Relation graphs of an association scheme based on attenuated spaces

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Abstract The set of subspaces with a given dimension in an attenuated space has a structure of a symmetric association scheme, which is a generalization of both Grassmann schemes and bilinear forms schemes. In this paper, we focus on two families of relation graphs. Their full automorphism groups are completely determined. As a consequence, the classical results of the automorphism groups of Grassmann graphs and bilinear forms graphs are generalized.

Keywords Association scheme · Attenuated space · Relation graph · Full automorphism group

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1 Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. For two non-negative integers n and l , suppose $\mathbb{F}_q^{(n+l)}$ denotes an $(n+l)$ -dimensional row vector space over

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$\mathbb{F}_q$. The set of all matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are non-singular $n \times n$ and $l \times l$ matrices, respectively, forms a group under matrix multiplication, called the *singular general linear group* of degree $n + l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$.

Let P be an m -dimensional subspace of $\mathbb{F}_q^{(n+l)}$, denote also by P an $m \times (n + l)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P . The group $GL_{n+l,n}(\mathbb{F}_q)$ acts on $\mathbb{F}_q^{(n+l)}$ by the vector matrix multiplication. This action induces an action on the set of subspaces of $\mathbb{F}_q^{(n+l)}$, i.e., a subspace P is carried by $T \in GL_{n+l,n}(\mathbb{F}_q)$ into the subspace PT . The vector space $\mathbb{F}_q^{(n+l)}$ together with this group action is called the $(n + l)$ -dimensional *singular linear space* over $\mathbb{F}_q$. This concept was introduced in [12, 13].

For $1 \leq i \leq n + l$, let e_i be the vector in $\mathbb{F}_q^{(n+l)}$ whose i th coordinate is 1 and all other coordinates are 0. Denote by E the l -dimensional subspace of $\mathbb{F}_q^{(n+l)}$ generated by $e_{n+1}, e_{n+2}, \dots, e_{n+l}$. An m -dimensional subspace P of $\mathbb{F}_q^{(n+l)}$ is called a subspace of *type* (m, k) if $\dim(P \cap E) = k$. The collection of all the subspaces of type $(m, 0)$ in $\mathbb{F}_q^{(n+l)}$ is the *attenuated space*, denoted by $A_q(n, l)$.

Denote by X_m the set of all subspaces of type $(m, 0)$ in $\mathbb{F}_q^{(n+l)}$. For $0 \leq i \leq \min\{m, n - m\}$ and $0 \leq j - i \leq \min\{m - i, j, l\}$, let

$$R_{i,j-i} = \{(P, Q) \in X_m \times X_m \mid \dim(P' \cap Q') = m - i, \dim(P \cap Q) = m - j\},$$

where

$$P = \begin{pmatrix} n & l \\ P' & P'' \end{pmatrix}, \quad Q = \begin{pmatrix} n & l \\ Q' & Q'' \end{pmatrix}. \quad (1)$$

Then the configuration

$$\mathfrak{X}_m = (X_m, \{R_{i,j-i} \mid 0 \leq i \leq \min\{m, n-m\}, 0 \leq j-i \leq \min\{m-i, j, l\}\}) \quad (2)$$

is a symmetric association scheme, and its intersection numbers are computed in [12]. Recently, Kurihara [8] computed its character table. This scheme is a common generalization of Grassmann schemes and bilinear forms schemes. We refer readers to [1, 3] for the general theory of association schemes.

If $l = 0$, the relation graph $(X_m, R_{1,0})$ of $\mathfrak{X}_m$ is the Grassmann graph $J_q(n, m)$. In 1949, Chow determined its full automorphism group.

Theorem 1.1 ([4]) *Let $1 < m < n - 1$. Then*

$$\text{Aut}(J_q(n, m)) = \begin{cases} P\Gamma L(n, \mathbb{F}_q), & \text{if } n \neq 2m, \\ P\Gamma L(n, \mathbb{F}_q).2, & \text{if } n = 2m. \end{cases}$$

In [6], Fujisaki et al. determined the full automorphism group of the twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$. This graph has the same spectrum as the Grassmann graph $J_q(2e + 1, e)$; see [9].

The relation graph $(X_m, R_{0,1})$ is $\begin{bmatrix} n \\ m \end{bmatrix}$ copies of the bilinear forms graph $H_q(m, l)$. In particular, if $m = n$, the relation graph $(X_m, R_{0,1})$ is $H_q(m, l)$. In 1951, Hua determined its full automorphism group.

Theorem 1.2 ([7]) *Let n and l be two integers not less than 2. Then*

$$\text{Aut}(H_q(n, l)) = \begin{cases} P\Gamma L(n + l, \mathbb{F}_q)_E, & \text{if } n \neq l, \\ P\Gamma L(n + l, \mathbb{F}_q)_{E,2}, & \text{if } n = l. \end{cases}$$

In this paper, we focus on the two graphs $(X_m, R_{1,0})$ and $(X_m, R_{1,0} \cup R_{0,1})$, and determine their full automorphism groups.

2 The graph $(X_m, R_{1,0})$

Since $(X_1, R_{1,0})$ is the complete multipartite graph $K_{[n]}(q^l)$, its full automorphism group is the wreath product $S_{q^l} \wr S_{[n]}$.

In this section we always assume that $1 < m \leq n - 1$ and $l > 0$. For each integer k with $2 \leq k \leq m$, let $\Gamma^{(k)}$ denote the relation graph $(X_k, R_{1,0})$ of $\mathfrak{X}_k$, where

$$R_{1,0} = \{(P, Q) \in X_k \times X_k \mid \dim(P' \cap Q') = \dim(P \cap Q) = k - 1\}.$$

By the proof of Lemma 3.1 in [12], $(X_k, R_{1,0})$ is an arc-transitive graph.

Note that an m -dimensional subspace P of form (1) in $\mathbb{F}_q^{(n+l)}$ is a vertex of $\Gamma^{(m)}$ if and only if $\text{rank}(P') = m$. Therefore, two vertices P, Q of $\Gamma^{(m)}$ are adjacent if and only if their sum $P + Q$ is a subspace of type $(m + 1, 0)$.

Lemma 2.1 *Let P and Q be two distinct vertices as in (1) of $\Gamma^{(m)}$. If $\dim(P' \cap Q') = m - i$ and $\dim(P \cap Q) = m - j$, then the distance of P and Q is*

$$\partial(P, Q) = \begin{cases} j & \text{if } i > 0, \\ j + 1 & \text{if } i = 0. \end{cases}$$

In particular, the diameter of $\Gamma^{(m)}$ is $m + 1$ if $l \geq m$, or $\min\{m, n - m + l\}$ if $l < m$.

Proof In the Grassmann graph $J_q(n + l, m)$, two vertices x and y are at distance j if and only if $\dim(x \cap y) = m - j$. Since $\Gamma^{(m)}$ is a subgraph of $J_q(n + l, m)$, by

$\dim(P \cap Q) = m - j$ we have $\partial(P, Q) \geq j$. Write $P \cap Q = W = \begin{pmatrix} n & l \\ W' & W'' \end{pmatrix}$ and

$$P = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad Q = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \beta''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \vdots & \vdots \\ \gamma'_i & \gamma''_i \end{pmatrix}. \quad (3)$$

Case 1 $i > 0$.

If $j = i$, write

$$P_1 = \begin{pmatrix} W' & W'' \\ \gamma'_1 & \gamma''_1 \\ \delta'_2 & \delta''_2 \\ \delta'_3 & \delta''_3 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad P_2 = \begin{pmatrix} W' & W'' \\ \gamma'_1 & \gamma''_1 \\ \gamma'_2 & \gamma''_2 \\ \delta'_3 & \delta''_3 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad \dots, \quad P_{j-1} = \begin{pmatrix} W' & W'' \\ \gamma'_1 & \gamma''_1 \\ \gamma'_2 & \gamma''_2 \\ \vdots & \vdots \\ \gamma'_{i-1} & \gamma''_{i-1} \\ \delta'_i & \delta''_i \end{pmatrix},$$

then $(P, P_1, \dots, P_{j-1}, Q)$ is a path of length j from P to Q . Therefore $\partial(P, Q) = j$.

If $j > i$, write

$$P_1 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i-2} & \alpha''_{j-i-2} \\ \alpha'_{j-i-1} & \alpha''_{j-i-1} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix},$$

$$P_2 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i-2} & \alpha''_{j-i-2} \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad \dots, \quad P_{j-i} = \begin{pmatrix} W' & W'' \\ \alpha'_2 & \beta''_2 \\ \vdots & \vdots \\ \alpha'_{j-i-1} & \beta''_{j-i-1} \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix},$$

then $(P, P_1, \dots, P_{j-i})$ is a path of length $j-i$. Since $\dim(P'_{j-i} \cap Q') = \dim(P_{j-i} \cap Q) = m-i$, we have $\partial(P_{j-i}, Q) = i$. It follows that $\partial(P, Q) = j$.

Case 2 $i = 0$.

The neighborhood of P consists of vertices

$$R = \begin{pmatrix} \bar{P}' & \bar{P}'' \\ \xi' & \xi'' \end{pmatrix},$$

where $\bar{P} = (\bar{P}' \ \bar{P}'')$ is an $(m - 1)$ -dimensional subspace of P and $\xi' \in \mathbb{F}_q^{(n)} \setminus P'$.

If $W \subseteq \bar{P}$, then $\dim(R \cap Q) = m - j$ and $\dim\left(\begin{pmatrix} \bar{P}' \\ \xi' \end{pmatrix} \cap Q'\right) = m - 1$. By Case 1, one obtains $\partial(R, Q) = j$. If $W \not\subseteq \bar{P}$, similarly we have $\partial(R, Q) = j + 1$. Hence, $\partial(P, Q) = j + 1$. $\square$

Write

$$M = \begin{pmatrix} m & n - m & l \\ I & 0 & 0 \end{pmatrix} = \begin{pmatrix} n & l \\ M' & M'' \end{pmatrix}.$$

Then M is a vertex of $\Gamma^{(m)}$. Next we shall study the first and second subconstituents of $\Gamma^{(m)}$ with respect to M , denoted by $\Gamma_1(M)$ and $\Gamma_2(M)$, respectively.

Let $\Gamma_{i,j-i}(M)$ be the set of vertices P of $\Gamma^{(m)}$ satisfying $(M, P) \in R_{i,j-i}$. By Lemma 3.1 the second subconstituent $\Gamma_2(M)$ has a partition

$$\Gamma_{0,1}(M) \dot{\cup} \Gamma_{1,1}(M) \dot{\cup} \Gamma_{2,0}(M).$$

For simplicity, write

$$P(U; \alpha, \beta, \gamma) = \begin{pmatrix} m & n - m & l \\ U & 0 & 0 \\ \alpha & \beta & \gamma \end{pmatrix} \begin{matrix} t \\ 1 \end{matrix},$$

$$P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2) = \begin{pmatrix} m & n - m & l \\ U & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix} \begin{matrix} t \\ 1 \\ 1 \end{matrix},$$

where $\text{rank}(U) = t$, $\text{rank}(\begin{smallmatrix} \beta & \gamma \end{smallmatrix}) = 1$ and $\text{rank}\left(\begin{smallmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{smallmatrix}\right) = 2$. Then

$$\begin{aligned} \Gamma_1(M) &= \{P(W; \alpha, \beta, \gamma) \mid \text{rank}(W) = m - 1, \beta \neq 0\}, \\ \Gamma_{0,1}(M) &= \{P(W; \alpha, 0, \gamma) \mid \text{rank}(W) = m - 1, \alpha \notin W, \gamma \neq 0\}, \\ \Gamma_{1,1}(M) &= \{P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \mid \text{rank}(U) = m - 2, \alpha_2 \notin U, \beta_1 \neq 0, \gamma_2 \neq 0\}, \\ \Gamma_{2,0}(M) &= \{P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2) \mid \text{rank}(U) = m - 2, \text{rank}\left(\begin{smallmatrix} \beta_1 \\ \beta_2 \end{smallmatrix}\right) = 2\}. \end{aligned}$$

Lemma 2.2 ([11, Corollary 1.9]) *Let $0 \leq k \leq m \leq n$. Then the number of m -dimensional subspaces containing a given k -dimensional subspace in $\mathbb{F}_q^{(n)}$ is equal to $\begin{bmatrix} n-k \\ m-k \end{bmatrix}$.*

- Lemma 2.3** (i) *Let $P(W; \alpha, 0, \gamma) \in \Gamma_{0,1}(M)$ and $P(W'; \alpha', \beta', \gamma') \in \Gamma_1(M)$. Then the two vertices are adjacent in $\Gamma^{(m)}$ if and only if $W = W'$. In particular, each vertex in $\Gamma_{0,1}(M)$ has $q^{l+1} \begin{bmatrix} n-m \\ 1 \end{bmatrix}$ neighbors in $\Gamma_1(M)$;*
- (ii) *Each vertex in $\Gamma_{1,1}(M)$ has q^2 neighbors in $\Gamma_1(M)$. Moreover, the induced subgraph on these q^2 vertices is the Cartesian product $K_q \square K_q$;*
- (iii) *If $\Gamma_{2,0}(M) \neq \emptyset$, then each vertex in $\Gamma_{2,0}(M)$ has $(q+1)^2$ neighbors in $\Gamma_1(M)$.*

Proof (i) Suppose $P(W; \alpha, 0, \gamma)$ is adjacent to $P(W'; \alpha', \beta', \gamma')$. If $W \neq W'$, by $\beta' \neq 0$ and $\gamma \neq 0$ the dimension of $P(W; \alpha, 0, \gamma) + P(W'; \alpha', \beta', \gamma')$ is $m+2$, a contradiction. The converse is immediate from $\alpha \notin W$ and $\beta' \neq 0$. Therefore, the first statement is valid. Since $P(W; \alpha', \beta', \gamma')$ is of type $(m, 0)$, the rank of $\begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix}$ is m . By Lemma 2.2 the subspace $\begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix}$ with $\beta' \neq 0$ has $q \begin{bmatrix} n-m \\ 1 \end{bmatrix}$ choices. For a given subspace $\begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix}$, there are q^l choices for γ' . Hence, (i) holds.

- (ii) Given a vertex $P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \in \Gamma_{1,1}(M)$. We claim that this vertex is adjacent to $P(W; \alpha, \beta, \gamma) \in \Gamma_1(M)$ if and only if $U \subseteq W$ and $P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$ is a subspace of type $(m+1, 0)$ containing $P(W; \alpha, \beta, \gamma)$. Suppose $P(W; \alpha, \beta, \gamma)$ is adjacent to $P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$. If $U \not\subseteq W$, by $\beta_1 \neq 0$ and $\gamma_2 \neq 0$, the dimension of $P(W; \alpha, \beta, \gamma) + P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$ is at least $m+2$, a contradiction. It follows that $U \subseteq W$. Then

$$\begin{aligned} P(W; \alpha, \beta, \gamma) + P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \\ = P(W; \alpha, \beta, \gamma) + P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2), \end{aligned}$$

and so $P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$ is a subspace of type $(m+1, 0)$ containing $P(W; \alpha, \beta, \gamma)$. The converse is immediate. Hence, our claim is valid.

By Lemma 2.2 the number of $(m-1)$ -subspaces W in $\mathbb{F}_q^{(m)}$ containing U is $q+1$. Observe that $P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$ is of type $(m+1, 0)$ if and only if $W \neq \begin{pmatrix} U \\ \alpha_2 \end{pmatrix}$. Then W has q choices. For a fixed W , by Lemma 2.2 again $P(W; \alpha, \beta, \gamma)$ has q choices, and each vertex is uniquely determined by the subspace (β, γ) . Moreover, the induced subgraph on these q vertices is K_q . Pick two distinct vertices $P(W_1; \alpha, \beta, \gamma)$ and $P(W_2; \alpha', \beta', \gamma')$ in the common neighborhood of M and $P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2)$, $W_1 \neq W_2$. Since the two vertices are adjacent if and only if (β, γ) and (β', γ') are the same subspace. Therefore, (ii) holds.

The proof of (iii) is similar to that of (ii), and will be omitted. $\square$

For a subspace W of type $(m-1, 0)$ in $\mathbb{F}_q^{(n+l)}$, let $C(W)$ be the set of all vertices of $\Gamma^{(m)}$ containing W . For convenience we denote by Δ the induced subgraph of $\Gamma^{(m)}$ on $C(W)$.

Lemma 2.4 *Let W be a subspace of type $(m-1, 0)$ in $\mathbb{F}_q^{(n+l)}$. Then Δ is isomorphic to the complete multipartite graph $K_{[n-m+1]}(q^l)$.*

Proof Let $W = \begin{pmatrix} n & l \\ W' & W'' \end{pmatrix}$. Then $C(W)$ consists of vertices

$$\begin{pmatrix} W' & W'' \\ \alpha' & \alpha'' \end{pmatrix}, \quad (4)$$

where $\text{rank} \begin{pmatrix} W' \\ \alpha' \end{pmatrix} = m$. By Lemma 2.2 the number of m -dimensional subspaces $\begin{pmatrix} W' \\ \alpha' \end{pmatrix}$ in $\mathbb{F}_q^{(n)}$ is $[n-m+1]$. Since α'' has q^l choices, the subgraph Δ has $q^l [n-m+1]$ vertices.

For a given m -dimensional subspace $\begin{pmatrix} W' \\ \alpha' \end{pmatrix}$ in $\mathbb{F}_q^{(n)}$, the vertices of form (4) form an independent set with q^l vertices. Note that all these independent sets form a partition of $C(W)$. Since each vertex in an independent set is adjacent to any vertex in the remaining independent sets, the desired result follows. $\square$

Lemma 2.5 *Let $1 < m \leq n-1$ and $l > 0$ with $(m, l, q) \neq (n-1, 1, 2)$. If an induced subgraph Δ of $\Gamma^{(m)}$ is isomorphic to $K_{[n-m+1]}(q^l)$, then Δ is a subgraph $C(W)$, where W is a subspace of type $(m-1, 0)$ in $\mathbb{F}_q^{(n+l)}$.*

Proof Since $\Gamma^{(m)}$ is arc-transitive, we may assume that Δ contains M . Pick $X \in \Delta \cap \Gamma_1(M)$, and write $M \cap X = W$. Then

$$W = \begin{pmatrix} m & n-m & l \\ W' & 0 & 0 \end{pmatrix}, \quad X = P(W'; \alpha', \beta', \gamma').$$

We claim that $\Delta \cap (\Gamma_{1,1}(M) \cup \Gamma_{2,0}(M)) = \emptyset$. Suppose for the contradiction that this set is not empty.

Case 1. $m \neq n-1$ or $l \neq 1$.

Pick a vertex $P \in \Delta \cap (\Gamma_{1,1}(M) \cup \Gamma_{2,0}(M))$. Then P and M have $q^{l+1} [n-m]$ common neighbors in Δ , a contradiction to Lemma 2.3 (ii), (iii).

Case 2. $m = n-1$ and $l = 1$.

Note that $\Gamma_{2,0}(M) = \emptyset$. Pick a vertex $P \in \Delta \cap \Gamma_{1,1}(M)$. Then the induced subgraph on the common neighbors of M and P is $K_q(q)$, contradicting to Lemma 2.3 (ii).

Hence, our claim is valid. So $\Delta \cap \Gamma_{0,1}(M) \neq \emptyset$. Pick a vertex $P(W''; \alpha'', 0, \gamma'') \in \Delta \cap \Gamma_{0,1}(M)$. Since Δ is a complete multipartite graph, the vertices X and $P(W''; \alpha'', 0, \gamma'')$ are adjacent. By Lemma 2.3 (i), one obtains $W' = W''$. It follows that $\Delta \cap \Gamma_{0,1}(M) \subseteq C(W)$. Similarly, we have $\Delta \cap \Gamma_1(M) \subseteq C(W)$. Hence,

$\Delta \subseteq C(W)$. By Lemma 2.4 the subgraphs Δ and $C(W)$ have the same number of vertices, so $\Delta = C(W)$. $\square$

If $(m, l, q) = (n - 1, 1, 2)$, then $\Gamma_{1,1}(M) \neq \emptyset$. Pick a vertex P from this set. Then the induced subgraph on the common neighbors of P and M together with P and M is not the subgraph $C(W)$. But this subgraph is $K_3(2)$, so Lemma 2.5 does not hold in this special case.

Theorem 2.6 *Let $1 < m \leq n - 1$ and $l > 0$. Then*

$$\text{Aut}(\Gamma^{(m)}) = \begin{cases} \text{Sym}(2^n), & \text{if } (m, l, q) = (n - 1, 1, 2), \\ P\Gamma L(n + l, \mathbb{F}_q)_E, & \text{if } (m, l, q) \neq (n - 1, 1, 2). \end{cases}$$

Proof If $(m, l, q) = (n - 1, 1, 2)$, by [8, Theorem 4.1] the graph $\Gamma^{(n-1)}$ is isomorphic to the graph whose vertices are the lines in the affine space $AG(n, \mathbb{F}_2)$ and two distinct vertices are adjacent if their intersection is not the empty set. Since each line in $AG(n, \mathbb{F}_2)$ consists of two distinct points, $\Gamma^{(n-1)}$ is isomorphic to the Johnson graph $J(2^n, 2)$. Hence, its full automorphism group is the symmetric group $\text{Sym}(2^n)$.

Now suppose $(m, l, q) \neq (n - 1, 1, 2)$. We first prove that the result holds for $m = 2$. Pick any automorphism τ of $\Gamma^{(2)}$. Then τ is a permutation on the set of lines and permutes the points of the attenuated space $A_q(n, l)$. By Deng and Li's result in [5], the automorphism τ can be extended to a collineation fixing E of the projective space $PG(n + l, \mathbb{F}_q)$. By the fundamental theorem of the projective geometry (cf. [2] and [10, Theorem 2.23]), we have $\tau \in P\Gamma L(n + l, \mathbb{F}_q)_E$. Thus, $\text{Aut}(\Gamma^{(2)}) \subseteq P\Gamma L(n + l, \mathbb{F}_q)_E$. On the other hand, $P\Gamma L(n + l, \mathbb{F}_q)_E \subseteq \text{Aut}(\Gamma^{(2)})$. Hence, $\text{Aut}(\Gamma^{(2)}) = P\Gamma L(n + l, \mathbb{F}_q)_E$.

Now let $m \geq 3$ and σ_m be an automorphism of the graph $\Gamma^{(m)}$. By Lemmas 2.4 and 2.5, the automorphism σ_m induces a permutation on the set $\{C(W) \mid W \in X_{m-1}\}$, and further induces a permutation σ_{m-1} on X_{m-1} . For any two vertices W_{m-1}, W'_{m-1} of $\Gamma^{(m-1)}$, we can check that W_{m-1} and W'_{m-1} are adjacent if and only if $C(W_{m-1}) \cap C(W'_{m-1}) \neq \emptyset$. If W_{m-1} and W'_{m-1} are adjacent, then $\sigma_m(C(W_{m-1}) \cap C(W'_{m-1})) = C(\sigma_{m-1}(W_{m-1})) \cap C(\sigma_{m-1}(W'_{m-1})) \neq \emptyset$, which implies that the two vertices $\sigma_{m-1}(W_{m-1})$ and $\sigma_{m-1}(W'_{m-1})$ are adjacent in $\Gamma^{(m-1)}$. Hence, $\sigma_{m-1} \in \text{Aut}(\Gamma^{(m-1)})$.

By induction, for each $3 \leq k \leq m$, the map $f_k : \sigma_k \mapsto \sigma_{k-1}$ is a homomorphism from $\text{Aut}(\Gamma^{(k)})$ to $\text{Aut}(\Gamma^{(k-1)})$. We claim that f_k is injective. Suppose $\sigma_{k-1} = \iota$, the identity permutation on X_{k-1} . For each vertex W_k of $\Gamma^{(k)}$, there exist two vertices W_{k-1} and W'_{k-1} of $\Gamma^{(k-1)}$ such that $W_k = W_{k-1} + W'_{k-1}$. Since $\{W_k\} = C(W_{k-1}) \cap C(W'_{k-1})$, we have $\{\sigma_k(W_k)\} = C(\sigma_{k-1}(W_{k-1})) \cap C(\sigma_{k-1}(W'_{k-1})) = \{W_k\}$. Hence our claim is valid. It follows that $|\text{Aut}(\Gamma^{(m)})| \leq |\text{Aut}(\Gamma^{(2)})|$. Since $P\Gamma L(n + l, \mathbb{F}_q)_E$ is a subgroup of $\text{Aut}(\Gamma^{(m)})$, the desired result follows. $\square$

3 The graph $(X_m, R_{1,0} \cup R_{0,1})$

Let $\Gamma_w^{(m)}$ denote the graph $(X_m, R_{0,1} \cup R_{1,0})$ of the scheme $\mathfrak{X}_m$. Then $\Gamma_w^{(m)}$ is a subgraph of the Grassmann graph $J_q(n + l, m)$. If $l = 0$, $\Gamma_w^{(m)}$ is the Grassmann graph $J_q(n, m)$;

If $m = n$ and $l > 0$, $\Gamma_w^{(m)}$ is the bilinear forms graph $H_q(m, l)$. Note that $\Gamma_w^{(m)}$ and $\Gamma^{(m)}$ have the same vertex set.

Lemma 3.1 *Let P and Q be two vertices of $\Gamma_w^{(m)}$. Then $\partial(P, Q) = m - \dim(P \cap Q)$. In particular, the diameter of $\Gamma_w^{(m)}$ is $\min\{m, n - m + l\}$.*

Proof Suppose $(P, Q) \in R_{i,j-i}$. Then $\dim(P \cap Q) = m - j$. Write $P \cap Q = \begin{pmatrix} n & l \\ W' & W'' \end{pmatrix}$. Then P and Q are of form (3). Write

$$P_1 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \delta'_2 & \delta''_2 \\ \delta'_3 & \delta''_3 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, P_2 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \gamma'_2 & \gamma''_2 \\ \delta'_3 & \delta''_3 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \dots, P_i = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \gamma'_2 & \gamma''_2 \\ \vdots & \vdots \\ \gamma'_{i-1} & \gamma''_{i-1} \\ \gamma'_i & \gamma''_i \end{pmatrix},$$

$$P_{i+1} = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \beta''_1 \\ \alpha'_2 & \beta''_2 \\ \alpha'_3 & \beta''_3 \\ \vdots & \vdots \\ \alpha'_{j-i-1} & \alpha''_{j-i-1} \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \vdots & \vdots \\ \gamma'_i & \gamma''_i \end{pmatrix}, P_{i+2} = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \beta''_1 \\ \alpha'_2 & \beta''_2 \\ \alpha'_3 & \alpha''_3 \\ \vdots & \vdots \\ \alpha'_{j-i-1} & \alpha''_{j-i-1} \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \vdots & \vdots \\ \gamma'_i & \gamma''_i \end{pmatrix}, \dots, P_{j-1} = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \beta''_1 \\ \alpha'_2 & \beta''_2 \\ \alpha'_3 & \beta''_3 \\ \vdots & \vdots \\ \alpha'_{j-i-1} & \beta''_{j-i-1} \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \vdots & \vdots \\ \gamma'_i & \gamma''_i \end{pmatrix}.$$

Then $(P, P_1, \dots, P_{j-1}, Q)$ is a path of length j . Since $\Gamma_w^{(m)}$ is a subgraph of $J_q(n + l, m)$, the distance between P and Q is j . $\square$

Suppose $1 < m \leq n - 1$ and $l > 0$. In order to determine the full automorphism group of $\Gamma_w^{(m)}$, we need to study its maximal cliques. It is direct to check that there are the following three types of maximal cliques:

- (I) The set of m -dimensional subspaces in a given subspace of type $(m + 1, 0)$. Its size is $\begin{bmatrix} m+1 \\ 1 \end{bmatrix}$.
- (II) The set of subspaces of type $(m, 0)$ in a given subspace of type $(m + 1, 1)$. By [13, Lemma 2.2] its size is q^m .
- (III) The set $C(W)$ of subspaces of type $(m, 0)$ containing a given subspace W of type $(m - 1, 0)$. By [13, Lemma 2.3] its size is $q^l \begin{bmatrix} n-m+1 \\ 1 \end{bmatrix}$.

Theorem 3.2 *Let $1 < m \leq n - 1$ and $l > 0$. Then the full automorphism group of the graph $(X_m, R_{0,1} \cup R_{1,0})$ is $P\Gamma L(n + l, \mathbb{F}_q)_E$.*

Proof Let f be an automorphism of the graph $\Gamma_w^{(m)}$. Then f induces a permutation on the set of maximal cliques. Since the sizes of these three types of maximal cliques are pairwise distinct, f induces a permutation on the set of maximal cliques of type III, and further induces a permutation g on the set X_{m-1} such that $f(C(W)) = C(g(W))$, where W is a subspace of type $(m-1, 0)$.

Next, we shall show that g is an automorphism of $\Gamma^{(m-1)}$. For any two vertices W_1 and W_2 of $\Gamma^{(m-1)}$, we can check that W_1 and W_2 are adjacent if and only if $C(W_1) \cap C(W_2) \neq \emptyset$. If W_1 and W_2 are adjacent, then

$$f(C(W_1) \cap C(W_2)) = f(C(W_1)) \cap f(C(W_2)) = C(g(W_1)) \cap C(g(W_2)) \neq \emptyset,$$

which implies that $g(W_1)$ and $g(W_2)$ are adjacent in $\Gamma^{(m-1)}$. So $g \in \text{Aut}(\Gamma^{(m-1)})$.

Similar to the proof of Theorem 2.6, $\text{Aut}(\Gamma_w^{(m)})$ is isomorphic to a subgroup of $\text{Aut}(\Gamma^{(m-1)})$. By Theorem 2.6, the desired result follows. $\square$

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