Young walls and graded dimension formulas for finite quiver Hecke algebras of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$

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Abstract We study graded dimension formulas for finite quiver Hecke algebras $R^{\Lambda_0}(\beta)$ of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ using combinatorics of Young walls. We introduce the notion of standard tableaux for proper Young walls and show that the standard tableaux form a graded poset with lattice structure. We next investigate Laurent polynomials associated with proper Young walls and their standard tableaux arising from the Fock space representations consisting of proper Young walls. Then, we prove the graded dimension formulas described in terms of the Laurent polynomials. When evaluating at $q = 1$, the graded dimension formulas recover the dimension formulas for $R^{\Lambda_0}(\beta)$ described in terms of standard tableaux of strict partitions.

Keywords Fock space representations · Graded dimension formulas · Quiver Hecke algebras · Standard tableaux · Young walls

1 Introduction

The quiver Hecke algebras $R = \bigoplus_{\beta \in \mathbf{Q}^+} R(\beta)$, introduced by Khovanov–Lauda [20, 21] and Rouquier [23], provide categorifications of the negative parts of quantum Kac–Moody algebras. They have $\mathbb{Z}$ -grading and can be defined for all symmetrizable Cartan datums. Moreover, their certain graded quotients, $R^\Lambda = \bigoplus_{\beta \in \mathbf{Q}^+} R^\Lambda(\beta)$,

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called cyclotomic quiver Hecke algebras, categorify integrable highest weight modules $V(\Lambda)$ over quantum Kac–Moody algebras [14]. When $\Lambda = \Lambda_0$, we call the algebras $R^{\Lambda_0}(\beta)$ *finite quiver Hecke algebras*. The results in [5, 23] inform that the block algebras of the cyclotomic Hecke algebras are isomorphic to cyclotomic quiver Hecke algebras of type $A_\ell^{(1)}$. In particular, when $\Lambda = \Lambda_0$, the finite quiver Hecke algebra of type $A_\ell^{(1)}$ is isomorphic to the Iwahori–Hecke algebra associated with the symmetric group. In this sense, finite quiver Hecke algebras can be understood as vast generalizations of Iwahori–Hecke algebras. Via the isomorphism in [5, 23], well-established theories on cyclotomic Hecke algebras can be applicable to R^Λ of type $A_\ell^{(1)}$ with additional grading structures [6, 7, 11]. However, only a few results about the algebraic structure of R^Λ for affine types except $A_\ell^{(1)}$ were known [1, 2]. For example, there exist no known homogeneous bases for R^Λ of affine types except type $A_\ell^{(1)}$.

The study of partitions (or Young diagrams) is one of the most important research areas in combinatorial representation theory. Combinatorics of partitions are deeply related to representations of Lie algebras and symmetric groups. Among proper subsets of partitions, strict partitions (or shifted Young diagrams) have its own meaning in combinatorial representation theory. In particular, they realize the Fock space representations of Kac–Moody algebras of types $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ [8, 12]. In [13], Kang introduced new combinatorial objects, called *Young walls*, to give combinatorial realizations of level 1 crystals of various quantum affine Kac–Moody algebras, which are deeply related to the theory of perfect crystals [16, 17]. In [19], Kashiwara et al. constructed the Fock space representation of quantum affine algebras using perfect crystals. In [18], Kang and Kwon proved that the spaces spanned by *proper* Young walls indeed form the Fock space representations of quantum affine Kac–Moody algebras. In particular, proper Young walls of type $A_{2\ell}^{(2)}$ (resp. $D_{\ell+1}^{(2)}$) can be identified with $(2\ell + 1)$ -strict partitions (resp. $(\ell + 1)$ -strict partitions) which are defined as follows: For $h \in \mathbb{Z}_{\geq 1} \sqcup \{\infty\}$,

$\lambda = (\lambda_0, \lambda_1, \lambda_2, \dots)$ is an h -strict partition if $\lambda_i = \lambda_{i+1} \neq 0$ implies $h|\lambda_i$.

An h -strict partition can be viewed as *intermediate* one between a partition and a strict partition since a partition is a 1-strict partition, and a strict partition is an ∞ -partition. Moreover, there exist the following inclusions:

$$\{\text{strict partitions}\} \subsetneq \cdots \subsetneq \{2h\text{-strict partitions}\} \subsetneq \{h\text{-strict partitions}\} \subsetneq \{\text{partitions}\}.$$

Thus, the classical limit of the Fock space representation in [18] consisting of proper Young walls of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ is *strictly larger* than the Fock space representation in [8, 12] consisting of strict partitions. The differences and connections for type $A_{2\ell}^{(2)}$ were discussed in [19, Appendix D], [22, Sect. 2]. It is explained in [24, Introduction] why proper Young walls of types $A_\ell^{(1)}$, $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ may be regarded as partitions in the viewpoint of perfect crystals. Type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ also appear in categorification of the Hecke–Clifford superalgebras [4, 24] and the quiver Hecke superalgebras [9, 15].

This paper aims for computing *graded* dimension formulas of $R^{\Lambda_0}(\beta)$ of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ using combinatorics of proper Young walls. To achieve the goal, we introduce and investigate new combinatorial objects, *standard tableaux of proper Young walls*. We define a standard tableau T of a proper Young wall Y as a sequences of certain proper Young walls, and also give an equivalent definition for standard tableaux by considering its entries (Proposition 4.2). As h -strict partitions do, standard tableaux of a proper Young wall can be understood as intermediate one between standard tableaux of a partition and those of a strict partition. As a combinatorial object itself, the set $\text{ST}(Y)$ of standard tableaux on a proper Young wall Y has a partial order induced from the weak order of the symmetric group, which is an analog of the partial order appeared in [7, Sect. 3.3]. It turns out that $w(\text{ST}(Y))$ is a *generalized quotient* which is introduced in [3] for generalizing distinguished coset representatives of parabolic subgroups of Coxeter groups (Theorem 4.4). Theorem 4.4 tells that $\text{ST}(Y)$ is a graded poset with lattice structure having a unique minimum and a unique maximum and that, for $T, T' \in \text{ST}(Y)$ with $T < T'$, there exist simple transpositions $s_{i_1}, s_{i_2}, \dots, s_{i_t}$ such that,

$$T' = s_{i_t} \cdots s_{i_1} T, \quad s_{i_j} \cdots s_{i_1} T \in \text{ST}(Y) \text{ for all } 1 \leq j \leq t,$$

where wT is the tableau obtained from T by replacing entry k with $w(k)$ for $k = 1, \dots, n$.

Standard tableaux for proper Young walls naturally arise from the actions $f_{i_1} \cdots f_{i_n}$ and $e_{j_1} \cdots e_{j_n}$ on the Fock space representations $\mathcal{F}(\Lambda_0)$ consisting of proper Young walls. Observing the action of Chevalley generators f_i and e_i on $\mathcal{F}(\Lambda_0)$, we get the Laurent polynomials $\mathfrak{E}_q(T)$ and $\mathfrak{F}_q(T)$ associated with standard tableaux T of proper Young walls Y . They may be viewed as $A_{2\ell}^{(2)}$ -type and $D_{\ell+1}^{(2)}$ -type analogs of $q^{\deg(T)}$ and $q^{-\text{codeg}(T)}$ which appear in graded dimension formulas for finite quiver Hecke algebra of the type $A_\ell^{(1)}$. Here, $\deg(T)$ and $\text{codeg}(T)$ are the degree and codegree for standard tableaux appeared in [7, Sect. 3.5]. Interestingly, the ratio of $\mathfrak{E}_q(T)$ and $\mathfrak{F}_q(T)$ becomes a Laurent polynomial again, which depends only on Y (Corollary 5.5). For any pair of standard tableaux (T, T') of Y with $T < T'$ and $s_i T = T'$, we compute $\mathfrak{E}_q(T')/\mathfrak{E}_q(T)$ and $\mathfrak{F}_q(T')/\mathfrak{F}_q(T)$ explicitly and show that they are same (Proposition 5.4). Moreover, Proposition 5.6 informs us that $m_q(T) := \mathfrak{E}_q(T)/\mathfrak{E}_q(T^Y) = \mathfrak{F}_q(T)/\mathfrak{F}_q(T^Y)$ is a Laurent polynomial, where T^Y is the canonical tableau defined in (4.2).

Our graded dimension formulas for $R^{\Lambda_0}(\beta)$ are described in terms of the Laurent polynomials associated with standard tableaux for proper Young walls. Considering the Fock space representations $\mathcal{F}(\Lambda_0)$ with the categorification theorem, we prove the graded dimension formulas for $R^{\Lambda_0}(\beta)$ (Theorem 6.3). Our graded dimension formulas are graded versions of the dimension formulas given in [1, 2] described in terms of standard tableaux on strict partitions. When evaluating at $q = 1$, the dimension of $R^{\Lambda_0}(\beta)$ can be computed from a proper subset $\text{ST}_\infty(Y)$ of the standard tableaux $\text{ST}(Y)$ which corresponds bijectively to the set of standard tableaux for strict partitions (Theorem 6.6). Thus, our graded dimension formulas recover the dimension formulas in [1, 2] described in terms of standard tableaux of strict partitions.

2 Fock space representations

2.1 Quantum affine algebras

Let $I = \{0, 1, \dots, \ell\}$ ($\ell \geq 2$) be an index set. The affine Cartan matrices of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ are given as follows:

$$A(A_{2\ell}^{(2)}) = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}, \quad A(D_{\ell+1}^{(2)}) = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

Throughout this paper, we set $\mathbf{X} = A_{2\ell}^{(2)}$ or $D_{\ell+1}^{(2)}$, and let $A(\mathbf{X}) = (a_{ij})_{i,j \in I}$ denote the affine Cartan matrix of type $\mathbf{X}$.

A realization of the affine Cartan matrix $A(\mathbf{X})$ is given as follows:

- (1) $\mathbf{P}$ is a free abelian group of rank $\ell + 2$, called the *weight lattice*,
- (2) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathbf{P}$, called the set of *simple roots*,
- (3) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathbf{P}^\vee := \text{Hom}(\mathbf{P}, \mathbb{Z})$, called the set of *simple coroots*,

which satisfy

- (a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
- (b) Π and $\Pi^\vee$ are linearly independent sets.

There is a symmetric bilinear pairing $(\mid)$ on $\mathbf{P}$ satisfying

$$(\alpha_i \mid \Lambda) = d_i \langle h_i, \Lambda \rangle \quad \text{for all } \Lambda \in \mathbf{P},$$

where

$$(d_0, d_1, \dots, d_\ell) = \begin{cases} (1, 2, \dots, 2, 4) & \text{if } \mathbf{X} = A_{2\ell}^{(2)}, \\ (1, 2, \dots, 2, 1) & \text{if } \mathbf{X} = D_{\ell+1}^{(2)}. \end{cases} \quad (2.1)$$

We denote by $\mathbf{P}^+ = \{\Lambda \in \mathbf{P} \mid \langle h_i, \Lambda \rangle \geq 0, i \in I\}$ the set of *dominant integral weight*. For $i \in I$, let Λ_i denote the i th *fundamental weight*, i.e., $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for $j \in I$. We call $\mathbf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the *root lattice*, and $\mathbf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ the *positive cone* of the root lattice. For $\beta = \sum_{i \in I} k_i \alpha_i \in \mathbf{Q}^+$, the *height* of β is defined by $|\beta| = \sum_{i \in I} k_i$.

Let q be an indeterminate and $m, n \in \mathbb{Z}_{\geq 0}$. For $i \in I$, let $q_i = q^{d_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

Definition 2.1 The *quantum affine algebra* $U_q(\mathbf{X})$ associated with $(A(\mathbf{X}), \mathbf{P}, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by e_i, f_i ($i \in I$) and q^h ($h \in \mathbf{P}^\vee$) satisfying following relations:

- (1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in \mathbf{P}^\vee$,
- (2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in \mathbf{P}^\vee, i \in I$,
- (3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$ where $K_i = q_i^{h_i}$,
- (4) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $U_{\mathbb{A}}(X)$ the $\mathbb{A}$ -form of $U_q(X)$. For a dominant integral weight $\Lambda \in \mathbf{P}^+$, let $V(\Lambda)$ be the irreducible highest weight $U_q(X)$ -module with highest weight Λ and $V_{\mathbb{A}}(\Lambda)$ be its $\mathbb{A}$ -form.

2.2 Young walls

A Young wall of type X is a wall consisting of blocks with unit width, unit thickness, and

: half-unit height, : unit height.

The walls are built on the ground-state wall by the following rules:

- (1) Blocks should be built in the pattern given below.
- (2) There should be no free space to the right of any block except the rightmost column.

	2	2	2	2	2
	1	1	1	1	1
	0	0	0	0	0
	0	0	0	0	0
	1	1	1	1	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	ℓ	ℓ	ℓ	ℓ	ℓ
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1	1	1	1	1
	0	0	0	0	0
	0	0	0	0	0

$$X = A_{2\ell}^{(2)}$$

	1	1	1	1	1
	0	0	0	0	0
	0	0	0	0	0
	1	1	1	1	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\ell-1$	$\ell-1$	$\ell-1$	$\ell-1$	$\ell-1$
	ℓ	ℓ	ℓ	ℓ	ℓ
	ℓ	ℓ	ℓ	ℓ	ℓ
	$\ell-1$	$\ell-1$	$\ell-1$	$\ell-1$	$\ell-1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1	1	1	1	1
	0	0	0	0	0
	0	0	0	0	0

$$X = D_{\ell+1}^{(2)}$$

For example, when $X = A_4^{(2)}$, the following are Young walls:

$$Y_1 \qquad Y_2 \qquad Y_3 \qquad (2.2)$$

For a Young wall Y , we write $Y = (y_k)_{k=0}^\infty = (\dots, y_2, y_1, y_0)$ as a sequence of its columns. Let $|y_k|$ be the number of blocks in y_k for $k \in \mathbb{Z}_{\geq 0}$. We denote by $\ell(Y)$ the number of nonzero columns of Y , i.e., $\ell(Y) = \#\{y_k \mid |y_k| \neq 0, k \geq 0\}$, and set

$$|Y| = \sum_{k \in \mathbb{Z}_{\geq 0}} |y_k|.$$

Note that $|y_k|$ are weakly decreasing from right to left. The numbers appeared in the above pattern belong to the index set I , and they are called *residues* or *colors* of Young walls. When a block b in a Young wall Y , simply $b \in Y$, has residue $i \in I$, we say that b is an i -block in Y .

Definition 2.2 (1) A column of a Young wall is called a *full column* if its height is a multiple of the unit length.

- (2) A Young wall is said to be *proper* if none of the full columns has the same heights.
- (3) An i -block of a proper Young wall Y is called a *removable i -block* if Y remains a proper Young wall after removing the block.
- (4) A place in a proper Young wall Y is called an *admissible* or *addable i -slot* if Y remains a proper Young wall after adding an i -block at the place.

If b is an i -block or an i -slot in Y and j (resp. k) is the residue directly above b (resp. directly below b), then we say

$$\text{res}(b) = i, \quad \text{res}_+(b) = j, \quad \text{res}_-(b) = k. \qquad (2.3)$$

For $i \in I$, we set

$$a_i = \begin{cases} 1 & \text{if } i = \ell \text{ and } X = A_{2\ell}^{(2)}, \\ 2 & \text{otherwise.} \end{cases}$$

Note that the weight $\sum_{i \in I} a_i \alpha_i$ (resp. $\frac{1}{2} \sum_{i \in I} a_i \alpha_i$) is the null root δ of $U_q(A_{2\ell}^{(2)})$ (resp. $U_q(D_{\ell+1}^{(2)})$).

- Definition 2.3** (1) The part of a column in a proper Young wall is called a δ -column if it contains a_0 -many 0-blocks, a_1 -many 1-blocks, ..., a_ℓ -many ℓ -blocks in some cyclic order.
- (2) A δ -column in a proper Young wall Y is *removable* if one can remove the δ -column from Y , and the resulting is a proper Young wall.
- (3) A proper Young wall is said to be *reduced* if it has no removable δ -column.

For example, considering the Young walls in (2.2), Y_1 is not proper, Y_2 is proper but not reduced, and Y_3 is proper and reduced.

Let $\mathcal{Y}(\Lambda_0)$ be the set of all reduced proper Young walls. Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ can be defined by considering the combinatorics of Young walls, which yield that $\mathcal{Y}(\Lambda_0)$ is a $U_q(\mathbf{X})$ -crystal [10, 13].

Theorem 2.4 [13, Thm.7.1] *The crystal $\mathcal{Y}(\Lambda_0)$ is isomorphic to the crystal of the irreducible highest weight $U_q(\mathbf{X})$ -module $V(\Lambda_0)$ with highest weight Λ_0 .*

2.3 Fock space representations

Let $\mathcal{Z}(\Lambda_0)$ be the set of all proper Young walls, and let

$$\mathcal{F}(\Lambda_0) = \bigoplus_{Y \in \mathcal{Z}(\Lambda_0)} \mathbb{Q}(q)Y$$

be the $\mathbb{Q}(q)$ -vector space generated by $\mathcal{Z}(\Lambda_0)$. We now define a $U_q(\mathbf{X})$ -module structure on $\mathcal{F}(\Lambda_0)$ by following [18].

Let $Y = (y_k)_{k=0}^\infty \in \mathcal{Z}(\Lambda_0)$, and let

$$\text{wt}(y_k) = \sum_{i \in I} t_{ki} \alpha_i, \quad \text{wt}(Y) = \Lambda_0 - \sum_{k \in \mathbb{Z}_{\geq 0}} \text{wt}(y_k),$$

where t_{ki} is the number of i -blocks in the k th column y_k of Y . If b is a block or a slot in the r th row in Y , then we set

$$\begin{aligned} \delta_-(b; Y) &= \begin{cases} 1 & \text{if } \text{res}(b) = \text{res}_-(b), \ r > 1, \\ 0 & \text{otherwise.} \end{cases} \\ \delta_+(b; Y) &= \begin{cases} 1 & \text{if } \text{res}(b) = \text{res}_+(b), \ r > 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.4)$$

Note that the ground-state wall is the 0th row. For $k \in \mathbb{Z}_{\geq 0}$ and $i \in I$, we set

$$d_i(Y, k) = -\text{wt}(y_k)(h_i) \quad \text{for } k \in \mathbb{Z}_{\geq 0}. \quad (2.5)$$

Note that

$$d_i(Y, k) = \begin{cases} -2 & \text{if } i = \text{res}(b) = \text{res}_-(b), \\ -1 & \text{if } i = \text{res}(b) \neq \text{res}_-(b), \ i \neq \text{res}_+(b), \\ 2 & \text{if } i = \text{res}(s) = \text{res}_+(s), \\ 1 & \text{if } i = \text{res}(s) \neq \text{res}_+(s), \ i \neq \text{res}_-(s), \\ 0 & \text{otherwise,} \end{cases} \quad (2.6)$$

where b is the block at the top of the k th column y_k of Y , and s is the slot on the top of the k th column y_k of Y .

If b is either a removable i -block or an admissible i -slot in the p th column y_p of Y , then we set

$Y \nearrow b =$ proper Young wall obtained by removing b from Y if b is removable,
 $Y \swarrow b =$ proper Young wall obtained by adding an i -block at b if b is admissible,

and

$$l(b; Y) = \#\{y_k \mid |y_k| = |y_p|\},$$

$$L_i(b; Y) = \sum_{k > p} d_i(Y, k) + \Lambda_0(h_i), \quad R_i(b; Y) = \sum_{p > k} d_i(Y, k). \quad (2.7)$$

Then, we define the action of Chevalley generators f_i, e_i on $\mathcal{F}(\Lambda_0)$ as follows: For $Y \in \mathcal{Z}(\Lambda_0)$,

$$f_i Y = \sum_b q_i^{L_i(b; Y)} \left(\frac{1 - (-q^2)^{l(b; Y)}}{q} \right)^{\delta_-(b; Y)} (Y \swarrow b),$$

$$e_i Y = \sum_{b'} q_i^{-R_i(b'; Y)} \left(\frac{1 - (-q^2)^{l(b'; Y)}}{q} \right)^{\delta_+(b'; Y)} (Y \nearrow b'), \quad (2.8)$$

where b and b' run over all admissible i -slots and all removable i -blocks in Y , respectively. Note that the above description of the Chevalley generators actions is simpler than that in [18, Sect. 5] since we only focus on the case of type $X = A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}$.

Theorem 2.5 [18, Thm.5.5] *The Fock space $\mathcal{F}(\Lambda_0)$ is a $U_q(X)$ -module. Moreover, the submodule of $\mathcal{F}(\Lambda_0)$ generated by $\emptyset$ is isomorphic to $V(\Lambda_0)$.*

3 Quiver Hecke algebras

We now recall the definition of the cyclotomic quiver Hecke algebra. Throughout this section, we denote by $U_q(\mathfrak{g})$ a quantum Kac–Moody algebra associated with a symmetrizable Cartan matrix. Let $\mathbf{k}$ be a field, and, for $i, j \in I$, let $\mathcal{Q}_{i,j}(u, v) \in \mathbf{k}[u, v]$ be polynomials of the form

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \sum_{p(\alpha_i|\alpha_i) + q(\alpha_j|\alpha_j) + 2(\alpha_i|\alpha_j) = 0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where (a_{ij}) is the Cartan matrix of $U_q(\mathfrak{g})$, $t_{i,j;p,q} \in \mathbf{k}$ are such that $t_{i,j;-a_{ij},0} \neq 0$, and $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$. Note that the symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \dots, n-1 \rangle$ acts on I^n by place permutations.

Definition 3.1 The *cyclotomic quiver Hecke algebra* $R^\Lambda(n)$ associated with polynomials $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I}$ and a dominant integral weight $\Lambda \in \mathbf{P}^+$ is the $\mathbb{Z}$ -graded $\mathbf{k}$ -algebra defined by three sets of generators

$$\{e(v) \mid v = (v_1, \dots, v_n) \in I^n\}, \{x_k \mid 1 \leq k \leq n\}, \{\tau_l \mid 1 \leq l \leq n-1\},$$

subject to the following relations:

$$\begin{aligned} e(v)e(v') &= \delta_{v,v'}e(v), \quad \sum_{v \in I^n} e(v) = 1, \quad x_k e(v) = e(v)x_k, \quad x_k x_l = x_l x_k, \\ \tau_l e(v) &= e(s_l(v))\tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \text{ if } |k-l| > 1, \quad \tau_k^2 e(v) = \mathcal{Q}_{v_k, v_{k+1}}(x_k, x_{k+1})e(v), \\ (\tau_k x_l - x_{s_k(l)} \tau_k) e(v) &= \begin{cases} -e(v) & \text{if } l = k \text{ and } v_k = v_{k+1}, \\ e(v) & \text{if } l = k+1 \text{ and } v_k = v_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(v) &= \delta_{v_k, v_{k+2}} \frac{\mathcal{Q}_{v_k, v_{k+1}}(x_k, x_{k+1}) - \mathcal{Q}_{v_k, v_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(v), \\ x_1^{\langle h_{v_1}, \Lambda \rangle} e(v) &= 0. \end{aligned}$$

The $\mathbb{Z}$ -grading on $R^\Lambda(n)$ is given as follows:

$$\deg(e(v)) = 0, \quad \deg(x_k e(v)) = (\alpha_{v_k} | \alpha_{v_k}), \quad \deg(\tau_l e(v)) = -(\alpha_{v_l} | \alpha_{v_{l+1}}).$$

For $\beta \in \mathbf{Q}^+$ with $|\beta| = n$, let

$$I^\beta = \{v = (v_1, \dots, v_n) \in I^n \mid \alpha_{v_1} + \dots + \alpha_{v_n} = \beta\}.$$

As I^β is invariant under the action of $\mathfrak{S}_n$, the element $e(\beta) = \sum_{v \in I^\beta} e(v)$ is a central idempotent of $R^\Lambda(n)$. We set

$$R^\Lambda(\beta) = R^\Lambda(n)e(\beta).$$

In particular, when $\Lambda = \Lambda_0$, we call the algebras $R^{\Lambda_0}(\beta)$ *finite quiver Hecke algebras*.

We denote the direct sum of the Grothendieck groups of the categories $\text{Proj}(R^\Lambda(\beta))$ of finitely generated projective graded $R^\Lambda(\beta)$ -modules by

$$[\text{Proj}(R^\Lambda)] = \bigoplus_{\beta \in \mathbf{Q}^+} [\text{Proj}(R^\Lambda(\beta))].$$

Note that $[\text{Proj}(R^\Lambda)]$ has a free $\mathbb{A}$ -module structure induced from the $\mathbb{Z}$ -grading on $R^\Lambda(\beta)$, i.e., $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$. Let $e(v, i)$ be

the idempotent corresponding to the concatenation of v and (i) , and set $e(\beta, i) = \sum_{v \in I^\beta} e(v, i)$ for $\beta \in \mathbb{Q}^+$. Define functors

$$\begin{aligned} E_i &: \text{Mod}(R^\Lambda(\beta + \alpha_i)) \longrightarrow \text{Mod}(R^\Lambda(\beta)), \\ F_i &: \text{Mod}(R^\Lambda(\beta)) \longrightarrow \text{Mod}(R^\Lambda(\beta + \alpha_i)), \end{aligned}$$

between categories $\text{Mod}(R^\Lambda(\beta))$ and $\text{Mod}(R^\Lambda(\beta + \alpha_i))$ of finitely generated graded modules by

$$E_i(N) = q_i^{1-\langle h_i, \Lambda - \beta \rangle} (e(\beta, i)N) \quad \text{and} \quad F_i(M) = R^\Lambda(\beta + \alpha_i) e(\beta, i) \otimes_{R^\Lambda(\beta)} M,$$

for $M \in \text{Mod}(R^\Lambda(\beta))$ and $N \in \text{Mod}(R^\Lambda(\beta + \alpha_i))$, respectively. Then, E_i and F_i give a $U_{\mathbb{A}}(\mathfrak{g})$ -module structure to $[\text{Proj}(R^\Lambda)]$.

Theorem 3.2 ([14, Thm.6.2]) *There exists a $U_{\mathbb{A}}(\mathfrak{g})$ -module isomorphism between $[\text{Proj}(R^\Lambda)]$ and $V_{\mathbb{A}}(\Lambda)$.*

For a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$, the *graded dimension* of M is defined by

$$\dim_q M = \sum_{k \in \mathbb{Z}} \dim(M_k) q^k.$$

Note that $\dim_q(q^t M) = q^t \dim_q M$. For $\Lambda \in \mathbb{P}^+$ and $\beta \in \mathbb{Q}^+$, set

$$\text{def}(\Lambda, \beta) = (\beta | \Lambda) - \frac{1}{2}(\beta | \beta).$$

The following proposition will be used crucially in proving the graded dimension formula:

Proposition 3.3 *Let $v = (v_1, \dots, v_n)$, $v' = (v'_1, \dots, v'_n) \in I^\beta$.*

(1) Let v_Λ be the highest weight vector of the highest weight $U_q(\mathfrak{g})$ -module $V(\Lambda)$. Then, we have

$$e_{v_1} \cdots e_{v_n} f_{v'_n} \cdots f_{v'_1} v_\Lambda = q^{-\text{def}(\Lambda, \beta)} (\dim_q e(v) R^\Lambda(\beta) e(v')) v_\Lambda.$$

(2) Let $- : \mathbb{A} \rightarrow \mathbb{A}$ be the homomorphism defined by $\bar{q} = q^{-1}$. Then,

$$\overline{q^{-\text{def}(\Lambda, \beta)} (\dim_q e(v) R^\Lambda(\beta) e(v'))} = q^{-\text{def}(\Lambda, \beta)} (\dim_q e(v) R^\Lambda(\beta) e(v'))$$

Proof (1) Note that $\beta = \alpha_{v_1} + \cdots + \alpha_{v_n} = \alpha_{v'_1} + \cdots + \alpha_{v'_n}$. For $k = 1, \dots, n$, let

$$t_k = \frac{1}{2}(\alpha_{v_k} | \alpha_{v_k}) - (\alpha_{v_k} | \Lambda - \beta + \alpha_{v_n} + \alpha_{v_{n-1}} + \cdots + \alpha_{v_k}).$$

Then, we have

$$\begin{aligned} t_1 + \cdots + t_n &= \frac{1}{2} \sum_{k=1}^n (\alpha_{v_k} | \alpha_{v_k}) - \sum_{k=1}^n ((\alpha_{v_k} | \Lambda - \beta) + (\alpha_{v_k} | \alpha_{v_n} + \alpha_{v_{n-1}} + \cdots + \alpha_{v_k})) \\ &= - \sum_{k=1}^n (\alpha_{v_k} | \Lambda - \beta) - \sum_{k=1}^n \left(\frac{1}{2} (\alpha_{v_k} | \alpha_{v_k}) + (\alpha_{v_k} | \alpha_{v_n} + \alpha_{v_{n-1}} + \cdots + \alpha_{v_{k+1}}) \right) \\ &= -(\beta | \Lambda - \beta) - \frac{1}{2} \sum_{k,l=1,\dots,n} (\alpha_{v_k} | \alpha_{v_l}) \\ &= -(\beta | \Lambda) + \frac{1}{2} (\beta | \beta) \\ &= -\text{def}(\Lambda, \beta). \end{aligned}$$

Therefore, it follows from $q_i^{1-\langle h_i, \Lambda - \gamma \rangle} = q^{(\alpha_i | \alpha_i)/2 - (\alpha_i | \Lambda - \gamma)}$ that

$$\begin{aligned} E_{v_1} \cdots E_{v_n} F_{v'_n} \cdots F_{v'_1} R^\Lambda(0) &= E_{v_1} \cdots E_{v_n} R^\Lambda(\beta) e(v') \\ &= q^{t_n} E_{v_1} \cdots E_{v_{n-1}} e(\beta - \alpha_{v_n}, v_n) R^\Lambda(\beta) e(v') \\ &\quad \vdots \\ &= q^{t_1 + \cdots + t_n} e(v) R^\Lambda(\beta) e(v') \\ &= q^{-\text{def}(\Lambda, \beta)} e(v) R^\Lambda(\beta) e(v'), \end{aligned}$$

which implies the assertion by Theorem 3.2.

(2) The assertion follows immediately from (1) and the fact that $e_{v_1} \cdots e_{v_n} f_{v'_n} \cdots f_{v'_1} v_\Lambda$ is bar-invariant. $\square$

4 Standard tableaux for proper Young walls

4.1 Standard tableaux

Let $Y \in \mathcal{Z}(\Lambda_0)$ be a proper Young wall of type X and let $n = |Y|$.

Definition 4.1 A *standard tableau* of shape Y is a sequence of proper Young walls

$$\emptyset = Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y^{(n-1)} \subset Y^{(n)} = Y,$$

where $|Y^{(k+1)}| - |Y^{(k)}| = 1$ for $k = 0, 1, \dots, n-1$.

For a standard tableau $T = (Y^{(0)}, \dots, Y^{(n)})$ of shape Y , let $T_{\leq k}$ (resp. $T_{< k}$) denote the standard tableau $(Y^{(0)}, \dots, Y^{(k)})$ (resp. $(Y^{(0)}, \dots, Y^{(k-1)})$) consisting of the first $k+1$ (resp. k) proper Young walls of T . Let

$\text{ST}(Y) =$ the set of all standard tableaux of shape Y .

A standard tableau also can be described by labeling blocks with numbers. For a standard tableau $T = (Y^{(0)}, \dots, Y^{(n)})$ of shape Y , if we label the block of $Y^{(k)} \setminus Y^{(k-1)}$ with the number k for all k , then the resulting is a proper Young wall blocks of which are labeled with entries $1, 2, \dots, n$. For example, when $X = A_4^{(2)}$, if we have the following standard tableau T :

then, the expression of T using numbers is given as follows:

$$T = \begin{array}{|c|c|c|} \hline & & 7 \\ \hline & & 5 \\ \hline & & 3 \\ \hline & 6 & 2 \\ \hline 8 & 4 & 1 \\ \hline \end{array} \quad (4.1)$$

We will use the above expression for standard tableaux if there is no confusion.

Just as usual standard tableaux for partitions, standard tableaux for proper Young walls also can be defined by considering their entries. A *tableau* T of shape Y is a labeling of blocks in Y with numbers, and let

$$\text{sh}(T) = Y.$$

The *canonical tableau* T^Y is the tableau in which the number $1, 2, \dots, n$ appear in order from bottom to top and from right to left. For example, the following is a canonical tableau:

$$T^Y = \begin{array}{|c|c|c|} \hline & & 5 \\ \hline & & 4 \\ \hline & & 3 \\ \hline & 7 & 2 \\ \hline 8 & 6 & 1 \\ \hline \end{array} \quad (4.2)$$

Note that canonical tableaux are standard. For a tableau T , $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}_{>0}$, let

$$T(i, j) = \text{the entry of the block at } (i, j)\text{-position in } T,$$

where (i, j) -position is the position in $\text{sh}(T)$ at the i th column and the j th row. If there is no block at (i, j) -position, then we set $T(i, j) = \infty$. For example, if T is the

tableau of (4.1), then $T(0, 1) = 1$, $T(0, 4) = 5$, and $T(2, 1) = 8$. The following is another description for standard tableaux. Since the proof is straightforward, we leave it to the reader.

Proposition 4.2 *Let $Y \in \mathcal{Z}(\Lambda_0)$ be a proper Young wall, and let T be a tableau of shape Y . T is standard if and only if, for all possible $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}_{>0}$,*

$$T(i, j) < T(i, j + 1), \quad T(i, j) < T(i + 1, j - \delta(i, j)),$$

where, if b is the block of Y at (i, j) -position, then

$$\delta(i, j) = \begin{cases} 0 & \text{if } \text{res}(b) = \text{res}_-(b), \\ 1 & \text{otherwise.} \end{cases}$$

For a proper Young wall $Y = (y_k)_{k=0}^\infty \in \mathcal{Z}(\Lambda_0)$, the associated partition $\lambda_Y = (|y_0|, |y_1|, |y_2|, \dots)$ of Y is an h -strict partition, where

$$h = \begin{cases} 2\ell + 1 & \text{if } X = A_{2\ell}^{(2)}, \\ \ell + 1 & \text{if } X = D_{\ell+1}^{(2)}. \end{cases} \quad (4.3)$$

In the cases of $h = 1, \infty$, Proposition 4.2 may give a definition for usual standard tableaux of Young diagrams and shifted Young diagrams. The case $h = 1$ may be viewed as the case $\ell = 0$. In that case, all blocks have half-unit height and all residue of Young walls are 0, so $\delta(i, j) = 0$ for all i, j . Thus, entries in rows and columns of a standard tableau of Y increase from right to left and from bottom to top, respectively. The case $h = \infty$ may be regarded as the case $\ell = \infty$. In that case, all residue in each column are different, so $\delta(i, j) = 1$ for all i, j . If, for each k , we move up the k -column of a standard tableau of shape Y by k , then the resulting tableau is a tableau of a shifted Young diagram. Moreover, entries in rows and columns of the resulting tableau increase from right and left and from bottom to top, respectively.

Let Y be a proper Young wall. If the associated partition λ_Y of Y is strict, then we define $\text{ST}_\infty(Y)$ to be the set of all standard tableaux of shape Y such that

$$T(i, j) < T(i, j + 1), \quad T(i, j) < T(i + 1, j - 1) \quad \text{for all possible } i, j. \quad (4.4)$$

We set $\text{ST}_\infty(Y) = \emptyset$ if the associated partition of Y is not strict. By definition, we have

$$\text{ST}_\infty(Y) \subset \text{ST}(Y).$$

Moreover, if the associated partition λ_Y is strict, then the set $\text{ST}_\infty(Y)$ is bijective to the set of all standard tableaux of shape λ_Y via the correspondence for $h = \infty$ in the above paragraph.

4.2 Weak order on $\text{ST}(Y)$

Let $\mathfrak{S}_n$ be the symmetric group of permutations of $\{1, 2, \dots, n\}$, which is generated by simple transposition $s_i := (i, i + 1)$ for $i = 1, \dots, n - 1$. For $w \in \mathfrak{S}_n$, let $l(w)$ be the *length* of w , i.e., the number of simple transpositions in its reduced expression. As a permutation $w \in \mathfrak{S}_n$ may be regarded as the word $(w(1)w(2) \cdots w(n))$, we think of $\mathfrak{S}_n$ as the set of words with n distinct letters $1, 2, \dots, n$. Note that, for a permutation $w = (w_1 \cdots w_i w_{i+1} \cdots w_n)$,

$$ws_i = (w_1 \cdots w_{i+1} w_i \cdots w_n), \quad (4.5)$$

$$l(ws_i) = l(w) + 1 \quad \text{if and only if} \quad w_i < w_{i+1}. \quad (4.6)$$

For $u, w \in \mathfrak{S}_n$, we say $u < w$ in *weak order* if there exist $s_{i_1}, s_{i_2}, \dots, s_{i_t}$ such that

$$l(s_{i_k} \cdots s_{i_2} s_{i_1} u) = l(u) + k \quad (1 \leq k \leq t), \quad w = s_{i_t} \cdots s_{i_1} u.$$

For a subset $V \subset \mathfrak{S}_n$, the set

$$\mathfrak{S}_n / V = \{w \in \mathfrak{S}_n \mid l(wv) = l(w) + l(v) \text{ for all } v \in V\}$$

is called a *generalized quotient* for $\mathfrak{S}_n$.

Theorem 4.3 [3, Thm.4.1] *Let $\mathfrak{S}_n / V$ be a generalized quotient.*

(1) *There exists a permutation $v \in \mathfrak{S}_n$ such that*

$$\mathfrak{S}_n / V = \{w \in \mathfrak{S}_n \mid \text{id} \leq w \leq v\}.$$

(2) *$\mathfrak{S}_n / V$ is a lattice under the weak order.*

We now define a partial order on $\text{ST}(Y)$ induced from the weak order on $\mathfrak{S}_n$. Let Y be a proper Young wall with n blocks. For a tableau T of shape Y , let $w(T)$ be the word obtained by reading entries of T column by column from bottom to top and from right to left. Since, for $T \in \text{ST}(Y)$, $w(T)$ is a word of n distinct letters $1, 2, \dots, n$, it can be viewed as a permutation of $\mathfrak{S}_n$. Thus, the set

$$w(\text{ST}(Y)) = \{w(T) \mid T \in \text{ST}(Y)\}$$

can be regarded as a subset of $\mathfrak{S}_n$. For example, $w(T^Y) = (12 \cdots n)$ and, if T is the tableau (4.1), $w(T)$ is given as

$$w(T) = (12357468) \in \mathfrak{S}_n.$$

For $T, T' \in \text{ST}(Y)$, we define

$$T < T' \quad (4.7)$$

if $w(T) < w(T')$ in the weak order of $\mathfrak{S}_n$, which gives a partial order on $\text{ST}(Y)$. Note that the canonical tableau T^Y is a unique smallest tableau in $\text{ST}(Y)$. The following theorem is a generalization of [3, Thm. 7.2] to standard tableaux for Young walls:

Theorem 4.4 *Let Y be a proper Young wall with n blocks.*

- (1) *The subset $w(\text{ST}(Y))$ is a generalized quotient for $\mathfrak{S}_n$.*
- (2) *The poset $\text{ST}(Y)$ is a graded poset with a unique maximum element.*
- (3) *The poset $\text{ST}(Y)$ is a lattice.*
- (4) *For $T, T' \in \text{ST}(Y)$ with $T < T'$, there exist simple transpositions $s_{i_1}, s_{i_2}, \dots, s_{i_t}$ such that,*

$$T' = s_{i_t} \cdots s_{i_1} T, \quad s_{i_j} \cdots s_{i_1} T \in \text{ST}(Y) \text{ for all } 1 \leq j \leq t,$$

where wT is the tableau obtained from T by replacing entry k with $w(k)$ for $k = 1, \dots, n$.

Proof For $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}_{>0}$, if Y has a block at (i, j) -position, let t_{ij} be the entry of the top block of the canonical tableau T^Y in the i th column and

$$o_{ij} = T^Y(i, j), \quad l_{ij} = T^Y(i + 1, j - \delta(i, j)),$$

where $\delta(i, j)$ is given in Proposition 4.2. Note that $o_{ij} \leq t_{ij} \leq l_{ij}$. For each i, j with $l_{ij} \neq \infty$, let

$$v_{ij} = (o_{ij}, o_{ij} + 1)(o_{ij} + 1, o_{ij} + 2) \cdots (t_{ij} - 1, t_{ij}) \\ (l_{ij} - 1, l_{ij})(l_{ij} - 2, l_{ij} - 1) \cdots (t_{ij} + 1, t_{ij} + 2)(t_{ij}, t_{ij} + 1).$$

Note that, for a tableau T , $w(T)v_{ij}$ is obtained from $w(T)$ by first moving $T(i, j)$ to the top block of the i th column, and then moving $T(i + 1, j - \delta(i, j))$ to the bottom block of $(i + 1)$ th column, and lastly interchanging these two entries. We now define

$$V = J \cup \{v_{ij} \mid l_{ij} \neq \infty\} \subset \mathfrak{S}_n,$$

where $J = \{(k, k + 1) \mid k = 1, 2, \dots, n - 1, k \neq t_{ij} \text{ for all possible } i, j\}$. Let T be a tableau of shape Y . By Proposition 4.2, (4.5) and (4.6), we have

$$T \in \text{ST}(Y) \Leftrightarrow \begin{cases} l(w(T)u) = l(w(T)) + l(u) & \text{for all } u \in J, \\ l(w(T)v_{ij}) = l(w(T)) + l(v_{ij}) & \text{for all } i, j. \end{cases}$$

Thus, $w(\text{ST}(Y)) = \{w \in \mathfrak{S}_n \mid l(wv) = l(w) + l(v) \text{ for all } v \in V\}$, which is a generalized quotient.

The remaining assertions follow from (1) and Theorem 4.3. □

5 Laurent polynomials associated with proper Young walls

Let $Y \in \mathcal{Z}(\Lambda_0)$ be a proper Young wall with n blocks. For a standard tableau $T \in \text{ST}(Y)$ of shape Y , we set $\deg(\emptyset) = 0$ and $\text{codeg}(\emptyset) = 0$, and, if b denotes the block of Y with entry n and $i = \text{res}(b)$, then we define inductively

$$\deg(T) = d_i \cdot L_i(b; Y) + \deg(T_{<n}), \quad \text{codeg}(T) = d_i \cdot R_i(b; Y) + \text{codeg}(T_{<n}), \quad (5.1)$$

where d_i are the integers defined by (2.1). In a similar manner, we set $\mathfrak{E}_q(\emptyset) = 1$ and $\mathfrak{F}_q(\emptyset) = 1$, and, if b denotes the i -block of Y with entry n , then we define inductively

$$\begin{aligned} \mathfrak{F}_q(T) &= q_i^{L_i(b; Y^-)} \left(\frac{1 - (-q^2)^{l(b; Y^-)}}{q} \right)^{\delta_-(b; Y^-)} \mathfrak{F}_q(T_{<n}), \\ \mathfrak{E}_q(T) &= q_i^{-R_i(b; Y)} \left(\frac{1 - (-q^2)^{l(b; Y)}}{q} \right)^{\delta_+(b; Y)} \mathfrak{E}_q(T_{<n}), \end{aligned} \quad (5.2)$$

where $Y^- = Y \nearrow b$, i.e., $Y^- = \text{sh}(T_{<n})$, $L_i(b; Y^-)$, $R_i(b; Y)$, $\delta_-(b; Y)$, $\delta_+(b; Y)$, and $l(b; Y)$ are given as (2.4) and (2.7). Since $L_i(b; Y) = L_i(b; Y^-)$, one can easily check that

$$\mathfrak{F}_q(T) = q^{\deg(T)} \mathfrak{F}'_q(T), \quad \mathfrak{E}_q(T) = q^{-\text{codeg}(T)} \mathfrak{E}'_q(T)$$

where $\mathfrak{F}'_q(T)$ and $\mathfrak{E}'_q(T)$ are defined inductively as follows:

$$\mathfrak{F}'_q(T) = \left(\frac{1 - (-q^2)^{l(b; Y^-)}}{q} \right)^{\delta_-(b; Y^-)} \mathfrak{F}'_q(T_{<n}), \quad \mathfrak{E}'_q(T) = \left(\frac{1 - (-q^2)^{l(b; Y)}}{q} \right)^{\delta_+(b; Y)} \mathfrak{E}'_q(T_{<n}).$$

Here $\mathfrak{F}'_q(\emptyset) = 1$ and $\mathfrak{E}'_q(\emptyset) = 1$.

Recall the number h defined as (4.3). We set

$$\varpi_Y = (\varpi_1, \varpi_2, \dots), \quad |\varpi_Y| = \sum_{j=1}^{\infty} \varpi_j, \quad (5.3)$$

where ϖ_j is the number of columns y_k of Y heights of which are hj , i.e., $\varpi_j = \#\{y_k \mid |y_k| = hj\}$. Note that, if a is the block at the top of the k th column y_k , then

$$h \mid |y_k| \quad \text{if and only if} \quad \text{res}(a) = \text{res}_+(a). \quad (5.4)$$

The following lemma can be understood as an analog of [7, Lemma 3.12]:

Lemma 5.1 *Let Y be a proper Young wall with n blocks and let $\beta = \Lambda_0 - \text{wt}(Y)$. Then, for $T \in \text{ST}(Y)$, we have*

$$\deg(T) + \text{codeg}(T) = \text{def}(\Lambda_0, \beta) - |\varpi_Y|.$$

Proof Let b be the block in T with entry n in the k th column and

$$i = \text{res}(b), \quad Y^- = \text{sh}(T_{<n}).$$

Since $d_i = 1$ if either $i = 0$ ($X = A_{2\ell}^{(2)}$) or $i = 0, \ell$ ($X = D_{\ell+1}^{(2)}$), it follows from (2.6) that

$$d_i(1 + d_i(Y; k)) = \begin{cases} -1 & \text{if } i = \text{res}(b) = \text{res}_-(b), \\ 1 & \text{if } i = \text{res}(b) = \text{res}_+(b), \\ 0 & \text{otherwise,} \end{cases}$$

which yields, by (5.4),

$$|\varpi_Y| = d_i(1 + d_i(Y; k)) + |\varpi_{Y^-}|.$$

We use induction on n . If $n = 0$, then there is nothing to prove. Suppose that the assertion holds for all proper Young walls Y' with $|Y'| < n$. Then, since

$$\begin{aligned} \text{def}(\Lambda_0, \beta - \alpha_i) &= \text{def}(\Lambda_0, \beta) - (\Lambda_0 - \beta|\alpha_i) - (\alpha_i|\alpha_i)/2, \\ &= \text{def}(\Lambda_0, \beta) - d_i(\Lambda_0 - \beta)(h_i) - d_i, \end{aligned}$$

we have

$$\begin{aligned} \deg(T) + \text{codeg}(T) &= d_i(L_i(b; Y) + R_i(b; Y)) + \deg(T_{<n}) + \text{codeg}(T_{<n}) \\ &= d_i((\Lambda_0 - \beta)(h_i) - d_i(Y; k)) + \text{def}(\Lambda_0, \beta - \alpha_i) - |\varpi_{Y^-}| \\ &= \text{def}(\Lambda_0, \beta) - d_i(1 + d_i(Y; k)) - |\varpi_{Y^-}| \\ &= \text{def}(\Lambda_0, \beta) - |\varpi_Y|, \end{aligned}$$

which complete the proof. $\square$

Definition 5.2 (1) For a proper Young wall Y , let

$$\begin{aligned} C_-(Y) &= \{b \in Y \mid \delta_-(b; Y) = 1\}, & c_-(Y) &= |C_-(Y)|, \\ C_+(Y) &= \{b \in Y \mid \delta_+(b; Y) = 1\}, & c_+(Y) &= |C_+(Y)|. \end{aligned} \quad (5.5)$$

Note that $C_-(Y) \cap C_+(Y) = \emptyset$.

- (2) Let b be a block in a proper Young wall Y in the i th column and the j th row.
- (a) If b is in $C_-(Y)$, then we define

$$l(b; T) = \max\{k \in \mathbb{Z}_{\geq 0} \mid T(i + k, j - 1) < T(i, j)\}$$

- (b) If b is in $C_+(Y)$, then we define

$$l(b; T) = \max\{k \in \mathbb{Z}_{\geq 0} \mid T(i - k, j + 1) > T(i, j)\}.$$

Note that $T(k, l) = \infty$ if there is no block at the (k, l) -position in T .

The following theorem gives us explicit descriptions for $\mathfrak{F}_q(T)$ and $\mathfrak{E}_q(T)$:

Theorem 5.3 Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall with n blocks.

(1) For any $T \in \text{ST}(Y)$, we have

$$\begin{aligned}\mathfrak{F}'_q(T) &= q^{-c_-(Y)} \prod_{b \in C_-(Y)} (1 - (-q^2)^{l(b;T)+1}), \\ \mathfrak{E}'_q(T) &= q^{-c_+(Y)} \prod_{b \in C_+(Y)} (1 - (-q^2)^{l(b;T)+1}),\end{aligned}$$

and hence

$$\begin{aligned}\mathfrak{F}_q(T) &= q^{\deg(T)-c_-(Y)} \prod_{b \in C_-(Y)} (1 - (-q^2)^{l(b;T)+1}), \\ \mathfrak{E}_q(T) &= q^{-\text{codeg}(T)-c_+(Y)} \prod_{b \in C_+(Y)} (1 - (-q^2)^{l(b;T)+1}),\end{aligned}$$

where $C_-(Y)$, $C_+(Y)$, $c_-(Y)$ and $c_+(Y)$ are given as (5.5).

(2) In particular, we have

$$\begin{aligned}\mathfrak{F}_q(T^Y) &= q^{o(Y)-c_-(Y)} (1 + q^2)^{c_-(Y)}, \\ \mathfrak{E}_q(T^Y) &= q^{d(Y)-c_+(Y)} \prod_{b \in C_+(Y)} (1 - (-q^2)^{l_R(b;Y)}),\end{aligned}$$

where $o(Y) = (\Lambda_0 | \Lambda_0 - \text{wt}(Y))$, $d(Y) = \sum_{1 \leq k < l \leq \ell(Y)} (\text{wt}(y_l) | \text{wt}(y_k))$, and, if b is in the p th column, then

$$l_R(b; Y) = \begin{cases} \#\{y_k \mid p \geq k, |y_k| = |y_p|\} & \text{if } b \text{ is at the top of its column,} \\ 1 & \text{otherwise.} \end{cases} \quad (5.6)$$

Proof (1) It can be proved easily from definitions by induction on $n = |Y|$.

(2) By the definition of the canonical tableaux T^Y , it follows from (2.5) and (2.7) that

$$\deg(T^Y) = o(Y), \quad \text{codeg}(T^Y) = -d(Y).$$

The configuration of entries of T^Y gives

$$l(b; T^Y) = 0 \text{ if } b \in C_-(b), \quad l(b; T^Y) = l_R(b; Y) - 1 \text{ if } b \in C_+(b),$$

which implies the assertion. $\square$

Theorem 4.4 tells us that, for any standard tableau $T \in \text{ST}(Y)$, there exist a sequence $\{T_k\}_{k=0}^t$ of standard tableaux such that

$$\begin{aligned}T_0 &= T^Y, \quad T_t = T, \\ T_k &= s T_{k-1} \text{ with } T_k > T_{k-1} \text{ for some simple transposition } s \in \mathfrak{S}_n.\end{aligned}$$

The following proposition with Theorem 4.4 allows to compute the Laurent polynomials $\mathfrak{F}_q(T)$ and $\mathfrak{E}_q(T)$ inductively.

Proposition 5.4 *Let $T, T' \in \text{ST}(Y)$ be standard tableaux of shape Y such that $T < T'$ and $s_k T = T'$ for some simple transposition $s_k \in \mathfrak{S}_n$. We denote by b_1 (resp. b_2) the block with entry k (resp. $k+1$) in T , and set*

$$i = \text{res}(b_1), \quad j = \text{res}(b_2), \quad Y_0 = \text{sh}(T_{<k}).$$

If b_1 (resp. b_2) is in the p th row (resp. the q th row) of T , then we have

$$\frac{\mathfrak{F}_q(T')}{\mathfrak{F}_q(T)} = \frac{\mathfrak{E}_q(T')}{\mathfrak{E}_q(T)} = \begin{cases} q^{-(\alpha_i, \alpha_j)} \left(\frac{1 - (-q^2)^{l(b_1; Y_0) + 1}}{1 - (-q^2)^{l(b_1; Y_0)}} \right) & \text{if } \text{res}(b_1) = \text{res}(b_2), \quad p = q + 1, \\ q^{-(\alpha_i, \alpha_j)} & \text{otherwise.} \end{cases}$$

Proof Since $T < T'$, b_2 appears to the left of b_1 in T by (4.6). Note that, when the case $\text{res}(b_1) = \text{res}(b_2)$ and $p = q + 1$, it can be viewed pictorially as follows:

For $t = 1, \dots, n$, let

$$Y_T(t) = \text{sh}(T_{\leq t}), \quad b_T(t) = \text{the block with entry } t \text{ in } T, \\ Y_{T'}(t) = \text{sh}(T'_{\leq t}), \quad b_{T'}(t) = \text{the block with entry } t \text{ in } T'.$$

Then, the condition $s_k T = T'$ tells that $b_T(t)$ and $b_{T'}(t)$ are placed at the same position in Y if $t \neq k, k+1$, and $Y_T(t) = Y_{T'}(t)$ if $t \neq k$. Set

$$Y_1 = Y_T(k), \quad Y_2 = Y_{T'}(k), \quad Y_3 = Y_T(k+1).$$

It follows from (5.2) that

$$\frac{\mathfrak{F}_q(T')}{\mathfrak{F}_q(T)} = \frac{q_j^{L_j(b_2; Y_0)} \left(\frac{1 - (-q^2)^{l(b_2; Y_0)}}{q} \right)^{\delta_-(b_2; Y_0)} q_i^{L_i(b_1; Y_2)} \left(\frac{1 - (-q^2)^{l(b_1; Y_2)}}{q} \right)^{\delta_-(b_1; Y_2)}}{q_i^{L_i(b_1; Y_0)} \left(\frac{1 - (-q^2)^{l(b_1; Y_0)}}{q} \right)^{\delta_-(b_1; Y_0)} q_j^{L_j(b_2; Y_1)} \left(\frac{1 - (-q^2)^{l(b_2; Y_1)}}{q} \right)^{\delta_-(b_2; Y_1)}}.$$

It follows from (2.5) and the fact that b_2 appears to the left of b_1 in T that

$$\begin{aligned} q_j^{L_j(b_2; Y_0)} &= q_j^{L_j(b_2; Y_1)}, & q_i^{-\alpha_j(h_i)} q_i^{L_i(b_1; Y_0)} &= q_i^{L_i(b_1; Y_2)} \\ \delta_-(b_2; Y_0) &= \delta_-(b_2; Y_1), & \delta_-(b_1; Y_0) &= \delta_-(b_1; Y_2), \\ l(b_2; Y_0) &= l(b_2; Y_1), \\ l(b_1; Y_2) &= \begin{cases} l(b_1; Y_0) + 1 & \text{if } \text{res}(b_1) = \text{res}(b_2), \ p = q + 1, \\ l(b_1; Y_0) & \text{otherwise,} \end{cases} \end{aligned}$$

which yields

$$\frac{\mathfrak{F}_q(T')}{\mathfrak{F}_q(T)} = \begin{cases} q_i^{-\alpha_j(h_i)} \left(\frac{1 - (-q^2)^{l(b_1; Y_0)+1}}{1 - (-q^2)^{l(b_1; Y_0)}} \right) & \text{if } \text{res}(b_1) = \text{res}(b_2), \ p = q + 1, \\ q_i^{-\alpha_j(h_i)} & \text{otherwise.} \end{cases}$$

In the same manner, (5.2) gives

$$\frac{\mathfrak{E}_q(T')}{\mathfrak{E}_q(T)} = \frac{q_j^{-R_j(b_2; Y_2)} \left(\frac{1 - (-q^2)^{l(b_2; Y_2)}}{q} \right)^{\delta_+(b_2; Y_2)} q_i^{-R_i(b_1; Y_3)} \left(\frac{1 - (-q^2)^{l(b_1; Y_3)}}{q} \right)^{\delta_+(b_1; Y_3)}}{q_i^{-R_i(b_1; Y_1)} \left(\frac{1 - (-q^2)^{l(b_1; Y_1)}}{q} \right)^{\delta_+(b_1; Y_1)} q_j^{-R_j(b_2; Y_3)} \left(\frac{1 - (-q^2)^{l(b_2; Y_3)}}{q} \right)^{\delta_+(b_2; Y_3)}}.$$

It follows from (2.5) and the fact that b_2 appears to the left of b_1 in T that

$$\begin{aligned} q_j^{\alpha_i(h_j)} q_j^{-R_j(b_2; Y_2)} &= q_j^{-R_j(b_2; Y_3)}, & q_i^{-R_i(b_1; Y_1)} &= q_i^{-R_i(b_1; Y_3)} \\ \delta_+(b_2; Y_3) &= \delta_+(b_2; Y_2), & \delta_+(b_1; Y_1) &= \delta_+(b_1; Y_3), \\ l(b_1; Y_1) &= l(b_1; Y_3), \\ l(b_2; Y_2) &= \begin{cases} l(b_2; Y_3) + 1 & \text{if } \text{res}(b_1) = \text{res}(b_2), \ p = q + 1, \\ l(b_2; Y_3) & \text{otherwise.} \end{cases} \end{aligned}$$

Since $l(b_1; Y_0) = l(b_2; Y_3)$ when $\text{res}(b_1) = \text{res}(b_2)$ and $p = q + 1$, we have

$$\frac{\mathfrak{E}_q(T')}{\mathfrak{E}_q(T)} = \begin{cases} q_j^{-\alpha_i(h_j)} \left(\frac{1 - (-q^2)^{l(b_1; Y_0)+1}}{1 - (-q^2)^{l(b_1; Y_0)}} \right) & \text{if } \text{res}(b_1) = \text{res}(b_2), \ p = q + 1, \\ q_j^{-\alpha_i(h_j)} & \text{otherwise.} \end{cases}$$

□

For $t \in \mathbb{Z}_{\geq 0}$ and $\underline{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$, we define

$$\langle t \rangle = \prod_{k=1}^t (1 - (-q^2)^k) \quad \text{and} \quad \langle \underline{m} \rangle = \prod_{k=1}^n \langle m_k \rangle.$$

The following corollary tells us that $\frac{\mathfrak{E}_q(T)}{\mathfrak{F}_q(T)}$ is a Laurent polynomial depending only on the proper Young wall $\text{sh}(T)$.

Corollary 5.5 *Let Y be a proper Young wall. For any $T \in \text{ST}(Y)$, we obtain*

$$\frac{\mathfrak{E}_q(T)}{\mathfrak{F}_q(T)} = q^{-\text{def}(\Lambda_0, \Lambda_0 - \text{wt}(Y))} \langle \varpi_Y \rangle.$$

where ϖ_Y is given as (5.3).

Proof It follows from Theorem 4.4 and Proposition 5.4 that

$$\frac{\mathfrak{E}_q(T)}{\mathfrak{F}_q(T)} = \frac{\mathfrak{E}_q(T^Y)}{\mathfrak{F}_q(T^Y)}.$$

Since $c_+(Y) - c_-(Y) = |\varpi_Y|$, the assertion follows from Lemma 5.1 and Theorem 5.3. $\square$

Consider a proper Young wall Y with n blocks. For $T \in \text{ST}(Y)$, set

$$\mathfrak{m}_q(T) = \mathfrak{F}_q(T) / \mathfrak{F}_q(T^Y).$$

Let $w = w(T) \in \mathfrak{S}_n$ and write its reduced expression $w = s_{i_t} \dots s_{i_2} s_{i_1}$, and set $T_0 = T^Y$ and $T_k = s_{i_k} T_{k-1}$ ($k = 1, \dots, t$). Note that $T = w T^Y$ and $T_{k-1} < T_k$. Since

$$\mathfrak{m}_q(T) = \mathfrak{F}_q(T) / \mathfrak{F}_q(T^Y) = \prod_{k=1}^t \mathfrak{F}_q(T_k) / \mathfrak{F}_q(T_{k-1}),$$

$\mathfrak{m}_q(T)$ may be computed directly by considering Theorem 5.3 and Proposition 5.4. Moreover, Proposition 5.4 tells

$$\mathfrak{m}_q(T) = \prod_{k=1}^t \mathfrak{F}_q(T_k) / \mathfrak{F}_q(T_{k-1}) = \prod_{k=1}^t \mathfrak{E}_q(T_k) / \mathfrak{E}_q(T_{k-1}) = \mathfrak{E}_q(T) / \mathfrak{E}_q(T^Y). \quad (5.7)$$

We define

$$\mathfrak{m}_q(Y) = \sum_{T \in \text{ST}(Y)} \mathfrak{m}_q(T). \quad (5.8)$$

It follows from (5.7) that

$$\mathfrak{F}_q(T^Y) \mathfrak{m}_q(Y) = \sum_{T \in \text{ST}(Y)} \mathfrak{F}_q(T), \quad \mathfrak{E}_q(T^Y) \mathfrak{m}_q(Y) = \sum_{T \in \text{ST}(Y)} \mathfrak{E}_q(T). \quad (5.9)$$

Proposition 5.6 *For any $T \in \text{ST}(Y)$, $\mathfrak{m}_q(T)$ is a Laurent polynomial and hence so is $\mathfrak{m}_q(Y)$.*

Proof Since $1 - (-t)^k$ is divided by $1 + t$ for all $k \in \mathbb{Z}_{>0}$, we have

$$\frac{1 - (-q^2)^{l(b;T)+1}}{1 + q^2} \in \mathbb{Z}[q]$$

for all $b \in C_-(Y)$. Thus, it follows from Theorem 5.3 that $\mathfrak{m}_q(T)$ is a Laurent polynomial. $\square$

6 Graded dimension formulas

6.1 Graded dimension formulas

Recall the Fock space representation $\mathcal{F}(\Lambda_0)$ of type X given in Sect. 2.3.

Definition 6.1 For a standard tableau $T \in \text{ST}(Y)$, we define the *residue sequence* of T by

$$\text{res}(T) = (\text{res}_1(T), \text{res}_2(T), \dots, \text{res}_n(T)) \in I^n,$$

where $\text{res}_k(T)$ is the residue of the block with entry k in T .

For $v \in I^n$, we set

$$\text{ST}(Y, v) = \{T \in \text{ST}(Y) \mid \text{res}(T) = v\}.$$

The action (2.8) of Chevalley generators f_i, e_i of the Fock space $\mathcal{F}(\Lambda_0)$ gives the following lemma.

Lemma 6.2 *Let $Y \in \mathcal{Z}(\Lambda_0)$ be a proper Young wall with n blocks and let $v = (v_1, v_2, \dots, v_n) \in I^\beta$. Then, we obtain*

$$\begin{aligned} e_{v_1} e_{v_2} \dots e_{v_n} Y &= \left(\sum_{T \in \text{ST}(Y, v)} \mathfrak{E}_q(T) \right) \emptyset, \\ f_{v_n} f_{v_{n-1}} \dots f_{v_1} \emptyset &= \sum_{Y' \in \mathcal{Z}(\Lambda_0), \text{wt}(Y') = \Lambda_0 - \beta} \left(\sum_{T \in \text{ST}(Y', v)} \mathfrak{F}_q(T) \right) Y', \end{aligned}$$

where $\mathfrak{E}_q(T)$ and $\mathfrak{F}_q(T)$ are the Laurent polynomials defined by (5.2).

We are now ready to prove our main theorem, graded dimension formulas for $R^{\Lambda_0}(\beta)$.

Theorem 6.3 *Let $\beta \in \mathbf{Q}^+$. For $v, v' \in I^\beta$, we have*

$$\begin{aligned} \dim_q e(v)R^{\Lambda_0}(\beta)e(v') &= q^{\text{def}(\Lambda_0, \beta)} \sum_{Y \in \mathcal{Z}(\Lambda_0)} \sum_{T \in \mathbf{ST}(Y, v), T' \in \mathbf{ST}(Y, v')} \mathfrak{E}_q(T) \mathfrak{F}_q(T'), \\ &= \sum_{Y \in \mathcal{Z}(\Lambda_0)} \langle \varpi_Y \rangle \mathfrak{F}_q(T^Y)^2 \sum_{T \in \mathbf{ST}(Y, v), T' \in \mathbf{ST}(Y, v')} \mathfrak{m}_q(T) \mathfrak{m}_q(T'). \end{aligned}$$

Moreover, we obtain

$$\dim_q R^{\Lambda_0}(\beta) = \sum_{Y \in \mathcal{Z}(\Lambda_0), \text{wt}(Y) = \Lambda_0 - \beta} \langle \varpi_Y \rangle \mathfrak{F}_q(T^Y)^2 \mathfrak{m}_q(Y)^2.$$

where ϖ_Y is given as (5.3), and $\mathfrak{E}_q(T)$, $\mathfrak{F}_q(T)$ and $\mathfrak{m}_q(Y)$ are the Laurent polynomials defined in Sect. 5.

Proof Let

$$v = (v_1, v_2, \dots, v_n), \quad v' = (v'_1, v'_2, \dots, v'_n).$$

By Theorem 2.5, the submodule of $\mathcal{F}(\Lambda_0)$ generated by $\emptyset$ is isomorphic to the irreducible highest weight module $V(\Lambda_0)$. Thus, it follows from Proposition 3.3, Corollary 5.5 and Lemma 6.2 that

$$\begin{aligned} &(\dim_q e(v)R^{\Lambda_0}(\beta)e(v')) \emptyset \\ &= q^{\text{def}(\Lambda_0, \beta)} e_{v_1} \cdots e_{v_n} f_{v'_n} \cdots f_{v'_1} \emptyset \\ &= q^{\text{def}(\Lambda_0, \beta)} e_{v_1} \cdots e_{v_n} \sum_{Y \in \mathcal{Z}(\Lambda_0)} \left(\sum_{T' \in \mathbf{ST}(Y, v')} \mathfrak{F}_q(T') \right) Y \\ &= q^{\text{def}(\Lambda_0, \beta)} \sum_{Y \in \mathcal{Z}(\Lambda_0)} \sum_{\substack{T \in \mathbf{ST}(Y, v), \\ T' \in \mathbf{ST}(Y, v')}} \mathfrak{E}_q(T) \mathfrak{F}_q(T') \emptyset \\ &= \sum_{Y \in \mathcal{Z}(\Lambda_0)} \langle \varpi_Y \rangle \sum_{\substack{T \in \mathbf{ST}(Y, v), \\ T' \in \mathbf{ST}(Y, v')}} \mathfrak{F}_q(T) \mathfrak{F}_q(T') \emptyset, \\ &= \sum_{Y \in \mathcal{Z}(\Lambda_0)} \langle \varpi_Y \rangle \mathfrak{F}_q(T^Y)^2 \sum_{T \in \mathbf{ST}(Y, v), T' \in \mathbf{ST}(Y, v')} \mathfrak{m}_q(T) \mathfrak{m}_q(T') \emptyset, \end{aligned}$$

which yields the first assertion.

On the other hand, it follows from (5.9) and $R^{\Lambda_0}(\beta) = \bigoplus_{v, v' \in I^\beta} e(v)R^{\Lambda_0}(\beta)e(v')$ that

$$\begin{aligned}
\dim_q R^{\Lambda_0}(\beta) &= \dim_q \bigoplus_{v, v' \in I^\beta} e(v) R^{\Lambda_0}(\beta) e(v') \\
&= q^{\text{def}(\Lambda_0, \beta)} \sum_{v, v' \in I^\beta} \sum_{Y \in \mathcal{Z}(\Lambda_0)} \sum_{\substack{T \in \text{ST}(Y, v), \\ T' \in \text{ST}(Y, v')}} \mathfrak{E}_q(T) \mathfrak{F}_q(T') \\
&= q^{\text{def}(\Lambda_0, \beta)} \sum_{Y \in \mathcal{Z}(\Lambda_0), \text{wt}(Y) = \Lambda_0 - \beta} \left(\sum_{T \in \text{ST}(Y)} \mathfrak{E}_q(T) \right) \left(\sum_{T' \in \text{ST}(Y)} \mathfrak{F}_q(T') \right) \\
&= q^{\text{def}(\Lambda_0, \beta)} \sum_{Y \in \mathcal{Z}(\Lambda_0), \text{wt}(Y) = \Lambda_0 - \beta} \mathfrak{E}_q(T^Y) \mathfrak{F}_q(T^Y) m_q(Y)^2 \\
&= \sum_{Y \in \mathcal{Z}(\Lambda_0), \text{wt}(Y) = \Lambda_0 - \beta} \langle \varpi_Y \rangle \mathfrak{F}_q(T^Y)^2 m_q(Y)^2,
\end{aligned}$$

which complete the proof. $\square$

6.2 Forgetting the grading

In this subsection, we study the dimensions for $e(v) R^{\Lambda_0}(\beta) e(v')$ and $R^{\Lambda_0}(\beta)$ which were investigated in [1, 2].

Recall the subset $\text{ST}_\infty(Y) \subset \text{ST}(Y)$ defined by (4.4).

Let

$$\text{ST}_\infty(Y, v) = \{T \in \text{ST}_\infty(Y) \mid \text{res}(T) = v\}.$$

The lemmas below follow immediately from the definition of $\text{ST}_\infty(Y)$ and Theorem 5.3.

Lemma 6.4 *For every $T \in \text{ST}(Y) \setminus \text{ST}_\infty(Y)$, there exist a block $b \in C_+(Y)$ with $l(b; T) > 0$. Moreover, if λ_T is a strict partition, there exist also a block $b' \in C_-(Y)$ with $l(b'; T) > 0$.*

Lemma 6.5 *For a standard tableau $T \in \text{ST}(Y)$, we have the following:*

- (1) $\mathfrak{E}_q(T)|_{q=1} = \begin{cases} 2^{c_+(Y)} & \text{if there is no } b \in C_+(Y) \text{ with } l(b; T) > 0, \\ 0 & \text{otherwise.} \end{cases}$
- (2) $\mathfrak{F}_q(T)|_{q=1} = \begin{cases} 2^{c_-(Y)} & \text{if there is no } b \in C_-(Y) \text{ with } l(b; T) > 0, \\ 0 & \text{otherwise.} \end{cases}$

We take an element $d_X \in P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\langle d_X, \Lambda_0 \rangle = 0, \quad \langle d_X, \alpha_i \rangle = \begin{cases} 1 & \text{if } (i = 0, X = A_{2\ell}^{(2)}) \text{ or } (i = 0, \ell, X = D_{\ell+1}^{(2)}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for a proper Young wall Y , $-\langle d_X, \text{wt}(Y) \rangle$ is the number of half-unit height blocks in Y . One can easily check that

$$-\langle d_X, \text{wt}(Y) \rangle - \ell(Y) = c_-(Y) + c_+(Y),$$

where $\ell(Y)$ is the number of columns in Y . Then, Theorem 6.3, Lemma 6.4 and Lemma 6.5 recover the dimension formulas given in [1, Thm. 3.4] and [2, Thm. 3.2] by evaluating the graded dimension formulas given in Theorem 6.3 at $q = 1$.

Theorem 6.6 For $\beta \in \mathbb{Q}^+$ and $v, v' \in I^\beta$, we have

$$\dim e(v)R^{\Lambda_0}(\beta)e(v') = \sum_{Y \in \mathcal{Z}(\Lambda_0)} \sum_{\substack{T \in \mathbf{ST}_\infty(Y, v), \\ T' \in \mathbf{ST}_\infty(Y, v')}} 2^{-(d_X, \text{wt}(Y)) - \ell(Y)} |\mathbf{ST}_\infty(Y, v)| |\mathbf{ST}_\infty(Y, v')|,$$

$$\dim R^{\Lambda_0}(\beta) = \sum_{Y \in \mathcal{Z}(\Lambda_0), \text{wt}(Y) = \Lambda_0 - \beta} 2^{-(d_X, \text{wt}(Y)) - \ell(Y)} |\mathbf{ST}_\infty(Y)|^2.$$

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