality of $\mathcal{A}_m(q) + q + q^{m^2-1}$, see [1].

The paper is structured as follows: We start with definitions and notations in Sect. 2. We then present the Main Lemma on unimodality of certain products of LR coefficients (Sect. 3). In Sect. 4 and 5, we apply the Main Lemma to derive all Theorem 1.1 and 1.2, respectively. In the following Sect. 6, we present a dual version of the Main Lemma and derive algebraically a weak version of Almkvist's theorem. We conclude with final remarks and open problems in Sect. 7.

2 Definitions, notation, and examples

We refer the reader to [14,31] for the background on symmetric functions and combinatorics of Young tableaux. Here we set the notations, recall the LR rule, and include an example of Theorem 1.1.

2.1 Partitions and Young diagrams

For any integer partition $\pi = (\pi_1, \ldots, \pi_k)$, let π' denote its *conjugate*, i.e., the partition whose Young diagram $[\pi']$ is the transpose of the Young diagram of π, or algebraically $\pi'_i = \#\{j : \pi_j \geq i\}$. Let $(a^b) = (a, \ldots, a)$, b times, denote the partition whose shape is a $b \times a$ rectangle. Assuming there is a fixed rectangle (a^b) in the context, we denote by $\bar{\pi}$, the *complement* of π within this rectangle, i.e., $\bar{\pi}_i = a - \pi_{b+1-i}$. For example, if $\pi = (5, 5, 3, 2)$, then $\pi' = (4, 4, 3, 2, 2)$, the complement of π within the (6^4) rectangle is $\bar{\pi} = (4, 3, 1, 1)$ (we assume that $\pi_j = 0$ for $j > k$).

2.2 Symmetric functions and the Kronecker product

Following [14,31], we use e_k and h_k to denote elementary and homogeneous symmetric functions, respectively, and let s_λ be the Schur functions. We use "$*$" to denote the *Kronecker product* in the ring of symmetric functions, so

$$s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu)\, s_\nu .$$

Here $g(\lambda, \mu, \nu)$ are the Kronecker coefficients as defined by (1) in § 1. Unlike the Littlewood-Richardson coefficients, explained in § 2.3, which have many nice properties like an easy combinatorial interpretation, no such properties are present for the Kronecker coefficients in the general case. The best current result along these lines is the combinatorial interpretation of Blasiak [5] in the case when one of the partitions λ, μ, ν is a hook.

2.3 The LR rule

The LR coefficients $c^{\lambda}_{\mu\nu}$ are originally defined as the multiplicity of the irreducible representation V_λ of $\mathrm{GL}(N, \mathbb{C})$ within the tensor product $V_\mu \otimes V_\nu$. For our purposes, we will recall their original combinatorial interpretation in terms of *semi-standard Young tableaux* (SSYT).

The *reading word* of a semi-standard Young tableaux T is the sequence obtained by successively recording the numbers appearing in T starting from the top row to the bottom row and reading each row from right to left. A *lattice permutation* (ballot sequence) is a sequence of positive integers $w = w_1 w_2, \ldots, w_n$, such that, for every k and i, among the first k letters of w there are at least as many i's as $(i + 1)$'s, or formally

$$\#\{j : w_j = i,\, j \le k\} \ \ge\ \#\{j : w_j = i+1,\, j \le k\} \qquad \text{for all} \quad 1 \le k \le n,\ i \ge 1.$$

We say that a sequence or a tableau is of *type* β, if it has β_i numbers equal to i.

The *Littlewood–Richardson rule* states that $c^{\lambda}_{\mu\nu}$ is equal to the number of SSYT's of shape λ/μ, of type ν, and whose reading word is a lattice permutation. We call such tableaux the *Littlewood–Richardson* (LR) *tableaux*.

For example, if $\lambda = (5, 5, 3, 2)$, $\mu = (2, 1)$, and $\nu = (4, 4, 3, 1)$, then the semi-standard tableau X below is an LR tableau of shape λ/μ, type ν, and whose reading word is 111222133243.

$X =$

		1	1	1
	1	2	2	2
2	3	3		
3	4			

2.4 Partitions in a rectangle

Let $\ell = m = 3$. Then $\mathcal{P}_n = \mathcal{P}_n(3,3)$ are as follows (here, for brevity and esthetics, we use a concise notation for the partitions, e.g., instead of $(3,2,2)$ we write 32^2):

$$\mathcal{P}_0 = \emptyset,\ \mathcal{P}_1 = \{1\},\ \mathcal{P}_2 = \{2, 1^2\},\ \mathcal{P}_3 = \{3, 21, 1^3\},\ \mathcal{P}_4 = \{31, 22, 21^2\},$$
$$\mathcal{P}_5 = \{32, 221, 31^2\},\ \mathcal{P}_6 = \{3^2, 321, 2^3\},\ \mathcal{P}_7 = \{3^2 1, 32^2\},\ \mathcal{P}_8 = \{3^2 2\},\ \mathcal{P}_9 = \{3^3\}.$$

Therefore,

$$\binom{6}{3}_q = \sum_n p_n(3,3)\, q^n = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9$$

and

$$\sum_n p_n(3,3,1)\, q^n = q + 2q^2 + 4q^3 + 5q^4 + 6q^5 + 5q^6 + 4q^7 + 2q^8 + q^9.$$

Note that even the symmetry of the last polynomial is not obvious. For example, term $2q^2$ comes from two partitions each with one corner, while $2q^8$ comes from one partition with two corners (cf. 7.4).

3 Main Lemma

For every two partitions, λ and μ, of size n, define

$$a_k(\lambda,\mu) = \sum_{\alpha \vdash k,\ \beta \vdash n-k} c^{\lambda}_{\alpha\beta}\, c^{\mu}_{\alpha\beta},$$

where $c^{\nu}_{\pi\theta}$ are the *Littlewood–Richardson coefficients*.

Lemma 3.1 (Main Lemma) *For any two partitions $\lambda, \mu \vdash n$, the sequence*

$$a_0(\lambda,\mu), \ldots, a_n(\lambda,\mu)$$

is symmetric and unimodal.

We refer the reader to §7.8 for additional references on this result.

Proof We start with *Littlewood's identity*:

$$(\circ) \qquad s_\lambda * (s_\pi s_\theta) = \sum_{\alpha \vdash k,\ \beta \vdash n-k} c^{\lambda}_{\alpha\beta} (s_\alpha * s_\pi)(s_\beta * s_\theta),$$

where $\lambda \vdash n$, $\pi \vdash k$ and $\theta \vdash n-k$ (see [13]).

If a is a positive integer, then $s_{(a)}$ corresponds to the trivial representation. So we have $s_\nu * s_{(a)} = s_\nu$, for all $\nu \vdash a$. For $\pi = (k)$ and $\theta = (n-k)$, we obtain

$$s_\lambda * (s_{(k)} s_{(n-k)}) = \sum_{\alpha \vdash k,\ \beta \vdash n-k} c^\lambda_{\alpha\,\beta}\, s_\alpha\, s_\beta = \sum_{\alpha \vdash k,\ \beta \vdash n-k,\ \nu \vdash n} c^\lambda_{\alpha\,\beta}\, c^\nu_{\alpha\,\beta}\, s_\nu .$$

Now let $\tau = (n-k, k)$, where $k \le n/2$. By the Jacobi–Trudi formula, we have

$$s_\tau = s_k\, s_{n-k} - s_{k-1}\, s_{n-k+1} .$$

We obtain:

$$s_\lambda * s_\tau = s_\lambda * (s_k\, s_{n-k}) - s_\lambda * (s_{k-1}\, s_{n-k+1}) = \sum_{\nu \vdash n} (\, a_k(\lambda, \nu) s_\nu - a_{k-1}(\lambda, \nu) s_\nu\,) .$$

Therefore, the Kronecker coefficient $g(\lambda, \mu, \tau)$, which is equal to the coefficient at s_μ in the expansion of $s_\lambda * s_\tau$ in terms of Schur functions, is given by

$$g(\lambda, \mu, \tau) = a_k(\lambda, \mu) - a_{k-1}(\lambda, \mu) .$$

Since $g(\lambda, \mu, \tau) \ge 0$, the unimodality follows. The symmetry is clear from the definition and the symmetry of the LR coefficients, i.e. the fact that $c^\tau_{\alpha\beta} = c^\tau_{\beta\alpha}$ for any τ, α, β. $\square$

4 Special cases of the Main Lemma

We begin with a few special cases which are obtained as corollaries to the Main Lemma when the LR coefficients are either 0 or 1. We present them in increasing order of complexity. This is done to simplify and streamline the exposition.

4.1 q-binomial coefficients

We first obtain the special case $r = 0$ in Theorem 1.1. In other words, we prove unimodality of the coefficients of q^n in $\binom{m+\ell}{m}_q$. See Sect. 7 for other generalizations, and § 4.3 below for another approach.

Corollary 4.1 *Let $p_n(\ell, m)$ be the number of partitions of n which fit in the $\ell \times m$ rectangle. Then the sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ is symmetric and unimodal.*

Proof Let $\lambda = \mu = (m^\ell)$. Recall that $c^{(m^\ell)}_{\alpha\beta} = 1$ if β is the complementary partition of α within the (m^ℓ) rectangle, and is 0 otherwise. This can be seen combinatorially as follows: The SSYT and lattice permutation property enforce that the first i rows of any skew LR tableau contains only the first i numbers. Since the rows in $(m^\ell)/\alpha$ are right-justified, filling them from top to bottom and right to left, we see by induction that the rightmost numbers in row i must be equal to i, and while the SSYT property forces them to be at least as many as the $(i-1)$'s above, the lattice permutation property requires them to be exactly as many, and hence sitting straight below the $(i-1)$'s. Continuing this way, the SSYT property enforces at least as many $(i-1)$'s in the i-th row as $(i-2)$'s above them, and the lattice permutation enforces them to be equally many, etc. This way we get a unique tableau, as in the example below, where $m=6$, $\ell=4$, and $\alpha=(4,3,1)$.

$$\begin{array}{cccccc} & & & & 1 & 1 \\ & & & 1 & 2 & 2 \\ & 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 4 & 4 \end{array}$$

Therefore, for any $\alpha \subset (m^\ell)$, there is a unique β giving a nonzero LR coefficient. This coefficient is equal to 1, so

$$a_n((m^\ell),(m^\ell)) = \sum_{\alpha \vdash n,\, \alpha \subset (m^\ell)} 1 = p_n(\ell,m).$$

Now Lemma 3.1 implies the result. □

4.2 Proof of Theorem 1.1

We proceed as in the case of the q-binomial coefficients. We choose shapes λ and μ such that the LR coefficients $c^\lambda_{\alpha\beta}$ and $c^\mu_{\alpha\beta}$ equal to 1 exactly when β differs from the complement of α within (m^ℓ) by r corners, and otherwise at least one of them is 0.

Let $\lambda=(m^\ell,1^r)$ and $\mu=(m+r,m^{\ell-1})$, i.e., a rectangle with a column of length r attached below and the same rectangle with a row of length r attached on its right. In order for both $c^\lambda_{\alpha\beta}$ and $c^\mu_{\alpha\beta}$ to be nonzero, we must have $\alpha,\beta \subset \lambda \cap \mu = (m^\ell)$.

To compute $c^\lambda_{\alpha\beta}$, note that the first ℓ rows of LR tableaux in λ/α are uniquely determined, by the same argument as in the proof of Corollary 4.1. The number of i's in the first ℓ rows of the LR tableaux λ/α is $m-\alpha_{\ell+1-i}=\bar{\alpha}_i$, where $\bar{\alpha}$ is the complement of α within (m^ℓ).

The remaining r rows in λ must be filled with r distinct numbers to preserve the SSYT property. Let these numbers be $i_1,\ldots,i_r$. The lattice permutation property is preserved up to row $(\ell+j)$ if and only if $1+\bar{\alpha}_{i_j} \le \bar{\alpha}_{i_j-1}$ if $i_{j-1} \ne i_j-1$ and $j>1$, and $1+\bar{\alpha}_{i_j} \le 1+\bar{\alpha}_{i_j-1}$ otherwise. The type β of the tableau should satisfy $\beta_i=\bar{\alpha}_i$ if $i \ne i_1,\ldots,i_r$, and $\beta_i=\bar{\alpha}_i+1$ otherwise. This is equivalent to saying that the type β of the LR tableaux is obtained from $\bar{\alpha}$ by adding a vertical strip of length r to it. As long as $\beta \subset (m^\ell)$, we have $c^\lambda_{\alpha\beta}=1$ in this case. For all other β, we have $c^\lambda_{\alpha\beta}=0$.

$$Y = \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 1 & 1 & 2 & 2 & 2 \\ & 2 & 2 & 3 & 3 & 3 \\ 1 & 3 & 3 & 4 & 4 & 4 \\ i_1 & & & & & \\ i_2 & & & & & \end{array}$$

For example, for the LR tableau Y in the figure above, we have $\alpha = (3, 1, 1)$, $m = 6$, $\ell = 4$, $r = 2$, and the reading word of Y is $111222113332244433 1 i_1 i_2$. In order for it to be a lattice permutation, we can have $i_1 = 2$ and $i_2 = 4$ or $i_1 = 2$ and $i_2 = 3$, so $\beta = (6, 6, 5, 4)$ or $\beta = (6, 6, 6, 3)$, and while $\bar{\alpha} = (6, 5, 5, 3)$, the vertical strip added to β consists of a box in row 2 and 4 in the first case, or in rows 2 and 3 in the second case.

Now let $\mu = (m + r, m^{\ell-1})$. It is well known and easy to see that for any μ, α, and β, we have $c^{\mu}_{\alpha\beta} = c^{\mu'}_{\alpha'\beta'}$ (see e.g., [10]). Note that $\mu' = (\ell^m, 1^r)$ has shape similar to λ, a rectangle plus a column at the bottom. The same argument as above applies and gives that $\beta' = \overline{\alpha'}$, where now $\overline{\alpha'}$ is the complement of α' within (ℓ^m), plus a vertical strip of size r. Note, however, that $\overline{\alpha'}$ is the conjugate of $\bar{\alpha}$, so applying the argument above we conclude that β' is $\bar{\alpha}'$ plus a vertical strip of size r. Conjugating again, this means that β is $\bar{\alpha}$ plus a horizontal strip of size r.

It follows that in order for both $c^{\lambda}_{\alpha\beta} \neq 0$ and $c^{\mu}_{\alpha\beta} \neq 0$ to hold, β should be $\bar{\alpha}$ plus a horizontal strip of size r, and at the same time $\bar{\alpha}$ plus a vertical strip of size r. This is possible if and only if the strips added are individual squares at distinct rows and columns. In other words, β is obtained from $\bar{\alpha}$ by adding r distinct corners of α and for each such β the LR coefficients are 1. Thus, fixing α and summing over all possible partitions β, we have

$$\sum_{\beta} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta} = \binom{\nu(\alpha)}{r},$$

the number of ways to select r distinct corners of α. Now Lemma 3.1 with $\lambda = (m^{\ell}, 1^r)$ and $\mu = (m + r, m^{\ell-1})$ implies the result. □

4.3 Partitions into distinct parts

Here we present yet another proof of Corollary 4.1, which we state in a different, but equivalent form (see Remark 4.3 below). The details of the proof are different, however.

Corollary 4.2 *Let $m > \ell$, and let $d_n(\ell, m)$ be the number of partitions of n into ℓ distinct parts $\leq m$. Then the sequence*

$$d_{\ell}(\ell, m),\ d_{\ell+1}(\ell, m),\ \ldots\ ,\ d_{mn}(\ell, m)$$

is symmetric and unimodal.

Proof Let $\lambda = (m^{\ell}, \ell)$ and $\mu = (m + 1)^{\ell}$. In order to have both LR coefficients $c^{\lambda}_{\alpha\beta} \neq 0$ and $c^{\mu}_{\alpha\beta} \neq 0$, the rectangular shape μ forces β to be the complementary of α within μ, denoted $\bar{\alpha}$. Then, $\beta_i = m + 1 - \alpha_{\ell+1-i}$, $1 \leq i \leq \ell$. In this case, $c^{\mu}_{\alpha\beta} = 1$. Moreover, for both LR coefficients to be nonzero, we must have $\alpha \subset \lambda \cap \mu = (m^{\ell})$.

To compute $c^{\lambda}_{\alpha\beta}$, we construct an LR tableau of shape λ/α and type β. As in the previous arguments, the first ℓ rows in λ/α are uniquely determined. It is easy to see that, for $i \leq \ell$, row i of this LR tableau has $\alpha_{i-r} - \alpha_{i-r+1}$ numbers equal to r for $r = 1, \ldots, i$, where we set $\alpha_0 = m$. Hence, in the first ℓ rows we have a total $m - \alpha_{\ell+1-r}$ numbers equal to r. As established in the previous paragraph, since the LR tableau of shape λ/α must have type β, it follows that the numbers $m+1-\alpha_{\ell+1-r}$ are equal to r.

				1	1
		1	1	2	2
	1	2	2	3	3
1	2	3			

Thus, the last, $(\ell+1)$-st row of λ/α (shaded in the figure above), must be exactly $1, 2, \ldots, \ell$. In order to preserve the SSYT property, the number in row ℓ and column r must be less than r, which is equivalent to $\alpha_{\ell-r+1} \geq r$ for each r. In order for the final reading word to be a ballot sequence, the part of the tableaux that lies in $(m^\ell)/\alpha$ must have strictly more r's than $(r+1)$'s, for $r = 1, \ldots, \ell-1$, which is equivalent to $\beta_r - 1 > \beta_{r+1} - 1$, i.e., that α has distinct parts. Finally, note that together with $\alpha_i > \ell - i$, these constraints are equivalent to α having ℓ nonzero distinct parts. Now Lemma 3.1 implies the result. □

Remark 4.3 Corollaries 4.2 and 4.1 are in fact equivalent, as can be shown by a natural bijection $\nu \leftrightarrow \alpha + (\ell, \ell-1, \ldots, 1)$. We omit the easy details.

5 Strict unimodality

5.1 The result

Consider a symmetric sequence $(a_1, a_2, \ldots, a_n)$. We say that it is *strictly unimodal*, if

$$\begin{array}{ll} a_1 < a_2 < \cdots < a_k = a_{k+1} > \cdots > a_n\,, & \text{for } n = 2k \\ a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_n\,, & \text{for } n = 2k-1 \end{array}$$

(cf. [15]). Strict unimodality of various partition functions was used in [22, § 6] to establish strict positivity of Kronecker coefficients in a similar context.[1] Of course, the Main Lemma (Lemma 3.1) does not imply strict unimodality.

In this section, we apply methods in [22] and reverse the logic of the Main Lemma to obtain Theorem 1.2 strict unimodality of the *diagonal q-binomial coefficients*:

$$\binom{2m}{m}_q = \sum_{n=0}^{m^2} p_n(m,m)\, q^n$$

Remark 5.1 A direct computation shows that strict unimodality easily fails for $m = 3, 4$, and 6 (see e.g. 2.1), but holds for $m = 2$ and 5. This implies that the bound $m \geq 7$ in Theorem 1.2 is tight.

[1] In fact, this paper grew out of our efforts to extend [22].

5.2 Partitions into distinct odd parts

We start with the following extension of Almkvist's theorem.

Theorem 5.2 *Consider the following product*

$$\mathcal{A}_m(q) = \prod_{i=1}^{m} \left(1 + q^{2i-1}\right) = \sum_{n=0}^{m^2} a_n q^n .$$

Then, for all $m \geq 27$, the sequence $(a_{26}, \ldots, a_{m^2-26})$ is symmetric and strictly unimodal.

Proof Fix $m \geq 27$. The symmetry is clear. It suffices to show that

$$a_n < a_{n+1} \quad \text{for all} \quad 26 \leq n < \frac{m^2-1}{2}.$$

We consider three special cases of n. First, for $n \geq 2m+1$, this was shown in [1, p. 122].

Denote by $\mathcal{Q}_n$, the set of partitions of n into distinct odd parts, and let $q(n) = |\mathcal{Q}_n|$. Observe that for $n \leq 2m$, we have $a_n = q(n)$. We define an injection $\varphi : \mathcal{Q}_n \to \mathcal{Q}_{n+1}$ as follows: For $\nu = (\nu_1, \ldots, \nu_\ell) \in \mathcal{Q}_n$, $n \geq 3$, let

$$\varphi(\nu) = \begin{cases} (\nu_1, \ldots, \nu_\ell, 1) & \text{if } \nu_\ell > 1, \\ (\nu_1 + 2, \nu_2, \ldots, \nu_{\ell-1}) & \text{if } \nu_\ell = 1. \end{cases}$$

This shows that $q(n+1) \geq q(n)$. Moreover, we have $\nu \in \mathcal{Q}_{n+1} \smallsetminus \varphi(\mathcal{Q}_n)$ for all partitions, s.t. $\nu_1 - \nu_2 = 2$ and the last part is at least 3, i.e. of the form $\nu = (2i+1, 2i-1, \ldots, j) \vdash n+1$, $j \geq 3$. For $n+1 > 26$, such a partition can be taken of the form $(2i+1, 2i-1)$, $(2i+1, 2i-1, 5)$, $(2i+1, 2i-1, 7, 3)$, $(2i+1, 2i-1, 3)$, depending on the residue of n modulo 4. This implies that $q(n+1) > q(n)$ for all $n \geq 26$.

Now, observe that $a_n = q(n)$ for all $n \leq 2m$, which implies that $a_{n+1} > a_n$ for all $26 \leq n \leq 2m-1$. The remaining inequality $a_{2m+1} > a_{2m}$ follows from $a_{2m+1} = q(2m+1) - 1$, and the additional partition

$$(2i+1, 2i-1, 9) \text{ or } (2i+1, 2i-1, 7) \in \mathcal{Q}_{2m+1} \smallsetminus \varphi(\mathcal{Q}_{2m}).$$

We omit the easy details. □

Remark 5.3 Note that $q(25) = q(26) = 12$ (see e.g. [27]), so for $m \geq 13$, we have $a_{25} = a_{26} = 12$. This implies that the constant 26 in the theorem cannot be improved.

5.3 Proof of Theorem 1.2

We follow the approach in the proof of Corollary 6.2 in [22], whose notation we adopt. Note that for $k \leq m$, we have $p_k(m, m) = \pi(k)$ which is the number of partitions of k. Since $\pi(k) - \pi(k-1)$ is equal to the number of partitions with no parts 1 (see e.g. [19]), we have

$$p_1(m,m) \; < \; p_2(m,m) \; < \; \cdots \; < \; p_m(m,m).$$

Assume $2 \le k \le n/2$. By Lemma 3.1 and Corollary 4.1, we have

$$p_k(m,m) - p_{k-1}(m,m) \; = \; g(m^m, m^m, \tau_k), \qquad \text{where} \;\; \tau_k = (n-k,k), \;\; 2 \le k \le m^2/2.$$

Therefore, reversing the logic of the proof, it suffices to show that

$$g(m^m, m^m, \tau_k) \; \ge \; 1, \quad \text{for} \;\; \tau_k = (n-k,k), \;\; m \le k \le m^2/2.$$

We prove this for $m \ge 27$. By Lemma 1.3 in [22], we have $g(m^m, m^m, \tau_k) \ge 1$ whenever the character value

$$\chi^{\tau_k}[2m-1, \ldots, 3, 1] \; \ne \; 0.$$

Following the logic of the proof of Lemma 6.1 in [22], this character is equal to the difference of partitions numbers:

$$\chi^{\tau_k}[2m-1, \ldots, 3, 1] \; = \; a_k \; - \; a_{k-1}\,,$$

where a_k is as in Theorem 5.2. By the theorem, for $k \ge 27$, we have $a_k - a_{k-1} > 0$. In summary, for $m \ge 27$ we obtain the strict unimodality both for $k \le m$ and $k > m$, as desired. Finally, for $7 \le m \le 26$, we check the result by a direct computation. □

6 Dual version

In this section, we apply our general approach of using Kronecker coefficients to prove unimodality. Here, we use *hooks* instead of two-row Young diagrams, and then apply the results to partitions which fit the rectangle.

6.1 New unimodality result

We prove the following version of Almkvist's theorem.

Theorem 6.1 *Consider a polynomial*

$$\mathcal{B}_m(q) \; = \; \left(1 + q^2 + q^4 + \cdots + q^N\right) \mathcal{A}_m(q)\,,$$

where $N = m^2 - 1$ if m is odd, and $N = m^2$ if m is even. Then the coefficients of $\mathcal{B}_m(q)$ are symmetric and unimodal.

6.2 Dual version of the Main Lemma

For partitions $\lambda, \mu \vdash n$ let

$$b_k(\lambda,\mu) \;=\; \sum_{\alpha\vdash k,\ \beta\vdash n-k} c^{\lambda}_{\alpha\beta}\, c^{\mu}_{\alpha'\beta} \qquad \text{and} \qquad B_k(\lambda,\mu) \;=\; \sum_{i=0}^{\lfloor k/2\rfloor} b_{k-2i}(\lambda,\mu)\,.$$

Lemma 6.2 *For any two partitions $\lambda, \mu \vdash n$ the sequence*

$$B_0(\lambda,\mu),\; B_1(\lambda,\mu),\; \ldots\,,\; B_n(\lambda,\mu)$$

is weakly increasing.

Proof We use again Littlewood's identity ($\circ$) from the proof of the Main Lemma, and apply it with $\pi = (1^k)$ and $\theta = (n-k)$ to obtain

$$s_\lambda * (s_{1^k} s_{n-k}) \;=\; \sum_{\alpha\vdash k,\ \beta\vdash n-k} c^{\lambda}_{\alpha\,\beta}\,(s_{1^k} * s_\alpha)\,(s_{n-k} * s_\beta)\,.$$

Recall that $s_m * s_\pi = s_\pi$ if $\pi \vdash m$, we have $s_{1^k} * s_\pi = s_{\pi'}$, where π' is the conjugate partition. So the above identity translates as

$$\begin{aligned} s_\lambda * \big(s_{1^k} s_{n-k}\big) \;&=\; \sum_{\alpha\vdash k,\ \beta\vdash n-k} c^{\lambda}_{\alpha\beta}\, s_{\alpha'}\, s_\beta \\ &=\; \sum_{\nu\vdash n,\ \alpha\vdash k,\ \beta\vdash n-k} c^{\lambda}_{\alpha\beta}\, c^{\nu}_{\alpha'\beta}\, s_\nu \;=\; \sum_{\nu\vdash n} b_k(\lambda,\nu)\, s_\nu\,. \end{aligned}$$

By *Pieri's rule*, we have

$$s_{1^k}\, s_{n-k} \;=\; e_k\, h_{n-k} \;=\; s_{(n-k,1^k)} \;+\; s_{(n-k+1,1^{k-1})}\,.$$

Using induction on k, we can express the Schur function for a hook as an alternating sum

$$s_{(n-k,1^k)} \;=\; e_k\, h_{n-k} \;-\; e_{k-1}\, h_{n-k+1} \;+\; e_{k-2}\, h_{n-k+2} \;-\; \ldots \;+\; (-1)^k e_0\, h_n\,.$$

Thus, we have

$$s_\lambda * s_{(n-k,1^k)} \;=\; \sum_{\nu\vdash n} \sum_{r=0}^{k} (-1)^r\, b_{k-r}(\lambda,\nu)\, s_\nu \;=\; \sum_{\nu\vdash n} \big(B_k(\lambda,\nu) \;-\; B_{k-1}(\lambda,\nu)\big)\, s_\nu\,.$$

We conclude

$$B_k(\lambda,\mu) \;-\; B_{k-1}(\lambda,\mu) \;=\; g\big(\lambda,\mu,(n-k,1^k)\big) \;\geq\; 0,$$

as desired. □

6.3 Proof of Theorem 6.1

We start with the following combinatorial result which follows from Lemma 6.2.

Corollary 6.3 *Let $w_n(m)$ be the number of self-conjugate partitions of size $(n-2i)$, for some i, which fit in the $m \times m$ square. Then the sequence*

$$w_0(m),\ w_1(m),\ \dots,\ w_{m^2}(m)$$

is weakly increasing.

Proof We apply Lemma 6.2 with $\lambda = \mu = (m^m)$. As noted in the the proof of Corollary 4.1, the LR coefficient $c^{(m^m)}_{\alpha\beta} = 1$ if β is the complementary partition of α within the $m \times m$ square, and 0 otherwise. In order for $c^{(m^m)}_{\alpha\beta} c^{(m^m)}_{\alpha'\beta} \neq 0$ we must have that the complements of α and α' within $m \times m$ are equal, which is equivalent to $\alpha = \alpha'$. Since for each self-conjugate α, there is a unique complementary $\beta = \bar{\alpha}$ for which $c^{(m^m)}_{\alpha\beta} \neq 0$, we have

$$\begin{aligned} w_n(m) &= \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\alpha \vdash n-2i,\ \alpha=\alpha',\ \alpha \subset (m^m)} 1 = \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\alpha \vdash n-2i} c^{(m^m)}_{\alpha\bar{\alpha}} c^{(m^m)}_{\alpha'\bar{\alpha}} \\ &= \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\alpha \vdash n-2i,\ \beta \vdash m^2-n+2i} c^{(m^m)}_{\alpha\beta} c^{(m^m)}_{\alpha'\beta} = B_n(m^m, m^m). \end{aligned}$$

Now the result follows from Lemma 6.2. □

Self-conjugate partitions of n with largest part $\leq m$ are in a classical bijection with partitions of n into distinct odd parts $\leq 2m-1$, (see e.g. [19]). Therefore, the Corollary implies unimodality of the following polynomials:

$$\begin{aligned} \left(1+q^2+q^4+\cdots+q^{m^2}\right) & \prod_{r=1}^{m} (1+q^{2r-1}) \\ &= \sum_{n=0}^{m^2} w_n(m) q^n + \sum_{n=1}^{m^2} w_{m^2-n}(m) q^{n+m^2} \end{aligned}$$

for even m, and

$$\begin{aligned} \left(1+q^2+q^4+\cdots+q^{m^2-1}\right) & \prod_{r=1}^{m} (1+q^{2r-1}) \\ &= \sum_{n=0}^{m^2} w_n(m) q^n + \sum_{n=1}^{m^2-1} w_{m^2-n}(m) q^{n+m^2} \end{aligned}$$

for odd m. This implies Theorem 6.1. □

7 Final remarks

7.1. A combinatorial proof of unimodality of q-binomial coefficients is given by O'Hara in [16] (see also [33,38]). It would be interesting to see if Theorem 1.1 can be proved by a direct combinatorial argument. Unfortunately, O'Hara's chain construction argument does not seem to imply the theorem even in the case $r = 1$ (cf. § 2.4). Indeed, the value of $v(\alpha)$ is not unimodal on the chains. For example, the fourth chain on p. 50 in [16] is

$$(2^2) \to (32) \to (42) \to (43) \to (43) \to (4^2) \to (4^2 1) \to (4^2 2) \to (4^2 3) \to (4^3),$$

and the number of corners dips in the middle.[2] Note also that O'Hara's construction does not give a symmetric chain decomposition of the poset $L(\ell, m)$ of partitions which fit the $\ell \times m$ rectangle (in other words, the difference between successive partitions is not always a corner). Existence of such decompositions remains an open problem (see e.g. [29,37] and references therein).

7.2. The fact that strict unimodality of q-binomial coefficients was open until now is perhaps a reflection on the lack of analytic proof of Sylvester's theorem, as all known proofs are either algebraic or combinatorial (see [23,30]). At the same time, our Theorem 1.1 is rather mysterious; it would be nice to see a truly conceptual explanation of this result. While on the subject, we are curious if there is a p-reduction of this result as discussed in [2].

7.3. Theorem 6.1 is somewhat weak, of course, and can be viewed as both a variation on Almkvist's result as well as a statement that the coefficients a_n in $\mathcal{A}_n(q)$ behave rather smoothly. Given the sharp asymptotic results by Almkvist, it can be derived by other means, as only unimodality of the first two and the middle coefficients does not follow from unimodality of $\mathcal{A}_n(q)$. We present it here as a partial triumph of algebraic methods, as until now the analytic proof was the only result of this kind.

We should note here that it may be too much to expect an algebraic proof of Almkvist's theorem, since $\mathcal{A}_n(q)$ is not fully unimodal, while $\mathcal{A}_n(q) + q + q^{m^2-1}$ is not combinatorially elegant. This makes it very different from *Hughes theorem* on unimodality of

$$\mathcal{H}(t) = \prod_{i=1}^{m} \left(1 + q^i\right),$$

which has both algebraic proofs [11,28] and an analytic proof [18]. In fact, Almkvist's proof is modeled on the Odlyzko–Richmond proof in [18].

7.4. In Theorem 1.1, the symmetry

$$p_n(\ell, m, r) = p_{\ell m - n + r}(\ell, m, r)$$

[2] Note that in [16], the author use subsets in place of partitions; the bijection is straightforward.

can be proved directly as follows: Simply note that $p_n(\ell, m, r)$ is the number of pairs of partitions (α, π) such that $\pi \subset \alpha \subset (m^\ell)$, $\alpha \vdash n$, and α/π consists of r squares which are all (inner) corners of α. They are then outer corners of π. By taking complementary partitions and reversing the order, we obtain pairs $(\overline{\pi}, \overline{\alpha})$ counting $p_{\ell m-n+r}(\ell, m, r)$.
7.5. An important generalization of q-binomial coefficients is given by $s_\lambda(1, q, \dots, q^m)$, which are also known to be unimodal [14, p. 137] (see also [9,12,24]). The proof goes back to Dynkin (see [30, p. 518]). When $\lambda = (\ell)$ or (1^ℓ), we get q-binomial coefficients back again.

It would be nice to find a common generalization of this result and Theorem 1.1. Note that the most straightforward generalizations $a_k(\lambda)$ = the number or partitions $\nu \vdash k$ which fit in the diagram $[\lambda]$, is *not* unimodal in general [32].
7.6. Theorem 1.1 suggests the following generalization. For $z \geq 1$, denote

$$A_k(\ell, m, z) = \sum_{\alpha \in \mathcal{P}_k(\ell,m)} \frac{\Gamma(\nu(\alpha) + z)}{\Gamma(\nu(\alpha) + 1)\,\Gamma(z)},$$

where $\Gamma(z)$ is the Gamma function. We conjecture that $A_n(m, \ell, z)$ is unimodal. Note that for $z \in \mathbb{N}$, we have $A_k(m, \ell, z) = a_k(m, \ell, z-1)$ and the claim follows from the theorem. See [35] for a different one-parametric generalization of Corollary 4.1.
7.7. Although there are several natural combinatorial interpretations of LR coefficients $c^\lambda_{\mu\nu}$ (see e.g., [14,31]), it is unlikely that Lemma 3.1 can be proved directly in full generality, by an explicit surjection. Indeed, this would give a combinatorial interpretation of Kronecker coefficients of $g(\lambda, \mu, \nu)$ for $\nu = (n-k, k)$, an important open problem whose solution is known only in a few special cases (see [3,4,25,26]).
7.8. After the paper was written, we learned that the formulas in the proof of the Main Lemma have independently appeared in a draft version of [36], since then revised and updated. The idea to apply these formulas to the present unimodality results, however, is new.

Most recently, Blasiak found a combinatorial interpretation of the Kronecker coefficients $g(\lambda, \mu, \nu)$, where $\nu = (n-k, 1^k)$ is a hook. This immediately gives a combinatorial interpretation of the difference $B_k(\lambda, \mu) - B_{k-1}(\lambda, \mu)$, as in Lemma 6.2. We use and extend this approach in [21].
7.9. There is yet another way to derive unimodality of q-binomial coefficients (see Corollary 4.1). Recall that the Kronecker product is related to the notion of *plethysm*, defined as a composition of two polynomial representations

$$\phi : \mathrm{GL}(V) \to \mathrm{GL}(W) \quad \text{and} \quad \psi : \mathrm{GL}(W) \to \mathrm{GL}(U),$$

giving a representation $\psi\phi : \mathrm{GL}(V) \to \mathrm{GL}(U)$, see e.g., [31, App. 2]. If the character of ϕ, denoted by f, is expressed as a sum of monomials via $f(x) = \sum_{\theta^i} x^{\theta^i}$ and the character of ψ is g, then the character of $\psi\phi$ is given by the plethysm $g[f] = g(x^{\theta^1}, x^{\theta^2}, \dots)$. Since $\psi\phi$ is a representation and thus decomposes into a direct sum of irreducible representations of $\mathrm{GL}(V)$, it follows that $g[f]$ is a nonnegative sum of Schur functions whenever f and g are themselves nonnegative sums of Schur functions.[3]

In particular, this gives the following recipe for producing unimodal sequences. Let $g = s_{(n-k,k)}$, and let f be any symmetric function that is a nonnegative sum of Schur functions. Let $pl_n(\lambda, f, k)$ be the coefficient of $s_\lambda(x)$ in the expansion of $h_{n-k}[f]h_k[f]$ in terms of Schur functions, i.e.,

$$h_{n-k}[f] \cdot h_k[f] = \sum_\lambda pl_n(\lambda, f, k) s_\lambda .$$

Observe that for $k \le n/2$, we have $\delta_k = pl_n(\lambda, f, k) - pl_n(\lambda, f, k-1)$ that is equal to the coefficient of s_λ in the expansion $s_{(n-k,k)}[f]$. This implies that $\delta_n \ge 0$, and thus the sequence

$$pl_n(\lambda, f, 0) , \ \ldots \ , \ pl_n(\lambda, f, n)$$

is symmetric and unimodal for any $\lambda \vdash n$.

For example, when $f = s_{(1,1)}$ and $\lambda = (m^{2\ell})$, this approach gives Corollary 4.1 again. We omit the details which are technical and somewhat involved.

7.10. In [20], we generalize Theorem 1.2 to all large enough q-binomial coefficients. Namely, we prove that

$$p_1(\ell, m) < \cdots < p_{\lfloor \ell m/2 \rfloor} = p_{\lceil \ell m/2 \rceil} > \cdots > p_{\ell m-1}(\ell, m) .$$

for all $\ell, m \ge 8$. We use a completely different approach, based on algebraic properties of Kronecker coefficients.

Most recently, Shareshian found another proof of our Theorem 1.1, which uses combinatorics of flags over $\mathbb{F}_q$ and reduces the result to Sylvester's theorem.[4]

7.11. The *log-concavity* is a stronger property than unimodality, which appears in many applications. A sequence $a_1, \ldots, a_N$ is called log-concave if $a_n^2 \ge a_{n-1}a_{n+1}$ for all $2 \le n \le N-1$. This property fails for q-binomial coefficients, but does hold in several related contexts. Let us single out [8] for q-log-concavity of a sequence

$$\binom{n}{0}_q, \binom{n}{1}_q, \ldots, \binom{n}{n}_q$$

viewed as polynomials, and [17] for log-concavity properties of certain LR coefficients. See [7,30] for the surveys.

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[3] Another standard notation for plethysm is $g \circ f$, see e.g. [14, § 1.8].

[4] Personal communication.

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