Vertex transitive graphs from injective linear mappings

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Abstract Based on a motivation coming from the study of the metric structure of the category of finite dimensional vector spaces over a finite field $\mathbb{F}$, we examine a family of graphs, defined for each pair of integers $1 \leq k \leq n$, with vertex set formed by all injective linear transformations $\mathbb{F}^k \rightarrow \mathbb{F}^n$ and edges corresponding to pairs of mappings, f and g , with $\lambda(f, g) = \dim \operatorname{Im}(f - g) = 1$. For $\mathbb{F} \cong \operatorname{GF}(q)$, this graph will be denoted by $\operatorname{INJ}_q(k, n)$. We show that all such graphs are vertex transitive and Hamiltonian and describe the full automorphism group of each $\operatorname{INJ}_q(k, n)$ for $k < n$. Using the properties of line-transitive groups, we completely determine which of the graphs $\operatorname{INJ}_q(k, n)$ are Cayley and which are not. The Cayley ones consist of three infinite families, corresponding to pairs $(1, n)$, $(n - 1, n)$, and (n, n) , with n and q arbitrary, and of two sporadic examples $\operatorname{INJ}_2(2, 5)$ and $\operatorname{INJ}_2(3, 5)$. Hence, the overwhelming majority of our graphs is not Cayley.

Keywords Vertex transitive graphs · Cayley graphs · Linear mappings

1 Introduction

All finite dimensional vector spaces over a fixed field $\mathbb{F}$, with $\mathbb{F}$ -linear mappings as morphisms, form a category. Given two morphisms $f, g: X \rightarrow Y$ the formula

$$\lambda(f, g) = \dim \operatorname{Im}(f - g) = \operatorname{codim} \operatorname{Ker}(f - g)$$

defines a metric on the set of all linear mappings $X \rightarrow Y$. The system $\{\lambda_{XY}\}$ of all these metrics is a *category metric* on the category of all finite dimensional $\mathbb{F}$ -linear spaces as defined in [7], that is, it satisfies the conditions

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$$\begin{aligned}\lambda_{XZ}(g \circ f_1, g \circ f_2) &\leq \lambda_{XY}(f_1, f_2), \\ \lambda_{XZ}(g_1 \circ f, g_2 \circ f) &\leq \lambda_{YZ}(g_1, g_2),\end{aligned}$$

for any $f, f_1, f_2: X \rightarrow Y$, and any $g, g_1, g_2: Y \rightarrow Z$. For simplicity, the subscripts in λ_{XY} will be omitted in most cases. If we represent linear mappings f and g by their matrices A_f and A_g , respectively, with respect to any pair of bases in X and Y , we see that $\lambda(f, g) = \text{rank}(A_f - A_g)$.

The general notion of a category metric was introduced in [7], where also the metric λ , as one of the particular and rather important examples, was examined. The interest in category metrics comes from the papers [9, 10], where defects of several properties of mappings, for example, injectivity, surjectivity, and homomorphism, in various concrete categories were quantitatively estimated. As shown in [7], category metrics enable a uniform approach to such estimations.

With the help of the category metric λ we can define a class of graphs as follows. Fix a finite field $\mathbb{F} = \text{GF}(q)$ for some prime power q . For any pair of integers k and n such that $1 \leq k \leq n$, we let $\text{INJ}_q(k, n)$ denote the set of all injective linear mappings $\mathbb{F}^k \rightarrow \mathbb{F}^n$. We endow $\text{INJ}_q(k, n)$ with incidence structure by taking the pairs $\{f, g\}$ satisfying $\lambda(f, g) = 1$ as edges. The resulting graph will still be denoted by $\text{INJ}_q(k, n)$. The present paper is devoted to the thorough examinations of the graphs $\text{INJ}_q(k, n)$, especially from the point of view of their symmetries. Note that the graph $\text{INJ}_q(k, n)$ is an induced subgraph of the graph of bilinear forms, consisting of all (not necessarily injective) linear mappings, see [3, 6].

The outline of the paper is as follows. After the preliminary Sect. 2, we list some elementary properties of the graphs $\text{INJ}_q(k, n)$ in Sect. 3. In Sect. 4 we show that with two trivial exceptions, all the graphs are Hamiltonian. The rather technical Sect. 5 is devoted to the study of cliques in our graphs. Its results are utilized in Sect. 6, where the complete description of their full automorphism groups is given, for $k < n$. Using the properties of line-transitive groups, we completely determine in the last Sect. 7 which of the graphs are Cayley graphs. The Cayley ones consist of three infinite families, corresponding to pairs $(1, n)$, $(n - 1, n)$, and (n, n) , with n and q arbitrary, and of two sporadic examples $\text{INJ}_2(2, 5)$ and $\text{INJ}_2(3, 5)$. Hence, the overwhelming majority of our graphs is not Cayley.

The results of the present article are, to a high extent, analogous to those contained in [8], where graphs on sets of injective mappings between finite sets, constructed in a similar way, were investigated. This is in accord with the frequent observation that *finite sets behave much like finite dimensional linear spaces over the “one-element field.”*

2 Preliminaries

In this section, we gather some facts on linear algebra and projective geometry which will be used in the sequel. For basic facts we refer to Artin [1] (our notation closely matches the one in [4]). We restrict ourselves to the case of projective spaces over finite fields.

Let $\mathbb{F} = \text{GF}(q)$ be a field with q elements. We will denote the *general semi-linear group*, *general linear group*, and the *special linear group* of $\mathbb{F}^n$ over $\mathbb{F}$ by $\Gamma\text{L}(n, q)$, $\text{GL}(n, q)$, and $\text{SL}(n, q)$, respectively. Similarly, we denote the *projective*

semilinear group, projective general linear group, and projective special linear group by $\text{P}\Gamma\text{L}(n, q)$, $\text{PGL}(n, q)$, and $\text{PSL}(n, q)$, respectively.

To every linear space U over $\mathbb{F}$ there corresponds the *projective space* (also called the projective geometry) associated with U whose elements are all linear subspaces of U . In the case $U = \mathbb{F}^n$, we denote the associated projective space by $\text{PG}(n - 1, q)$. We denote the set of j -dimensional linear subspaces of $\mathbb{F}^n$ by $\text{Sub}_q(j, n)$.

In subsequent sections, we need the classification of subgroups of $\text{P}\Gamma\text{L}(n, q)$ acting transitively on some $\text{Sub}_q(i, n)$ due to Cameron and Kantor [5, Proposition 8.4].

Theorem 2.1 *Let H be a subgroup of $\text{P}\Gamma\text{L}(n, q)$ acting transitively on $\text{Sub}_q(j, n)$ for some $2 \leq j \leq n - 2$. Then, for every $1 \leq i \leq n - 1$, H acts transitively on $\text{Sub}_q(i, n)$, and one of the following conditions holds:*

1. $\text{PSL}(n, q)$ is a normal subgroup of H ;
2. H is A_7 in $\text{PGL}(4, 2)$;
3. H is $\Gamma\text{L}(1, 2^5)$ in $\text{PGL}(5, 2)$, and $j = 2$ or 3 , in which cases the action of H on $\text{Sub}_2(j, 5)$ is even regular.

For every $n \geq 2$, there exists the so-called Singer cycle, that is, a certain cyclic subgroup of $\text{PGL}(n, q)$ of order $|\text{Sub}_q(1, n)|$. It can be obtained as follows: considering $(\text{GF}(q^n) \setminus \{0\}, \cdot)$ as a subgroup of $\text{GL}(n, q)$, its projection to $\text{PGL}(n, q)$ is a Singer cycle. Singer cycles act regularly both on $\text{Sub}_q(1, n)$ and $\text{Sub}_q(n - 1, n)$.

For any pair of linear mappings $\varphi \in \text{GL}(k, q)$ and $\psi \in \text{GL}(n, q)$, the assignment $f \mapsto \psi \circ f \circ \varphi^{-1}$ defines an automorphism of $\text{INJ}_q(k, n)$. This defines a group homomorphism $\text{GL}(n, q) \times \text{GL}(k, q) \rightarrow \text{Aut}(\text{INJ}_q(k, n))$ with kernel $\{(c \cdot \text{id}_n, c \cdot \text{id}_k); c \in F^* = \mathbb{F} \setminus \{0\}\}$. Representing f by its matrix $A_f = \|a_{ij}\|_{nk}$ with respect to the standard bases in $\mathbb{F}^n$ and $\mathbb{F}^k$, every automorphism ρ of the field $\mathbb{F}$ gives rise to an automorphism of $\text{INJ}_q(k, n)$ defined by $\|a_{ij}\|_{nk} \mapsto \|\rho(a_{ij})\|_{nk}$.

We can combine these two types of automorphisms. Let us denote by $\Gamma\text{L}_\rho(n, q)$ the set of all mappings $\varphi \in \Gamma\text{L}(n, q)$ that are semilinear with respect to the field automorphism ρ of $\mathbb{F}$, and put

$$H_q(k, n) = \bigcup_{\rho \in \text{Aut}(\mathbb{F})} \Gamma\text{L}_\rho(n, q) \times \Gamma\text{L}_\rho(k, q). \quad (1)$$

Then, one can easily verify the following result.

Proposition 2.2 *The set $H_q(k, n)$ is a subgroup of the direct product $\Gamma\text{L}(n, q) \times \Gamma\text{L}(k, q)$ and the mapping $\mathcal{E}: H_q(k, n) \rightarrow \text{Aut}(\text{INJ}_q(k, n))$, given by $\mathcal{E}(\psi, \varphi)(f) = \psi \circ f \circ \varphi^{-1}$, is a group homomorphism with kernel $\text{Ker}(\mathcal{E}) = \{(c \cdot \text{id}_n, c \cdot \text{id}_k); c \in \mathbb{F}^*\}$.*

Automorphisms of $\text{INJ}_q(k, n)$ of the above form will be called *natural*. If $k = n$, then there are also other automorphisms of $\text{INJ}_q(k, n)$, for example, $f \mapsto f^{-1}$, and $\|a_{ij}\|_{nk} \mapsto \|a_{ji}\|_{nk}$, that is, inverse and transpose functions, respectively. In Sect. 7 we show that the homomorphism $\mathcal{E}$ is even surjective, that is, $\text{Aut}(\text{INJ}_q(k, n))$ is isomorphic to a quotient group of $H_q(k, n)$.

By Proposition 2.2 any change of bases both in $\mathbb{F}^k$ and $\mathbb{F}^n$ gives rise to an automorphism of $\text{INJ}_q(k, n)$. As a consequence we obtain the following theorem.

Theorem 2.3 *The graph $\text{INJ}_q(k, n)$ is vertex transitive.*

3 Elementary properties and vertex transitivity

From now on, $\mathbb{F}$ denotes a finite field with q elements, and k and n denote a pair of integers such that $1 \leq k \leq n$. By $\mathbb{F}^n$ we denote the linear space of *columns* of ordered n -tuples of elements of $\mathbb{F}$. Vectors of the standard basis in $\mathbb{F}^n$ will be denoted by $\varepsilon_1, \dots, \varepsilon_n$.

If f and g are adjacent in $\text{INJ}_q(k, n)$, we usually write $f \sim g$; if $f \sim g$ and U is the (necessarily $(k-1)$ -dimensional) linear subspace of $\mathbb{F}^k$ such that $\text{Ker}(f-g) = U$, we write $f \sim_U g$.

In this section, we study elementary properties of these graphs. From the proof of Theorem 9.5.2 of [3], we deduce the order and degree.

Proposition 3.1 *The graph $\text{INJ}_q(k, n)$ has $(q^n - 1) \cdot (q^n - q) \cdots (q^n - q^{k-1})$ vertices and is regular of valency $\frac{q^k - 1}{q - 1}(q^n - q^{k-1} - 1)$.*

We proceed to show that the graph distance dist coincides with the original category metric λ . We start with a technical lemma.

Lemma 3.2 *Let f and g be distinct vertices in $\text{INJ}_q(k, n)$, with $V = \text{Ker}(f-g)$, and let U be a hyperplane of $\mathbb{F}^k$. Then, the following two properties hold:*

1. *If $V \subseteq U$, then there exists a mapping $h \in \text{INJ}_q(k, n)$ which satisfies $\lambda(g, h) = \lambda(f, g) - 1$ and $f \sim_U h$.*
2. *Let $h \in \text{INJ}_q(k, n)$ be arbitrary. If $f \sim_U h$ and $\lambda(h, g) = \lambda(f, g) - 1$, then $V \subseteq U$.*

Proof 1. Let $\alpha_1, \dots, \alpha_k$ be a basis of $\mathbb{F}^k$ such that $\alpha_1, \dots, \alpha_{k-1}$ is a basis of U . All the vectors $\alpha_k + \alpha_1, \dots, \alpha_k + \alpha_{k-1}, \alpha_k$ belong to $\mathbb{F}^k \setminus U$ and form a basis of $\mathbb{F}^k$. Since g is injective, for at least one of them, say α , we have $g(\alpha) \notin f(U)$. Define the mapping $h: \mathbb{F}^k \rightarrow \mathbb{F}^n$ by $h(\alpha_i) = f(\alpha_i)$, $i = 1, \dots, k-1$, and $h(\alpha) = g(\alpha)$.

As $\alpha_1, \dots, \alpha_{k-1}, \alpha$ obviously form a basis of $\mathbb{F}^k$, this h is a well-defined linear mapping. The conditions $\alpha \in \mathbb{F}^k \setminus U$ and $V \subseteq U$ imply that $\lambda(f, h) = 1$ and $\lambda(h, g) = \lambda(f, g) - 1$. Since $h(\alpha) = g(\alpha) \notin f(U)$, the mapping h is injective.

2. Let $l = \dim V = k - \lambda(f, g)$ and $W = \text{Ker}(g-h)$. Then, we have $\dim W = l + 1$ and $\dim(U \cap W) \geq l$. Since $W \cap U \subseteq V$, we have $V \subseteq U$. $\square$

Now, we are in the position to prove the following result.

Proposition 3.3 *For any vertices f and g of $\text{INJ}_q(k, n)$, we have $\text{dist}(f, g) = \lambda(f, g)$.*

Proof We proceed by induction on $i = \lambda(f, g)$.

For $i = 1$, the theorem holds by the definition of $\text{INJ}_q(k, n)$. Since λ is a metric, the inequality $\lambda(f, g) \leq \text{dist}(f, g)$ follows from the triangle inequality by an easy induction argument. Hence, it suffices to show that $\text{dist}(f, g) \leq \lambda(f, g)$. Suppose that the theorem holds for $i - 1$, and take f and g with $\lambda(f, g) = i$. By the previous lemma, there exists a vertex h adjacent to f such that $\lambda(h, g) = i - 1$. Therefore,

$$\text{dist}(f, g) \leq \text{dist}(f, h) + \text{dist}(g, h) = 1 + i - 1 = i = \lambda(f, g),$$

as required. $\square$

Corollary 3.4 *The diameter of $\text{INJ}_q(k, n)$ equals k , the only exceptions being $\text{INJ}_2(1, 1) \cong K_1$, which has diameter 0.*

Proof As $\lambda(f, g) \leq k$ for any $f, g \in \text{INJ}_q(k, n)$, it suffices to find mappings $f, g \in \text{INJ}_q(k, n)$ such that $\lambda(f, g) = k$.

In $\text{INJ}_q(k, k)$ we can take $f = \text{id}_k$ and any linear transformation of $\mathbb{F}^k$ whose minimal polynomial is coprime to $x(x - 1)$ for g . If $k < n$, take any pair f and g from $\text{INJ}_q(k, k)$ with $\lambda(f, g) = k$ take any $h \in \text{INJ}_q(k, n)$ and consider the mappings $h \circ f$ and $h \circ g$.

If $k = 1$, we have a complete graph. $\square$

It is easy to see that Lemma 3.2 can now be strengthened in the following way:

Theorem 3.5 *Let f and g be two mappings from $\text{INJ}_q(k, n)$. Let $l = \lambda(f, g)$ and $W = \text{Ker}(f - g)$. Then, for every sequence $\mathbb{F}^k = W_0 \supseteq W_1 \supseteq \dots \supseteq W_l = W$ of linear subspaces of $\mathbb{F}^k$, such that $\dim W_{i+1} = \dim W_i - 1$ for $i = 0, 1, \dots, l-1$, there exists a path $f = f_0, f_1, \dots, f_l = g$ in $\text{INJ}_q(k, n)$ satisfying condition $\text{Ker}(f_i - f) = W_i$ for $i = 0, 1, \dots, l$.*

Proof Suppose that we already have the mapping f_i . Take a hyperplane U such that $U \cap W_i = W_{i+1}$. By Lemma 3.2, there is a neighbor h of f_i such that $\text{Ker}(f_i - h) = U$ and $\lambda(g, h) + 1 = \lambda(g, f_i)$. From the triangle inequality for λ , it follows that $\text{Ker}(f - h) = W_{i+1}$, and we can put $f_{i+1} = h$. $\square$

Corollary 3.6 *Let $f \in \text{INJ}_q(k, n)$ and let V and W be two linear subspaces of $\mathbb{F}^k$. Then, $V \subseteq W$ if and only if for every $g \in \text{INJ}_q(k, n)$ such that $\text{Ker}(f - g) = V$, there exists a shortest path from g to f containing a vertex h which satisfies $\text{Ker}(f - h) = W$.*

4 Hamiltonian cycles

Recall that a path or cycle in a graph is *Hamiltonian* if it contains all its vertices. A graph is *Hamiltonian* if it has a Hamiltonian cycle. A graph is *Hamiltonian connected* if it is Hamiltonian and every two vertices are connected by a Hamiltonian path. In this section, we prove that apart from three small exceptions, all graphs $\text{INJ}_q(k, n)$ are Hamiltonian connected. The proof is based on an induction argument divided into four lemmas.

Lemma 4.1 *Let Γ be a Hamiltonian connected graph and let $n \geq 3$ be a positive integer. Then, the cartesian product $\Gamma \square K_n$ of Γ and the complete graph on n vertices is Hamiltonian connected, too.*

Proof Let (u, v) and (u', v') be a pair of vertices in $\Gamma \square K_n$.

Let w_1 and w_2 be a pair of vertices in K_n such that $v' \notin \{w_1, w_2\}$ and $v \neq w_2$. Finally, let P , P' , and P'' be Hamiltonian paths in K_n connecting v and w_1 , w_1 and w_2 , and w_2 and v' , respectively.

If $u \neq u'$, then there exists a Hamiltonian path $u = u_1 \sim u_2 \sim \dots \sim u_m = u'$ in Γ . Let $P_1 = \{u_1\} \times P$, let $P_m = \{u_m\} \times P''$ and let $P_i = \{u_i\} \times P'$

for $i \in \{2, 3, \dots, m-1\}$. It is easy to see that $(u, v) = (u_1, v)P_1(u_1, w_2) \sim (u_2, w_2)P_2(u_2, w_1) \sim (u_3, w_1)P_3 \dots P_m(u_m, v') = (u', v')$ is a Hamiltonian path in $\Gamma \square K_n$ connecting (u, v) and (u', v') (note that for even $i < m$ we use P_i in the opposite direction).

Now let $u = u'$. Let us take a Hamiltonian path $u = u_1 \sim u_2 \sim \dots \sim u_m$ in Γ such that $u \sim u_m$. If we replace the path P_1 from the previous case by a Hamiltonian path P'_1 in $\{u_1\} \times K_n \setminus \{v'\}$ between (u_1, v) and (u_1, w_2) , then $(u, v) = (u_1, v)P'_1(u_1, w_2) \sim (u_2, w_2)P_2(u_2, w_1) \sim (u_3, w_1)P_3 \dots P_m(u_m, v') \sim (u', v')$ is a Hamiltonian path in $\Gamma \square K_n$ connecting (u, v) and (u', v') , again. $\square$

For $\gamma \in \mathbb{F}^n \setminus \{0\}$, let $\text{INJ}_q^\gamma(k, n)$ be the subgraph of $\text{INJ}_q(k, n)$ induced by the vertex set $\{f \in \text{INJ}_q(k, n); f(\varepsilon_1) = \gamma\}$. Clearly, the graphs $\text{INJ}_q^\gamma(k, n)$ and $\text{INJ}_q^\delta(k, n)$ are isomorphic for any $\gamma, \delta \in \mathbb{F}^n \setminus \{0\}$.

Lemma 4.2 *If the graph $\text{INJ}_q(k, n)$ is Hamiltonian connected for some k and n , then so is the graph $\text{INJ}_q^\gamma(k+1, n+1)$, for each $\gamma \in \mathbb{F}^n \setminus \{0\}$.*

Proof Without loss of generality we can assume that $\gamma = \varepsilon_1$. Then, the matrix of $f \in \text{INJ}_q^\gamma(k+1, n+1)$ is of the form $A_f = \begin{pmatrix} 1 & \alpha^T \\ 0 & A \end{pmatrix}$, where 1 is the multiplicative identity in $\mathbb{F}$, $\alpha \in \mathbb{F}^n$, 0 is the zero vector in $\mathbb{F}^k$, and A represents some $f' \in \text{INJ}_q(k, n)$. Moreover, every matrix of this form is the matrix of some $f \in \text{INJ}_q^\gamma(k+1, n+1)$.

Take $f, g \in \text{INJ}_q^\gamma(k+1, n+1)$ and let $\begin{pmatrix} 1 & \alpha^T \\ 0 & A \end{pmatrix}, \begin{pmatrix} 1 & \beta^T \\ 0 & B \end{pmatrix}$ be their matrices, where A and B represent f' and $g' \in \text{INJ}_q(k, n)$, respectively. Then, f and g are adjacent if and only if one of the following conditions holds:

- (i) $A = B$ and $\alpha \neq \beta$,
- (ii) $\alpha = \beta$ and f' is adjacent to g' in $\text{INJ}_q(k, n)$,
- (iii) $A \neq B$, $\alpha \neq \beta$, and every row of $A - B$ is a scalar multiple of $\alpha^T - \beta^T$.

Thus, all edges of $\text{INJ}_q^\gamma(k+1, n+1)$ which satisfy (i) or (ii) are the Cartesian product $\text{INJ}_q(k, n) \square K_{q^k}$. Therefore, it is Hamiltonian connected by Lemma 4.1. $\square$

Lemma 4.3 *If $k > 1$ and graph $\text{INJ}_q^\gamma(k, n)$ is Hamiltonian connected, then so is $\text{INJ}_q(k, n)$.*

Proof Take any $h \neq g$ in $\text{INJ}_q(k, n)$. We can assume that $h(\varepsilon_1) \neq g(\varepsilon_1)$. Let us order the set $\mathbb{F}^n \setminus \{0\}$ into a sequence $\gamma_1, \gamma_2, \dots, \gamma_{q^n-1}$ in such a way that $\gamma_1 = h(\varepsilon_1)$ and $\gamma_{q^n-1} = g(\varepsilon_1)$. We claim that there is a sequence h_i, g_i ($i = 1, \dots, q^n-1$) of vertices in $\text{INJ}_q(k, n)$, such that $h_1 = h$, $h_i \neq g_i$ in $\text{INJ}_q^{\gamma_i}(k, n)$, g_i is adjacent to h_{i+1} and $g_{q^n-1} = g$. Taking into account that graphs $\text{INJ}_q^{\gamma_i}(k, n)$ are Hamiltonian connected, this results in a Hamiltonian path from h to g .

To this end, we estimate the number of neighbors of a vertex $f \in \text{INJ}_q^\gamma(k, n)$ belonging to $\text{INJ}_q^\delta(k, n)$, for $\delta \neq \gamma$. We consider three cases.

- Case 1. If $\delta \notin \text{Im}(f)$, then every mapping g defined by $g(\varepsilon_1) = \delta$, $g(\varepsilon_i) = f(\varepsilon_i) + c_i(\gamma - \delta)$, $i = 2, \dots, k$, $c_i \in \mathbb{F}$, belongs to $\text{INJ}_q^\delta(k, n)$ and is adjacent to f . Thus, f has at least q^{k-1} neighbors in $\text{INJ}_q^\delta(k, n)$.

- Case 2. If $\delta = c \cdot \gamma$, $c \neq 0, 1$, then every mapping g defined by $g(\varepsilon_1) = \delta$, $g(\varepsilon_i) = f(\varepsilon_i) + c_i \delta$, $i = 2, \dots, k$, $c_i \in \mathbb{F}$, belongs to $\text{INJ}_q^\delta(k, n)$ and is adjacent to f . Again, f has at least q^{k-1} neighbors in $\text{INJ}_q^\delta(k, n)$.
- Case 3. If $\delta \in \text{Im}(f) \setminus [\gamma]$, then we can assume $\delta = f(\varepsilon_2)$. Every mapping g defined by $g(\varepsilon_1) = \delta$, $g(\varepsilon_2) = \gamma$, $g(\varepsilon_i) = f(\varepsilon_i) + c_i(\gamma - \delta)$, $i = 3, \dots, k$, $c_i \in \mathbb{F}$, belongs to $\text{INJ}_q^\delta(k, n)$ and is adjacent to f . Thus, f has at least q^{k-2} neighbors in $\text{INJ}_q^\delta(k, n)$.

We see that a problem can arise if g_{q^n-2} has only one neighbor in $\text{INJ}_q^{\gamma_{q^n-1}}$. In such a case this neighbor may happen to be g . This is possible only if $q^{k-2} = 1$, that is, $k = 2$. Since $\text{INJ}_q^{\gamma_{q^n-2}}(k, n)$ is Hamiltonian connected, it has at least three distinct vertices, and we can choose g_{q^n-2} to be not adjacent with g in this case too.

As $\text{INJ}_2^\gamma(2, 2) \cong K_2$ is not Hamiltonian, it does not satisfy the assumption of the lemma. Thus, in spite of the fact that $\text{INJ}_2(2, 2) \cong K_{3,3}$ is not Hamiltonian connected, the lemma is still true. $\square$

Lemma 4.4 *The graph $\Gamma = \text{INJ}_2^\gamma(3, 3)$ is Hamiltonian connected.*

Proof By Lemma 4.2, every mapping $f \in \Gamma$ has the matrix of the form $A_f = \begin{pmatrix} 1 & \alpha \\ 0 & A \end{pmatrix}$, where α is a vector from the two dimensional row vector space over $\text{GF}(2)$ and A is a regular 2×2 matrix over $\text{GF}(2)$.

Therefore, vertex set of Γ decomposes into four subsets $\Gamma_\alpha = \{f \in \Gamma; A_f = \begin{pmatrix} 1 & \alpha \\ 0 & A \end{pmatrix}\}$, $\alpha \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ which we call *levels* here. Every level induces a subgraph of Γ isomorphic to the graph $\text{INJ}_2(2, 2)$ via the mapping $\begin{pmatrix} 1 & \alpha \\ 0 & A \end{pmatrix} \mapsto A$.

The isomorphism $\text{INJ}_2(2, 2) \cong K_{3,3}$, induces the partition of $\text{INJ}_2(2, 2)$ onto the sets $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$. Consequently, we have similar partition on each level Γ_α . In accord with the proof of Lemma 4.2, we distinguish three types of edges. Edges of Type 1 change level but do not change partition; edges of Type 2 change partition but do not change level; and edges of Type 3 change both level and partition.

Now take $f \neq g$ in Γ . We can assume that they belong to different levels of Γ (this can be achieved by a single elementary row operation on their matrices).

Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be an ordering of levels of Γ such that $f \in \Gamma_1$ and $g \in \Gamma_4$. We construct a Hamiltonian path from f to g as follows:

1. Starting in f we exhaust Γ_1 by a Hamiltonian path (with all the edges of Type 2).
2. Through an edge of Type 1 we go into Γ_2 and exhaust it by a Hamiltonian path again.
3. Through an edge of Type 1 we go into Γ_3 and exhaust it by a Hamiltonian path again.
4. If f and g belong to the same partition then we go to Γ_4 through an edge of Type 3. Otherwise, we pass to Γ_4 through an edge of Type 1.

5. Now we are in Γ_4 , in a partition not containing g and can make a Hamiltonian path in Γ_4 ending in g .

By taking f adjacent to g , we get a Hamiltonian cycle. $\square$

Next we can prove the main result of this section.

Theorem 4.5 *Every graph $\text{INJ}_q(k, n)$ is Hamiltonian connected, with the exceptions of $\text{INJ}_2(1, 1) \cong K_1$, $\text{INJ}_3(1, 1) \cong K_2$, and $\text{INJ}_2(2, 2) \cong K_{3,3}$. The last graph is Hamiltonian.*

Proof Assume that $\text{INJ}_q(k, n)$ is Hamiltonian connected. By Lemma 4.2 so is $\text{INJ}_q^y(k+1, n+1)$, and by Lemma 4.3 $\text{INJ}_q^y(k+1, n+1)$ is also Hamiltonian connected. This provides us with the necessary induction argument on $k \geq 1$ establishing the desired property for the class $\text{INJ}_q(k, k+l)$ ($l \geq 0$). Thus, it is enough to start the induction.

If $q^{l+1} \geq 4$, we can start with the graph $\text{INJ}_q(1, 1+l) \cong K_{q^{l+1}-1}$. If $q = 3$ and $l = 0$, then by Lemma 4.3 we can start with $\text{INJ}_3^y(2, 2) \cong K_6$. If $q = 2$ and $l = 0$, we start with $\text{INJ}_2^y(3, 3)$ which is Hamiltonian connected by Lemma 4.4. $\square$

5 Cliques

A set of vertices in a graph inducing a complete subgraph are called a *clique*. A *maximal clique* are one that is not properly contained in a larger clique. The property of being a clique is a characteristic one, that is, it is preserved by graph automorphisms. We use cliques in the graph $\text{INJ}_q(k, n)$ to describe its full automorphism group.

For a linear subspace $U \subseteq \mathbb{F}^k$ and a mapping, that is, a vertex $f \in \text{INJ}_q(k, n)$ we denote

$$\{f, U\} = \{g \in \text{INJ}_q(k, n); \text{Ker}(f - g) \supseteq U\}.$$

It can easily be seen that for a hyperplane U , the set $\{f, U\}$ is a clique in $\text{INJ}_q(k, n)$. We list some properties of the sets $\{f, U\}$ which will be used later.

Since $\text{INJ}_q(1, n) \cong K_{q^n-1}$, and the structure of complete graphs is well known, from now on we always assume $k \geq 2$.

In this section we frequently represent linear mappings by their images of some basis, depicted in a self-explanatory table form.

Lemma 5.1 *Let U and V be two distinct hyperplanes of $\mathbb{F}^k$ and let f and g be elements of $\text{INJ}_q(k, n)$ such that $f \sim_U g$. Then, g has at most q neighbors (including f) in $\{f, V\}$. More precisely, g has exactly $q - 1$ neighbors in $\{f, V\}$ whenever $\text{Im}(f - g) \subseteq \text{Im}(f) \setminus f(V)$ and q neighbors otherwise.*

Proof As $\dim U = \dim V = k - 1$ and $\mathbb{F}^k = U + V$, we have $\dim(U \cap V) = k - 2$. Let $\alpha_3, \dots, \alpha_k$ be a basis of $U \cap V$. Then, there are vectors α_1, α_2 such that $\alpha_2, \alpha_3, \dots, \alpha_k$ is a basis of U , $\alpha_1, \alpha_3, \dots, \alpha_k$ is a basis of V , and $\alpha_1, \alpha_2, \dots, \alpha_k$ is a basis of $\mathbb{F}^k$. Representing f and g by a table, any mapping $h \in \{f, V\}$ has the following form (for some $\delta \in \mathbb{F}^n$):

	α_1	α_2	α_3	$\dots$	α_k
f	β_1	β_2	β_3	$\dots$	β_k
g	$\beta_1 + \gamma$	β_2	β_3	$\dots$	β_k
h	β_1	$\beta_2 + \delta$	β_3	$\dots$	β_k

Now, h is adjacent to g if and only if $\delta \in [\gamma]$, so there are at most q possibilities for h .

Note that h is not injective if and only if $\delta = -\beta_2 + \sum_{i \neq 2} c_i \beta_i$ for some scalars c_i . A vector δ of this form is not contained in $[\gamma] = \text{Im}(f - g)$ if and only if $\gamma \notin \text{Im}(f)$ or $\gamma \in f(V)$. $\square$

Now, we are ready to characterize all cliques in $\text{INJ}_q(k, n)$ of maximum size.

Theorem 5.2 *Let U be a hyperplane of $\mathbb{F}^k$ and $f \in \text{INJ}_q(k, n)$. Then, the set $\{f, U\}$ is a maximal clique and it has exactly $q^n - q^{k-1}$ elements. Moreover, if $k < n$, then $\{f, U\}$ is a maximal clique of maximum size, and every clique with $q^n - q^{k-1}$ elements is of the form $\{g, V\}$ for some $g \in \text{INJ}_q(k, n)$ and some hyperplane $V \subseteq \mathbb{F}^k$.*

Proof Take a vector $\alpha \in \mathbb{F}^k \setminus U$. For each $\beta \in \mathbb{F}^n$, there is a unique linear mapping $g: \mathbb{F}^k \rightarrow \mathbb{F}^n$ such that $U \subseteq \text{Ker}(f - g)$ and $g(\alpha) = \beta$. If $\beta \in f(U) = g(U)$, then g is not injective, hence $\{f, U\}$ has $q^n - q^{k-1}$ elements.

By Lemma 5.1, a vertex $g \in \text{INJ}_q(k, n) \setminus \{f, U\}$ has at most q neighbors in $\{f, U\}$. As $k \geq 2$, we have $q^n - q^{k-1} > q$, and the set $\{f, U\} \cup \{g\}$ is not a clique; hence $\{f, U\}$ is maximal.

Let C be a clique in $\text{INJ}_q(k, n)$ containing f such that for at least two hyperplanes V of $\mathbb{F}^k$ we have $C \cap (f, V) \neq \emptyset$. By Lemma 5.1, each of these intersections has at most $q - 1$ elements, so

$$|C| \leq 1 + (q - 1)|\text{Sub}_q(k - 1, k)| = 1 + (q - 1)\frac{q^k - 1}{q - 1} = q^k.$$

Therefore, $k < n$ implies $|C| < |\{f, U\}|$, and the rest readily follows. $\square$

Let U_1, U_2 , and U_3 be hyperplanes in $\mathbb{F}^k$, such that $U_1 \neq U_2$ and $U_1 \neq U_3$. Take $f, g \in \text{INJ}_q(k, n)$ such that $f \sim_{U_1} g$. To the next two lemmas will enable us to infer, from the properties of the cliques $\{f, U_2\}$ and $\{g, U_3\}$, whether or not $U_2 = U_3$.

Let f and g be elements of $\text{INJ}_q(k, n)$ such that $f \sim_U g$.

Lemma 5.3 *Let U and V be two distinct hyperplanes in $\mathbb{F}^k$, and let f and g be elements of $\text{INJ}_q(k, n)$ such that $f \sim_U g$. If $k < n$, then the following two properties hold:*

1. $\{f, V\} \cap \{g, V\} = \emptyset$;
2. there exist $f' \in \{f, V\}$, $g' \in \{g, V\}$ such that $f' \sim_U g'$ and $f' \not\sim_V g$.

Proof 1. If there were a mapping $h \in \text{INJ}_q(k, n)$ such that $f \sim_V h \sim_V g$, then $\text{Ker}(f - g) = U + V = \mathbb{F}^k$ and $f = g$, a contradiction.

2. As in the proof of Lemma 5.1, there is a basis $\alpha_1, \alpha_2, \dots, \alpha_k$ of $\mathbb{F}^k$ such that $\alpha_2, \alpha_3, \dots, \alpha_k$ is a basis of U and $\alpha_1, \alpha_3, \dots, \alpha_k$ is a basis of V . Take linear mappings $f', g': \mathbb{F}^k \rightarrow \mathbb{F}^n$ such that $\text{Ker}(f - f') = \text{Ker}(g - g') = V$ and $f'(\alpha_2) = g'(\alpha_2)$ and consider the table:

	α_1	α_2	α_3	$\dots$	α_k
f	β_1	β_2	β_3	$\dots$	β_k
g	$\beta_1 + \gamma$	β_2	β_3	$\dots$	β_k
f'	β_1	$\beta_2 + \delta$	β_3	$\dots$	β_k
g'	$\beta_1 + \gamma$	$\beta_2 + \delta$	β_3	$\dots$	β_k

If $\beta_2 + \delta \notin [\beta_1, \gamma, \beta_3, \dots, \beta_k]$, then both f' and g' are injective. Now, $f' \not\sim g$ if and only if $\delta \notin [\gamma]$, so there are at least $q^n - q^k - q$ admissible possibilities for δ . However, $q^n - q^k - q > 0$ for $2 \leq k < n$. $\square$

Lemma 5.4 *Let U_1, U_2 , and U_3 be three distinct hyperplanes in $\mathbb{F}^k$ and let $W = U_1 \cap U_2 \cap U_3$. Let f and g be elements of $\text{INJ}_q(k, n)$ such that $f \sim_{U_1} g$. If $k < n$ and $\text{Im}(f) \neq \text{Im}(g)$, then the following holds:*

1. if $\dim W = k - 2$, then $\langle f, U_2 \rangle \cap \langle g, U_3 \rangle \neq \emptyset$;
2. if $\dim W = k - 3$, then $f' \sim g'$ implies $f' \sim g$ for each $f' \in \langle f, U_2 \rangle$, and each $g' \in \langle g, U_3 \rangle$.

Proof 1. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ be a basis of $\mathbb{F}^k$ such that $\alpha_3, \dots, \alpha_k$ is a basis of W , and let $U_1 = [\alpha_1] + W$, $U_2 = [\alpha_2] + W$, and $U_3 = [\alpha_1 + \alpha_2] + W$, and take a linear mapping $h: \mathbb{F}^k \rightarrow \mathbb{F}^n$ defined by the following table:

	α_1	α_2	$\alpha_1 + \alpha_2$	α_3	$\dots$	α_k
f	β_1	β_2	$\beta_1 + \beta_2$	β_3	$\dots$	β_k
g	β_1	$\beta_2 + \gamma$	$\beta_1 + \beta_2 + \gamma$	β_3	$\dots$	β_k
h	$\beta_1 + \gamma$	β_2	$\beta_1 + \beta_2 + \gamma$	β_3	$\dots$	β_k

Now, $\text{Ker}(f - h) = U_2$, $\text{Ker}(g - h) = U_3$, and as $\gamma \notin \text{Im}(f)$ we have $\beta_1 + \gamma \notin [\beta_2, \beta_3, \dots, \beta_k]$, that is, the mapping h is injective.

2. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ be a basis of $\mathbb{F}^k$, such that $\alpha_4, \dots, \alpha_k$ is a basis of W , and let $U_1 = [\alpha_2, \alpha_3] + W$, $U_2 = [\alpha_1, \alpha_3] + W$, and $U_3 = [\alpha_1, \alpha_2] + W$. Take $f' \in \langle f, U_2 \rangle$, $g' \in \langle g, U_3 \rangle$ such that $f' \sim g'$ and consider the table:

	α_1	α_2	α_3	α_4	$\dots$	α_k
f	β_1	β_2	β_3	β_4	$\dots$	β_k
g	$\beta_1 + \gamma$	β_2	β_3	β_4	$\dots$	β_k
f'	β_1	$\beta_2 + \delta$	β_3	β_4	$\dots$	β_k
g'	$\beta_1 + \gamma$	β_2	$\beta_3 + \varepsilon$	β_4	$\dots$	β_k

We have $f' \sim g'$ if and only if $\delta, \varepsilon \in [\gamma]$. This means that $f' \sim g$ and $g' \sim f$, too. $\square$

If $f \sim_U g$ for some hyperplane U in $\mathbb{F}^k$, then $\text{Im}(f - g) = [\gamma]$ for some $\gamma \in \mathbb{F}^n$, and either $\gamma \notin \text{Im } f$ or $\gamma \in \text{Im } f$. Strengthening Lemma 5.1, we show that the problem which of these possibilities actually takes place can be decided on the basis just of the properties of the vertex g in $\text{INJ}_q(k, n)$.

Theorem 5.5 *Let f and g be elements from $\text{INJ}_q(k, n)$ such that for some hyperplane $U \subseteq \mathbb{F}^k$ we have $f \sim_U g$. Take a vector $\gamma \in \mathbb{F}^n$ subject to $\text{Im}(f - g) = [\gamma]$. Then, the following two properties hold:*

1. *If $\gamma \notin \text{Im } f$, then for every hyperplane $V \neq U$ of $\mathbb{F}^k$ the vertex g has exactly q neighbors in $\{f, V\}$.*
2. *If $\gamma \in \text{Im } f$, then there exists a hyperplane $V \neq U$ of $\mathbb{F}^k$ such that g has exactly $q - 1$ neighbors in $\{f, V\}$.*

Proof The first statement was proved in Lemma 5.1.

2. Take $\alpha \in \mathbb{F}^k$ satisfying $f(\alpha) = \gamma$. By Lemma 5.1, it suffices to find a hyperplane $V \neq U$ such that $\alpha \notin V$.

If $\alpha \in U$, then there exists a basis $\alpha_1, \dots, \alpha_k$ of $\mathbb{F}^k$ such that $\alpha_k = \alpha$ and $\alpha_2, \dots, \alpha_k$ is a basis of U . We now see that the hyperplane V generated by the vectors $\alpha_1, \dots, \alpha_{k-1}$ has the required properties.

If $\alpha \notin U$ then there exists a basis $\alpha_1, \dots, \alpha_k$ of $\mathbb{F}^k$ such that $\alpha_k + \alpha_1 = \alpha$ and $\alpha_2, \dots, \alpha_k$ is a basis of U . Again we see that the hyperplane generated by the vectors $\alpha_1, \dots, \alpha_{k-1}$ has the required properties. $\square$

Corollary 5.6 *If $2 \leq k < n$, then the graph $\text{INJ}_q(k, n)$ is not edge transitive.*

Proof Since $2 \leq k < n$, every vertex $f \in \text{INJ}_q(k, n)$ has neighbors g and h such that $\text{Im}(g) = \text{Im}(f) \neq \text{Im}(h)$. Then, $\text{Im}(f - g) \subseteq \text{Im}(f)$ and $\text{Im}(f - h) \not\subseteq \text{Im}(f)$. By Theorem 5.5 no automorphism of $\text{INJ}_q(k, n)$ can map the edge (f, g) to the edge (f, h) . $\square$

6 Automorphism groups

As we have seen in Sect. 3, the graphs $\text{INJ}_q(k, n)$ are vertex transitive, that is, they display a high degree of symmetry. In order to get a deeper insight into their structure, we describe their full automorphism groups for $2 \leq k < n$. It turns out that in all these cases there exist only the natural automorphisms of $\text{INJ}_q(k, n)$. To make this statement precise, let us denote by $PH_q(k, n)$ the subgroup of $\text{PGL}(n, q) \times \text{PGL}(k, q)$ consisting of all the pairs (ψ, φ) induced by the elements of the group $H_q(k, n)$ defined by (1). Additionally, we denote by K the subgroup of $\text{Aut}(\text{INJ}_q(k, n))$ consisting of the automorphisms of the form $f \mapsto cf$ for $c \in \mathbb{F} \setminus \{0\}$.

Theorem 6.1 *Let $2 \leq k < n$. Then, for every automorphism $\Phi \in \text{Aut}(\text{INJ}_q(k, n))$ there exists a unique pair of permutations $\psi \in \text{PGL}(n, q)$, $\varphi \in \text{PGL}(k, q)$ such that for each $f \in \text{INJ}_q(k, n)$, $\alpha \in \mathbb{F}^k$, and $\beta \in \mathbb{F}^n$ we have*

$$f([\alpha]) = [\beta] \text{ if and only if } \Phi(f)(\varphi[\alpha]) = \psi[\beta]. \quad (2)$$

Moreover, the mapping $\mathcal{F}: \text{Aut}(\text{INJ}_q(k, n)) \rightarrow \text{PGL}(n, q) \times \text{PGL}(k, q)$ defined by $\Phi \mapsto (\psi, \varphi)$ is a group homomorphism with kernel K and image $PH_q(k, n)$.

We prove this theorem in a sequence of lemmas.

Lemma 6.2 *Let $\Phi \in \text{Aut}(\text{INJ}_q(k, n))$. Then, there exists a permutation η of hyperplanes of $\mathbb{F}^k$ such that for every $f \in \text{INJ}_q(k, n)$ and every hyperplane $U \subseteq \mathbb{F}^k$ one has*

$$\Phi\{f, U\} = \{\Phi(f), \eta(U)\}.$$

Proof By Lemma 5.2, for every $f \in \text{INJ}_q(k, n)$ there exists a permutation η_f of hyperplanes of $\mathbb{F}^k$, such that Φ maps each clique $\{f, U\}$ onto the clique $\{\Phi(f), \eta_f(U)\}$. We show that η_f is actually independent of f .

Let $f \sim_U g$ be two adjacent mappings with $\text{Im}(f) \neq \text{Im}(g)$. Then $\text{Im } \Phi(f) \neq \text{Im } \Phi(g)$ by Theorem 5.5. Obviously, $f \sim_U g$ implies that $\eta_f(U) = \eta_g(U)$. Hence taking a hyperplane $V \neq U$ in $\mathbb{F}^k$, we get $\eta_f(V) = \eta_g(V)$ by Lemmas 5.3 and 5.4.

If $f \sim g$ and $\text{Im}(f) = \text{Im}(g)$ holds, then f and g have a common neighbor h such that $\text{Im}(f) \neq \text{Im}(h)$. According to the previous paragraph this yields that $\eta_f = \eta_h = \eta_g$.

As $\text{INJ}_q(k, n)$ is connected, $\eta_f = \eta_g$ for any $f, g \in \text{INJ}_q(k, n)$. $\square$

Remark Lemma 6.2 is the main reason for the assumption $k < n$. For $k = n$ there are automorphisms of $\text{INJ}_q(k, n)$ for which the permutations η_f do not exist (for example, $A \mapsto A^T$), or they exist but are different (for example, $f \mapsto f^{-1}$). However, if $k = n$ and for $\Phi \in \text{Aut}(\text{INJ}_q(k, n))$ the permutation η exists, then all the subsequent lemmas still apply to Φ .

Lemma 6.3 *Let Φ and η be as in Lemma 6.2. Then, η can be extended to a collineation φ of $\text{PG}(k-1, q)$ such that the equivalence*

$$[\alpha] \subseteq \text{Ker}(f - g) \text{ if and only if } \varphi[\alpha] \subseteq \text{Ker}(\Phi(f) - \Phi(g))$$

holds for any $[\alpha] \in \text{PG}(k-1, q)$ and $f, g \in \text{INJ}_q(k, n)$.

Proof Fix a mapping $f \in \text{INJ}_q(k, n)$ and a linear subspace $V \subseteq \mathbb{F}^k$. Take a mapping $g \in \text{INJ}_q(k, n)$ such that $\text{Ker}(f - g) = V$, and set $\mathcal{U}(V) = \{U; V \subseteq U \text{ is a hyperplane in } \mathbb{F}^k\}$. By Theorem 3.5, $U \in \mathcal{U}(V)$ if and only if there exists a shortest path from g to f containing a vertex from (f, U) and

$$V = \bigcap_{U \in \mathcal{U}(V)} U.$$

Thus for $V' = \text{Ker}(\Phi(f) - \Phi(g))$, we have $\eta(\mathcal{U}(V)) = \mathcal{U}(V')$.

Since Φ is an automorphism of $\text{INJ}_q(k, n)$, we have

$$\dim \bigcap_{U \in \eta(\mathcal{U}(V'))} U = \dim V'$$

and V' does not depend on g . Hence, $V \mapsto V'$ is a well-defined mapping $\varphi : \text{PG}(k-1, q) \rightarrow \text{PG}(k-1, q)$. Since Φ is a bijection, so is φ . As η does not depend on f , neither does φ .

Now, Theorem 3.5 states that φ preserves the inclusion of subspaces, that is, φ is a collineation of $\text{PG}(k-1, q)$ inducing η and satisfying

$$[\alpha] \subseteq \text{Ker}(f - g) \Leftrightarrow \varphi[\alpha] \subseteq \text{Ker}(\Phi(f) - \Phi(g)).$$

□

Lemma 6.4 *Let Φ and φ be as in Lemma 6.3. Then, there exists a collineation ψ of $\text{PG}(n-1, q)$ such that the equivalence*

$$f([\alpha]) = [\beta] \text{ if and only if } \Phi(f)(\varphi[\alpha]) = \psi[\beta]$$

holds for any $f \in \text{INJ}_q(k, n)$, $[\alpha] \in \text{PG}(k-1, q)$, and $[\beta] \in \text{PG}(n-1, q)$.

Proof Let ϕ be a permutation on $\mathbb{F}^k \setminus \{0\}$ such that for every $\alpha \in \mathbb{F}^k \setminus \{0\}$ we have $\phi(\alpha) \in \varphi[\alpha]$. As the equality $f(\alpha) = g(\alpha)$ means that $\alpha \in \text{Ker}(f - g)$, Lemma 6.3 implies that for every nonzero $\alpha \in \mathbb{F}^k$ there exists a mapping $\psi_\alpha : \mathbb{F}^n \setminus \{0\} \rightarrow \mathbb{F}^n \setminus \{0\}$ such that for any $f \in \text{INJ}_q(k, n)$ the equivalence $f(\alpha) = \beta$ if and only if $\Phi(f)(\phi(\alpha)) = \psi_\alpha(\beta)$ holds. Since Φ and all mappings from $\text{INJ}_q(k, n)$ are injective, ψ_α is a permutation of $\mathbb{F}^n \setminus \{0\}$.

Suppose for a contradiction, that for some $[\alpha] \neq [\alpha']$ there exist $[\beta] \neq [\beta']$ such that $\psi_\alpha(\beta) = \psi_{\alpha'}(\beta')$. Take a mapping $f \in \text{INJ}_q(k, n)$ such that $f(\alpha) = \beta$ and $f(\alpha') = \beta'$. Then, $\Phi(f)(\phi(\alpha)) = \psi_\alpha(\beta) = \psi_{\alpha'}(\beta') = \Phi(f)(\phi(\alpha'))$, contradicting the injectivity of f . Hence the permutation ψ_α induces a permutation ψ'_α of $\text{Sub}_q(1, n)$ which does not depend on α . We can now put $\psi = \psi'_\alpha$.

To show that ψ preserves collinearity, take three distinct collinear points $[\beta_1], [\beta_2], [\beta_3]$ in $\text{PG}(n-1, q)$. Then, there exist distinct collinear points $[\alpha_1], [\alpha_2], [\alpha_3] \in \text{PG}(k-1, q)$ and a mapping $f \in \text{INJ}_q(k, n)$ such that $f(\alpha_i) = \beta_i$ for $i = 1, 2, 3$. The points $\varphi[\alpha_1], \varphi[\alpha_2], \varphi[\alpha_3]$ are also collinear, and from the linearity of f we get that $\psi[\beta_1], \psi[\beta_2], \psi[\beta_3]$ are collinear, as well. □

If Φ , ψ , and φ are as in Lemma 6.4, we write $\mathcal{F}(\Phi) = (\psi, \varphi)$.

Lemma 6.5 *Let $\Phi \in \text{Aut}(\text{INJ}_q(k, n))$ and $\mathcal{F}(\Phi) = (\psi, \varphi)$. Then, the following three statements hold:*

1. *if $\varphi = \text{id}_k$, and $\psi = \text{id}_n$ then there exists a scalar $c \in \mathbb{F}^*$ such that $\Phi(f) = c \cdot f$ for any $f \in \text{INJ}_q(k, n)$;*
2. *if $\psi = \text{id}_n$, then φ is induced by a linear transformation;*
3. *if $\varphi = \text{id}_k$, then ψ is linear.*

Proof 1. Let $f \in \text{INJ}_q(k, n)$ and let α , and α' be two linearly independent vectors in $\mathbb{F}^k$, that is, $[\alpha] \neq [\alpha']$. Set $\beta = f(\alpha)$, $\beta' = f(\alpha')$, $\gamma = \Phi(f)(\alpha)$, and $\gamma' = \Phi(f)(\alpha')$. As $\varphi = \text{id}_k$ and $\psi = \text{id}_n$, we have $d\beta = \gamma$ and $d'\beta' = \gamma'$ for some d , and $d' \in \mathbb{F}^*$.

We have $f(\alpha + \alpha') = \beta + \beta'$ and $\Phi(f)(\alpha + \alpha') = d\beta + d'\beta'$. Since $d\beta + d'\beta' \in [\beta + \beta']$, we get $d = d'$, that is, $d = c_f$ depends at most on f and does not depend on α . In other words, $\Phi(f) = c_f \cdot f$ and it remains to show that c_f does not depend on f . To this end if $f \sim_U g$, then $c_f \cdot f \sim_U c_g \cdot g$, and $c_f = c_g$. Since the graph $\text{INJ}_q(k, n)$ is connected, this proves the statement.

2. Let $f \in \text{INJ}_q(k, n)$. Because $\psi = \text{id}_n$, there exists a mapping $\xi: \mathbb{F}^k \rightarrow \mathbb{F}^k$ such that for $\alpha \in \mathbb{F}^k$ we have $f(\alpha) = \Phi(f)(\xi(\alpha))$. This can be written as $\Phi(f) = \text{id}_n \circ f \circ \xi^{-1}$.

Let $c, c' \in \mathbb{F}$ and $\alpha, \alpha' \in \mathbb{F}^k$. Then

$$\begin{aligned}\Phi(f)(\xi(c\alpha + c'\alpha')) &= f(c\alpha + c'\alpha') = c f(\alpha) + c' f(\alpha') \\ &= c \Phi(f)(\xi(\alpha)) + c' \Phi(f)(\xi(\alpha')) = \Phi(f)(c\xi(\alpha) + c'\xi(\alpha')).\end{aligned}$$

As $\Phi(f)$ is injective, ξ is linear. Since ξ clearly induces φ , this part is proved.

3. If $n > 2$, then ψ is induced by some semilinear transformation $\xi: \mathbb{F}^n \rightarrow \mathbb{F}^n$. It suffices to show that ξ is in fact linear.

Let $c, c' \in \mathbb{F}$ and $\beta, \beta' \in \mathbb{F}^n$ be two linearly independent vectors. Let f be an element of $\text{INJ}_q(k, n)$ such that $\beta, \beta' \in \text{Im}(f)$ and let $\alpha, \alpha' \in \mathbb{F}^k$ be such that $f(\alpha) = \beta$ and $f(\alpha') = \beta'$. Put $\gamma = \Phi(f)(\alpha)$ and $\gamma' = \Phi(f)(\alpha')$.

From the definition of ξ we have

$$\xi(c\beta + c'\beta') \in [c\gamma + c'\gamma'].$$

In particular, there exist $d, d' \in \mathbb{F} \setminus \{0\}$ such that $\xi(\beta) = d\gamma$ and $\xi(\beta') = d'\gamma'$. Without loss of generality we can put $d' = 1$. If we put $c' = 1$, then for some field automorphism ρ we have $\rho(c)d\gamma + \gamma' \in [c\gamma + \gamma']$, that is,

$$\rho(c)d = c,$$

for each $c \in \mathbb{F}$. This is only possible when $d = 1$ and ρ is the identity on $\mathbb{F}$, that is, ξ is linear.

If $n = 2$, then $k = 2$ and ψ is induced by the linear transformation $\Phi(\text{id}_2)$. $\square$

Lemma 6.6 *Let $(\psi, \varphi) \in PH_q(k, n)$ be induced by a pair $(\overline{\psi}, \overline{\varphi}) \in H_q(k, n)$. Then $\mathcal{F}(\mathcal{E}(\overline{\psi}, \overline{\varphi})) = (\psi, \varphi)$, where $\mathcal{E}: H_q(k, n) \rightarrow \text{Aut}(\text{INJ}_q(k, n))$ is the group homomorphism from Lemma 2.2.*

Proof Denote $\Phi = \mathcal{E}((\overline{\psi}, \overline{\varphi}))$, that is, $\Phi(f) = \overline{\psi} \circ f \circ \overline{\varphi}^{-1}$. If $f(\alpha) = \beta$, then

$$\Phi(f)(\overline{\varphi}(\alpha)) = \overline{\psi} \circ f \circ \overline{\varphi}^{-1}(\overline{\varphi}(\alpha)) = \overline{\psi}(f(\alpha)) = \overline{\psi}(\beta).$$

Conversely, if $\Phi(f)(\overline{\varphi}(\alpha)) = \overline{\psi}(\beta)$, then

$$\overline{\psi}(\beta) = \overline{\psi} \circ f \circ \overline{\varphi}^{-1}(\overline{\varphi}(\alpha)) = \overline{\psi}(f(\alpha))$$

and $f(\alpha) = \beta$. Hence $\mathcal{F}(\Phi) = (\psi, \varphi)$. $\square$

Proof of Theorem 6.1 Uniqueness. Let Φ be an automorphism of $\text{INJ}_q(k, n)$ and both pairs (ψ, φ) and (ψ', φ') satisfy condition (2) of Theorem 6.1. Let $\alpha \in \mathbb{F}^k \setminus \{0\}$, $\beta \in \mathbb{F}^n \setminus \{0\}$, and let f and $g \in \text{INJ}_q(k, n)$ be such that $\text{Ker}(f - g) = [\alpha]$ and $f(\alpha) = g(\alpha) = \beta$. Then, $\varphi([\alpha]) = \text{Ker}(\Phi(f) - \Phi(g)) = \varphi'([\alpha])$. Consequently, $\psi([\beta]) = \psi'([\beta'])$, and hence $(\psi, \varphi) = (\psi', \varphi')$.

Homomorphism. Let Φ_1, Φ_2 be automorphisms of $\text{INJ}_q(k, n)$ with $\mathcal{F}(\Phi_1) = (\psi_1, \varphi_1)$ and $\mathcal{F}(\Phi_2) = (\psi_2, \varphi_2)$. By (2) the following conditions are equivalent:

$$\begin{aligned} f([\alpha]) &= [\beta], \\ \Phi_1(f)(\varphi_1([\alpha])) &= \psi_1([\beta]), \\ \Phi_2(\Phi_1(f))(\varphi_2(\varphi_1([\alpha]))) &= \psi_2(\psi_1([\beta])). \end{aligned}$$

Therefore, $\mathcal{F}(\Phi_2 \circ \Phi_1) = (\psi_2 \circ \psi_1, \varphi_2 \circ \varphi_1)$, that is, $\mathcal{F}$ is a group homomorphism.

Kernel. The fact that $\text{Ker}(\mathcal{F}) = \{f \mapsto cf; c \in \mathbb{F} \setminus \{0\}\}$ has already been proved in the first part of Lemma 6.5.

Image. Let Φ be an automorphism of $\text{INJ}_q(k, n)$ with $\mathcal{F}(\Phi) = (\psi, \varphi)$. Since $n > 2$, the mapping ψ is induced by some semilinear transformation $\overline{\psi}: \mathbb{F}^n \rightarrow \mathbb{F}^n$. Take a semilinear transformation $\overline{\varphi'}: \mathbb{F}^k \rightarrow \mathbb{F}^k$ such that $(\overline{\psi}, \overline{\varphi'}) \in H_q(k, n)$. (That is, $\overline{\psi}$ and $\overline{\varphi'}$ are semilinear with a respect to the same field automorphism ρ .) Denote by φ' the collineation induced by $\overline{\varphi'}$. For $\Phi' = \mathcal{E}(\overline{\psi}, \overline{\varphi'})$ we have $\mathcal{F}(\Phi') = (\psi, \varphi')$. By the second part of Lemma 6.5 there exists a collineation ξ induced by a linear transformation $\overline{\xi}$ such that $\mathcal{F}(\Phi \circ \Phi'^{-1}) = (\text{id}_n, \xi)$. Therefore, $\varphi = \xi \circ \varphi'$ is induced by the mapping $\overline{\xi} \circ \overline{\varphi'}$, semilinear with a respect to the same field automorphism as $\overline{\varphi'}$ and $\overline{\psi}$. This proves that $(\psi, \varphi) \in PH_q(k, n)$. $\square$

Realizing that the size of $\text{Aut}(\text{INJ}_q(k, n))$ equals the product of sizes of $\text{Ker}(\mathcal{F})$ and $\text{Im}(\mathcal{F})$, one readily gets the following consequence.

Corollary 6.7 *If $2 \leq k < n$ and $q = p^e$ then*

$$|\text{Aut}(\text{INJ}_q(k, n))| = (q - 1)e |\text{PGL}(k, q)| |\text{PGL}(n, q)|.$$

Corollary 6.8 *If $k < n$, then the mapping $\mathcal{E}: H_q(k, n) \rightarrow \text{Aut}(\text{INJ}_q(k, n))$ from Proposition 2.2 is a surjective group homomorphism, hence*

$$\text{Aut}(\text{INJ}_q(k, n)) \cong H_q(k, n) / \text{Ker } \mathcal{E}.$$

Remark As we pointed out after Lemma 6.2, one can define $\mathcal{F}(\Phi)$ also in the case $k = n$. It is an open question how $\text{Im } \mathcal{F}$ looks like in the case $2 = k = n$.

7 Which of the graphs $\text{INJ}_q(k, n)$ are Cayley?

Let us recall that given a group G with a set $X \subseteq G$ of generators such that $1_G \notin X = X^{-1}$, the (undirected) *Cayley graph* $\text{CG}(G, X)$ has the set of vertices equal to G and the set of edges $\{(g, gx); g \in G, x \in X\}$.

Our main tool, enabling us to decide which of the graphs $\text{INJ}_q(k, n)$ are Cayley and which are not, will be the well-known Sabidussi's result [2]:

Lemma 7.1 *A graph Γ is a Cayley graph if and only if its automorphism group contains a subgroup acting regularly on the vertices of Γ .*

Thus, to answer the above question, it is enough either to find a regular subgroup of the full automorphism group of $\text{INJ}_q(k, n)$ or to show that such a subgroup does not exist.

If $k = 1$, then $\text{INJ}_q(k, n)$ is the complete graph on $q^n - 1$ vertices, and so it is Cayley.

If $k = n$, then $\text{GL}(n, q)$ acts regularly on $\text{INJ}_q(k, n)$ via the action $\psi(f) = \psi \circ f$, for $\psi \in \text{GL}(n, q)$.

So, let us assume that $2 \leq k < n$. By Theorem 6.1, there exists a homomorphism $\mathcal{F}: \text{Aut}(\text{INJ}_q(k, n)) \rightarrow \text{P}\Gamma\text{L}(n, q) \times \text{P}\Gamma\text{L}(k, q)$. Denote by π_1 the projection of the direct product onto $\text{P}\Gamma\text{L}(n, q)$ and put $\mathcal{F}_1 = \pi_1 \circ \mathcal{F}$. One can easily see that for $f \in \text{INJ}_q(k, n)$ for we have

$$\mathcal{F}_1(\Phi)(\text{Im } f) = \text{Im } \Phi(f).$$

This implies the following lemma.

Lemma 7.2 *Let R be a regular subgroup of $\text{Aut}(\text{INJ}_q(k, n))$. Then, $\mathcal{F}_1(R)$ is a subgroup of $\text{P}\Gamma\text{L}(n, q)$ which acts transitively on $\text{Sub}_q(k, n)$ and the order of $\mathcal{F}_1(R)$ divides the order of $\text{INJ}_q(k, n)$.*

In parts 1 and 2 of Theorem 2.1 we have $|H| > |\text{INJ}_q(k, n)|$. It remains to show that for any other choice of q , k , and n (that is, part 3 of Theorem 2.1 and the case $k = n - 1$), the graph $\text{INJ}_q(k, n)$ is Cayley. To this end, the following lemma will be useful.

Lemma 7.3 *Let $\mathbb{F} \cong \text{GF}(q)$ be a finite field, let $2 \leq k < n$, and let H be a subgroup of $\text{PGL}(n, q)$ acting regularly on $\text{Sub}_q(k, n)$. Denote by $\overline{H}$, the preimage of H in $\text{GL}(n, q)$ and take $H' = \overline{H} \times \text{GL}(k, q)$. Then, for $\mathcal{E}$ from Proposition 2.2, the group $\mathcal{E}(H')$ is a regular subgroup of $\text{Aut}(\text{INJ}_q(k, n))$.*

Proof As H acts regularly on the set $\text{Sub}_q(k, n)$, the groups H and $\overline{H}$ have $|\text{Sub}_q(k, n)|$ and $(q - 1) \cdot |\text{Sub}_q(k, n)|$ elements, respectively. Hence the number of elements of H' is

$$(q - 1) \cdot |\text{Sub}_q(k, n)| \cdot |\text{INJ}_q(k, k)| = (q - 1) \cdot |\text{INJ}_q(k, n)|.$$

Since $\text{Ker}(\mathcal{E}) \subseteq H'$, it follows that $|\mathcal{E}(H')| = |\text{INJ}_q(k, n)|$; thus, it suffices to show that H' acts transitively on $\text{INJ}_q(k, n)$.

Take $f, g \in \text{INJ}_q(k, n)$. As H is regular on $\text{Sub}_q(k, n)$, there exists an element $\psi \in \overline{H}$ such that $\psi(\text{Im}(f)) = \text{Im}(g)$. Then, for $h = \psi \circ f$ we have $\text{Im}(h) = \text{Im}(g)$.

Let $\beta_1, \dots, \beta_k$ be a basis of $\text{Im}(g)$. Take two bases $\alpha_1, \dots, \alpha_k$ and $\gamma_1, \dots, \gamma_k$ of $\mathbb{F}^k$ such that $h(\alpha_i) = g(\gamma_i) = \beta_i$, for $i \leq k$, and the linear transformation $\varphi \in \text{GL}(k, \mathbb{F})$ given by $\varphi(\alpha_i) = \gamma_i$. Then, $(\psi, \varphi) \in H'$ and

$$\mathcal{E}(\psi, \varphi)(f) = \psi \circ f \circ \varphi^{-1} = h \circ \varphi^{-1} = g.$$

Thus, H' acts transitively on $\text{INJ}_q(k, n)$. $\square$

By the previous lemma, it suffices to find a subgroup H of $\text{PGL}(n, q)$ which acts regularly on $\text{Sub}_q(k, n)$. If $k = n - 1$, then for H we can take a Singer cycle in the group $\text{PGL}(n, q)$.

If $\mathbb{F} = \text{GF}(2)$, $n = 5$, and $k = 2$ or 3 , then we can take $H = \Gamma\text{L}(1, \text{GF}(2^5))$.

Summing up, we have proved our main result.

Theorem 7.4 *Let $\mathbb{F} \cong \text{GF}(q)$ be a finite field and let $1 \leq k \leq n$. Then, $\text{INJ}_q(k, n)$ is a Cayley graph if and only if one of the following conditions holds:*

1. $\mathbb{F}$ and n are arbitrary, and k equals 1 , $n - 1$ or n ,
2. $\mathbb{F} \cong \text{GF}(2)$, $n = 5$ and $k = 2$ or 3 .

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