Semiregular automorphisms of edge-transitive graphs

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Abstract The polycirculant conjecture asserts that every vertex-transitive digraph has a semiregular automorphism: a non-trivial automorphism whose cycles all have the same length. In this paper, we investigate the existence of semiregular automorphisms of edge-transitive graphs. In particular, we show that any regular edge-transitive graph of valency three or four has a semiregular automorphism.

Keywords Edge-transitive graphs · Semiregular automorphism · Polycirculant conjecture

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1 Introduction

All graphs considered in this paper are finite and simple. A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\text{Aut}(\Gamma)$ acting transitively on $V(\Gamma)$. Similarly, Γ is said to be *G-arc-transitive* or *G-edge-transitive* if G acts transitively on the arcs or edges of Γ , respectively. (An *arc* is an ordered pair of adjacent vertices.) Furthermore, a graph is said to be *locally G-arc-transitive* provided that, for every vertex v of Γ , the permutation group $G_v^{\Gamma(v)}$ induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is transitive. When $G = \text{Aut}(\Gamma)$, the prefix G in the above notation is sometimes omitted.

For a group G that acts on a set Ω , we say that it is *semiregular* on Ω if the stabiliser G_ω is trivial for every $\omega \in \Omega$; note that this implies that the action is faithful. A non-identity element $g \in G$ is called *semiregular* (on Ω), provided that the group $\langle g \rangle$ generated by g is semiregular on Ω . Equivalently, a non-identity element $g \in G$ is semiregular provided that $\langle g \rangle$ acts faithfully on Ω and that all of the cycles of the permutation induced by g have the same length.

Existence of a graph automorphism that acts semiregularly on the vertex-set of the graph (we shall refer to such an automorphism simply as a *semiregular automorphism*) is often a desirable feature of a graph, from both theoretical and practical points of view. For example, they give nice representations of the graphs [2], are used in the enumeration of graphs of small order [8] and in the construction of Hamiltonian paths and cycles [1].

In 1981, Marušič conjectured [9] that every vertex-transitive digraph with at least 2 vertices has a semiregular automorphism. Even though the conjecture has been settled for some special classes of vertex-transitive graphs (for example, for vertex-transitive graphs of prime-power order [9] and order that is twice the square of a prime [10], for graphs of valency at most 4 [5, 10], for distance-transitive graphs [7], for locally quasiprimitive arc-transitive graphs [6], and for some other special families of arc-transitive graphs [16, 18]), the complete solution still seems to be beyond our reach.

Recently, the interest in semiregular automorphisms has spread to other areas of combinatorics. For example, Pisanski raised the question [12] whether every flag-transitive configuration of type (v_r) is polycyclic (see [3] for the definition of and basic facts about polycyclic configurations).

This motivates the following graph-theoretical question, whose affirmative answer would imply an affirmative answer to the question of Pisanski (see [11] for an overview of the connection between symmetry properties of combinatorial configurations and graphs).

Question 1.1 *Does every connected, regular, locally arc-transitive graph admit a semiregular automorphism? Moreover, if the graph is bipartite, can the semiregular automorphism be chosen so as to fix each part of the bipartition?*

Note that the word “regular” in Question 1.1 is required as the complete bipartite graph $K_{3,4}$ does not admit a semiregular automorphism. We prove in this paper that the answer to Question 1.1 is positive for regular graphs of valency at most 4. In fact, our results are stated in the slightly more general setting of edge-transitive graphs.

Theorem 1.2 *Let Γ be a connected G -edge-transitive 3-valent graph. Then G contains a semiregular element.*

Formulation of the analogous result for 4-valent graphs requires some definitions. The *wreath graph* $W(n, 2)$ is the lexicographic product of a cycle of length $n \geq 3$ and an edgeless graph on 2 vertices. The *subdivided double* $\mathbb{D}\Lambda$ of a graph Λ is the bipartite graph with parts $E(\Lambda)$ and $V(\Lambda) \times \mathbb{Z}_2$. An edge $e \in E(\Lambda)$ is adjacent in $\mathbb{D}\Lambda$ to a pair $(v, i) \in V(\Lambda) \times \mathbb{Z}_2$ whenever v is an end vertex of e in Λ .

Theorem 1.3 *Let Γ be a connected G -edge-transitive 4-valent graph. Then at least one of the following holds:*

- (1) G contains a semiregular element,
- (2) $\Gamma \cong W(2^n, 2)$ for some $n \geq 2$,
- (3) $\Gamma \cong \mathbb{D}\Lambda$ for some arc-transitive, 4-valent graph Λ of order a power of 2.

Note that in the 3-valent case, we can guarantee that any edge-transitive group contains a semiregular element, while in the 4-valent case, we only claim this for the full automorphism group. In fact, the exceptional cases (2) and (3) in Theorem 1.3 cannot be avoided, as will be shown in Section 4. Note also that the three cases in Theorem 1.3 are not mutually exclusive.

Using Theorems 1.2 and 1.3, it is not difficult to prove the following result, which answers Question 1.1 for graphs of valency at most 4.

Corollary 1.4 *Let Γ be a connected edge-transitive regular graph of valency at most 4. Then Γ admits a semiregular automorphism. Moreover, if Γ is bipartite and locally arc-transitive, the semiregular element can be chosen in such a way as to preserve each part of the bipartition.*

The proofs of Theorems 1.2 and 1.3 and Corollary 1.4 are given in Section 3.

2 Preliminaries

In this section, we prove a few auxiliary results. We start with the following lemma, the proof of which is easy and left to the reader.

Lemma 2.1 *A connected locally G -arc-transitive graph is G -edge-transitive.*

Next, we have the following well-known classification of G -edge-transitive graphs:

Lemma 2.2 *Let Γ be a connected G -edge-transitive graph. Then one of the following occurs:*

- (1) Γ is G -arc-transitive,
- (2) Γ is G -vertex-transitive and $G_v^{\Gamma(v)}$ has two orbits of equal size for every vertex v ,
- (3) Γ is bipartite and the two sets of the bipartition are exactly the orbits of G on $V(\Gamma)$.

Note that if (1) or (3) of Lemma 2.2 occurs, then Γ is locally G -arc-transitive, while if (2) occurs, then Γ is G -half-arc-transitive. Lemma 2.2 also has the following two immediate corollaries.

Corollary 2.3 *Let Γ be a connected G -edge-transitive graph. If there is a path of even length between two vertices, then they are in the same G -orbit.*

Corollary 2.4 *Let Γ be a connected G -edge-transitive graph. Then Γ is either locally G -arc-transitive or it is G -vertex-transitive, and $G_v^{\Gamma(v)}$ has two orbits of equal size (and, in particular, the valency of Γ is even).*

We also need the following folklore lemma:

Lemma 2.5 *Let p be a prime, let Γ be a connected graph and let $G \leq \text{Aut}(\Gamma)$. If, for every $v \in V(\Gamma)$, $|G_v^{\Gamma(v)}|$ is coprime to p , then so is $|G_u|$ for every $u \in V(\Gamma)$.*

Proof We prove this by contradiction. Let p be a prime dividing $|G_u|$ for some $u \in V(\Gamma)$ and let $g \in G_u$ have order p . Since g is non-trivial, it must move some vertex. Let w be a vertex of Γ moved by g at minimal distance from u . By the connectivity of Γ , there is a path $u, u_1, \dots, u_t, w$ such that g fixes each u_i . Then $g \in G_{u_t}$ and g acts non-trivially on $\Gamma(u_t)$. Thus, p divides $|G_{u_t}^{\Gamma(u_t)}|$ and the result follows. $\square$

Lemma 2.6 *Let Γ be a connected G -edge-transitive graph. Let $\{u, v\}$ be an edge of Γ and let N be a normal subgroup of G such that $N_v^{\Gamma(v)}$ is transitive. Let A be the set of vertices a of Γ such that there is a path of even length between u and a . Then N is transitive on A .*

Proof Let $a \in A$. Then there is a path of even length 2ℓ between u and a . We show that $a \in u^N$ by induction on ℓ . This is clearly true if $\ell = 0$. Assume that the statement is true for some $\ell \geq 0$. Let $y \in u^G$ such that there is a path P of length $2\ell + 2$ between u and y . Let x be the vertex before y on P , and let w be the vertex before x . There is a path of length 2ℓ between u and w and hence, by the induction hypothesis, $w \in u^N$. There is also a path of even length between v and x hence, by Corollary 2.3, $x \in v^G$ and hence $N_x^{\Gamma(x)}$ is transitive. It follows that there is an element of N mapping w to y and therefore $y \in u^N$, completing the induction. $\square$

Lemma 2.7 *Let p be a prime and let Γ be a connected G -edge-transitive regular graph. Let u be a vertex of Γ and let A be the set of vertices a of Γ such that there is a path of even length between u and a . Suppose that $|A|$ is a power of p . Then either G contains a semiregular element or Γ is bipartite and there exist p vertices of Γ each two of which have the same neighbourhood.*

Proof Let B be the set of vertices b of Γ such that there is a path of odd length between u and b . If Γ is not bipartite, then $A = B = V(\Gamma)$, while if Γ is bipartite, the parts are A and B and, since Γ is regular, $|A| = |B|$.

Let S be a Sylow p -subgroup of G . By [17, Theorem 3.4'], S is transitive on both A and B . Let z be an element of order p in the centre of S . Then, for each S -orbit, z acts either trivially or semiregularly on it. Since z has order p , it cannot act trivially

on both A and B . If z acts semiregularly on both A and B , then z is a semiregular automorphism contained in G and the result holds. Without loss of generality, we may thus assume that z acts trivially on A and semiregularly on B and hence $A \neq B$ and Γ is bipartite.

Let $b \in B$. All the neighbours of b are contained in A , and thus z fixes all of them. It follows that all the vertices in the $\langle z \rangle$ -orbit of b have the same neighbourhood. Since the order of z is p , it follows that $|b^{\langle z \rangle}| = p$, which completes the proof of the lemma. $\square$

A graph is called *unworthy* if two of its vertices have the same neighbourhood. Unworthy 4-valent edge-transitive graphs were classified in [13].

Lemma 2.8 ([13] Lemma 4.3) *Let Γ be a connected edge-transitive 4-valent unworthy graph. Then one of the following occurs:*

- (1) $\Gamma \cong W(n, 2)$ for some $n \geq 3$,
- (2) $\Gamma \cong \mathbb{D}\Lambda$ for some arc-transitive, 4-valent graph Λ .

Lemma 2.9 ([16] Lemma 2.6) *Let p be an odd prime and let Γ be a connected G -arc-transitive 4-valent graph such that p divides $|\mathcal{V}(\Gamma)|$. If G is solvable, then G contains a semiregular element of order p .*

Let G be a permutation group on a set Ω and K be a normal subgroup of G . Then the partition $\Omega/K = \{\omega^K : \omega \in \Omega\}$ into the set of K -orbits is G -invariant. There is thus a natural induced action of G on Ω/K . Clearly, K is always contained in the kernel of this action, and hence there is an induced action of G/K on Ω/K . The next lemma deals with the situation where the latter action is faithful; this is equivalent to saying that the kernel of the action of G on Ω/K equals K .

Lemma 2.10 ([16] Lemma 2.3) *Let G be a permutation group acting on a set Ω and let K be a normal subgroup of G such that the induced action of G/K on Ω/K is faithful. If G/K contains an element of some order r coprime to $|K|$ that acts semiregularly on Ω/K , then G contains an element of order r acting semiregularly on Ω .*

Let Γ be a graph and let $N \leq \text{Aut}(\Gamma)$. The quotient graph Γ/N is the graph the vertices of which are the N -orbits with two such N -orbits v^N and u^N adjacent whenever there is a pair of vertices $v' \in v^N$ and $u' \in u^N$ that are adjacent in Γ . Clearly, if Γ is connected, then so is Γ/N . We will need a few facts about quotient graphs which we collect in the following lemma (see for example [15, Lemmas 1.4, 1.5, 1.6 and Remark 1.8]).

Lemma 2.11 *Let Γ be a connected locally G -arc-transitive graph and N be a normal subgroup of G . Let K be the kernel of the action of G on N -orbits. Then*

- (1) G/K acts faithfully on Γ/N and Γ/N is locally G/K -arc-transitive, and
- (2) the valency of a vertex v^N in Γ/N divides the number of orbits of $K_v^{\Gamma(v)}$.
- (3) In particular, if $G_v^{\Gamma(v)}$ is 2-transitive and the valency of v^N is greater than 1, then the valency of v^N is the same as the valency of v and $K_v^{\Gamma(v)} = 1$.

(4) If $G_v^{\Gamma(v)}$ is 2-transitive, then $(G/K)_{v^N}^{(\Gamma/N)(v^N)}$ is also 2-transitive.

In the proof of the main results, we will need to consider the action of $\text{Aut}(\Gamma)$ not only on the vertex-set, but also on the edge-set of Γ . Let us state two easy results about this action.

Lemma 2.12 *If Γ is a connected graph with at least 3 vertices, then $\text{Aut}(\Gamma)$ acts faithfully on $E(\Gamma)$.*

The proof of Lemma 2.12 is an easy exercise.

Corollary 2.13 *Let p be a prime and let Γ be a connected graph with at least 3 vertices such that both $|V(\Gamma)|$ and $|E(\Gamma)|$ are powers of p . Let $G \leq \text{Aut}(\Gamma)$ such that G is transitive on both $V(\Gamma)$ and $E(\Gamma)$. Then G contains an element that acts semiregularly on each of $V(\Gamma)$ and $E(\Gamma)$.*

Proof Let S be a Sylow p -subgroup of G . By [17, Theorem 3.4'], S acts transitively on both $V(\Gamma)$ and $E(\Gamma)$. Let z be an element of order p in the centre of S . If z fixes a vertex, then, being central in a vertex-transitive group, it must fix all vertices, which is a contradiction. Similarly, if z fixes an edge, it must fix all edges and, by Lemma 2.12, this is a contradiction. We thus conclude that z is semiregular on both $V(\Gamma)$ and $E(\Gamma)$. $\square$

3 Proofs of the main Theorems

3.1 Proof of Theorem 1.2

Let Γ be a connected, G -edge-transitive, 3-valent graph. We must show that G contains a semiregular element.

Since the valency of Γ is odd, Corollary 2.4 implies that Γ is locally G -arc-transitive. Moreover, since Γ is connected and is 3-valent, it follows by Lemma 2.5 that G_v is a $\{2, 3\}$ -group for every vertex v . An element of G of order a prime greater than 3 is thus necessarily semiregular. We may thus assume that G contains no such element and hence G itself is a $\{2, 3\}$ -group. It follows by Burnside's Theorem that G is solvable. In particular, G contains a non-trivial normal elementary abelian p -group N for some $p \in \{2, 3\}$. If N is semiregular, then any of its non-trivial elements is semiregular, and the conclusion of the theorem holds. We may thus assume that N is not semiregular and hence $N_v^{\Gamma(v)} \neq 1$ for some vertex $v \in V(\Gamma)$. Since $N_v^{\Gamma(v)}$ is a normal subgroup of a transitive permutation group $G_v^{\Gamma(v)}$ of degree 3, it follows that $N_v^{\Gamma(v)}$ is transitive and $p = 3$. Let u be a neighbour of v and let A be the set of vertices a of Γ such that there is a path of even length between u and a . By Lemma 2.6, N is transitive on A . In particular, $|A|$ is a power of 3. In view of Lemma 2.7, we may assume that $\Gamma \cong K_{3,3}$. Since G is edge-transitive, 9 divides $|G|$ and hence G contains the unique Sylow 3-subgroup of $\text{Aut}(\Gamma)$, which is easily seen to contain a semiregular element. This concludes the proof.

3.2 Proof of Theorem 1.3

Let Γ be a connected, G -edge-transitive, 4-valent graph. By way of contradiction, let us assume that (Γ, G) is a counterexample to our theorem, that is G does not contain a semiregular element, Γ is not isomorphic to a wreath graph $W(2^n, 2)$ for any integer $n \geq 2$ and Γ is not isomorphic to $\mathbb{D}\Lambda$ for any arc-transitive 4-valent graph Λ of order a power of 2. We will deduce various facts about such a counterexample before eventually obtaining a contradiction.

Since Γ is 4-valent, it follows that for every $v \in V(\Gamma)$, $G_v^{\Gamma(v)}$ is a subgroup of $\text{Sym}(4)$ and thus a $\{2, 3\}$ -group. Lemma 2.5 then implies that G_v is also a $\{2, 3\}$ -group for every vertex $v \in V(\Gamma)$. An element of G of order a prime greater than 3 is thus necessarily semiregular. Hence, we may assume that G contains no such element. This establishes the following:

Fact 1: G is a $\{2, 3\}$ -group.

If Γ is bipartite, let A and B be the two parts. Otherwise, let $A = B = V(\Gamma)$. By Corollary 2.3, G is transitive on both A and B and $|A| = |B|$. Since G is a $\{2, 3\}$ -group, no prime other than 2 or 3 can divide $|A|$. Suppose that $|A|$ is a power of 2. Then, by Lemma 2.7, either G contains a semiregular element or Γ is unworthy. In the latter case, Lemma 2.8 implies that Γ is isomorphic to either a wreath graph or to $\mathbb{D}\Lambda$ for some arc-transitive, 4-valent graph Λ . Since the order of Γ is a power of 2, each of these cases contradicts our assumption that (Γ, G) is a counterexample. We have thus established the following:

Fact 2: 3 divides $|A|$.

If G_v is a 2-group for every $v \in V(\Gamma)$, then every element of order 3 in G is semiregular, contradicting our assumptions. We may thus assume that there exists a vertex w such that 3 divides $|G_w|$. By Lemma 2.5, there exists a vertex u such that 3 divides $|G_u^{\Gamma(u)}|$. By Corollary 2.4, it follows that Γ is locally G -arc-transitive. In particular, 4 also divides $|G_u^{\Gamma(u)}|$, and hence $G_u^{\Gamma(u)}$ must be 2-transitive. Hence

Fact 3: Γ is locally G -arc-transitive and there exists a vertex u such that $G_u^{\Gamma(u)}$ is 2-transitive.

Since G is a $\{2, 3\}$ -group, Burnside's Theorem implies that G is solvable and thus contains a non-trivial normal elementary abelian p -group N for some $p \in \{2, 3\}$.

Suppose that N is transitive on A or B . Then, since 3 divides $|A|$, $p = 3$ and $|A|$ is a power of 3. In view of Lemma 2.7, we may assume that Γ is bipartite and at least three vertices of Γ have the same neighbourhood. In particular, the graph contains a subgraph isomorphic to $K_{3,4}$. Since Γ is edge-transitive, every edge is contained in such a subgraph and it easily follows that $\Gamma \cong K_{4,4}$, which contradicts the fact that 3 divides $|V(\Gamma)|$. This allows us to assume that

Fact 4: N is intransitive on both A and B .

Together with Lemma 2.6, Fact 4 immediately implies that $N_v^{\Gamma(v)}$ is intransitive for every vertex $v \in V(\Gamma)$. An intransitive normal subgroup of a 2-transitive group is necessarily trivial, hence $N_u^{\Gamma(u)} = 1$. Let v be a neighbour of u . If $N_v^{\Gamma(v)} = 1$ then, by Lemma 2.5, N is semiregular. We may thus assume that $N_v^{\Gamma(v)} \neq 1$.

Thus, $N_v^{\Gamma(v)}$ is a non-trivial intransitive normal elementary abelian p -subgroup of $G_v^{\Gamma(v)}$, which is itself a transitive subgroup of $\text{Sym}(4)$. A simple examination of the

transitive subgroups of $\text{Sym}(4)$ yields that $p = 2$, $G_v^{\Gamma(v)}$ is a 2-group and $N_v^{\Gamma(v)}$ has two orbits. Moreover, since $G_v^{\Gamma(v)} \not\cong G_u^{\Gamma(u)}$, this also implies that Γ is bipartite. We thus have

Fact 5: $p = 2$, $G_v^{\Gamma(v)}$ is a 2-group, $N_u^{\Gamma(u)} = 1$, $N_v^{\Gamma(v)}$ has two orbits of length 2 and Γ is bipartite with parts A and B .

Let us now consider the quotient graph Γ/N . Since Γ is bipartite and N preserves the parts, Γ/N is bipartite with parts A/N and B/N . Since N is a 2-group while $|A| = |B|$ is divisible by 3, it follows that both $|A/N|$ and $|B/N|$ are divisible by 3. Let K be the kernel of the action of G on the N -orbits. By Lemma 2.11 (1), G/K acts faithfully on Γ/N and Γ/N is locally G/K -arc-transitive. We thus have:

Fact 6: Γ/N is a connected locally G/K -arc-transitive bipartite graph with parts having size divisible by 3.

In particular, Γ/N has minimal valency at least 2. On the other hand, since $N_v^{\Gamma(v)}$ has two orbits, the valency of v^N in Γ/N is, by Lemma 2.11 (2), a divisor of 2. In particular, the valency of v^N is exactly 2.

Since $G_u^{\Gamma(u)}$ is 2-transitive, it follows from Lemma 2.11 (3) that the valency of u^N is 4 and that $K_u^{\Gamma(u)} = 1$. Since $G_v^{\Gamma(v)}$ is a 2-group, $K_v^{\Gamma(v)}$ is also a 2-group. By Lemma 2.5, K_v is a 2-group and, since $K = NK_v$, K is also a 2-group. We have thus established the following:

Fact 7: v^N has valency 2, u^N has valency 4 and K is a 2-group.

Suppose that two vertices in the G/K -orbit of v^N have the same neighbourhood. Since $G_u^{\Gamma(u)}$ is 2-transitive, it follows from Lemma 2.11 (4) that $(G/K)_{u^N}^{\Gamma/N(u^N)}$ is 2-transitive and hence all neighbours of u^N have the same neighbourhood. It follows that $\Gamma/N \cong K_{2,4}$, which contradicts the fact that 3 divides the size of the parts of Γ/N . We may thus assume that vertices in the G/K -orbit of v^N have distinct neighbours and hence Γ/N is the subdivision of a connected simple 4-valent graph Λ . We thus have:

Fact 8: Γ/N is the subdivision of a connected simple 4-valent graph Λ of order divisible by 3.

By Lemma 2.12, G/K acts faithfully on Λ . Finally, since Γ/N is locally G/K -arc-transitive, Λ is G/K -arc-transitive, giving us:

Fact 9: G/K acts faithfully on Λ and Λ is G/K -arc-transitive.

Since 3 divides $|\text{V}(\Lambda)|$ and G/K is solvable, it follows from Lemma 2.9 that there exists an element $Kg \in G/K$ of order 3 acting semiregularly on $\text{V}(\Lambda)$. Since 3 is odd, Kg also acts semiregularly on $\text{E}(\Lambda)$ and hence on $\text{V}(\Gamma/N)$. Since 3 is coprime to $|K|$, Lemma 2.10 implies that G contains a semiregular element of order 3. This contradiction concludes the proof.

3.3 Proof of Corollary 1.4

Let Γ be a connected, edge-transitive graph of valency at most 4. Suppose first that $\Gamma \cong W(2^n, 2)$ for some $n \geq 2$. Then there is a natural action of $S_2 \wr D_{2^n}$ as a group of automorphisms of Γ with a system of imprimitivity with blocks of size two. Observe that the element of $S_2 \wr D_{2^n}$ that swaps each pair of elements in a block is semiregular and preserves the parts of the bipartition of Γ . This concludes the proof in this case.

Suppose now that $\Gamma \cong \mathbb{D}\Lambda$ for some arc-transitive, 4-valent graph Λ of order a power of 2. By Corollary 2.13, there exists $g \in \text{Aut}(\Lambda)$ such that g acts semiregularly on each of $V(\Lambda)$ and $E(\Lambda)$. This induces an automorphism of $\Gamma = \mathbb{D}\Lambda$ which maps a vertex (v, i) to the vertex (v^g, i) and a vertex $e \in E(\Lambda)$ to the vertex e^g . This automorphism of Γ is clearly semiregular on $V(\Gamma)$ and preserves the two parts of the bipartition of Γ .

We now assume that we are not in one of the two cases above. By Theorems 1.2 and 1.3, $\text{Aut}(\Gamma)$ contains a semiregular element. If Γ is bipartite and locally arc-transitive, let G be the subgroup of $\text{Aut}(\Gamma)$ preserving the two parts of the bipartition. By Lemma 2.1, Γ is G -edge-transitive and applying Theorems 1.2 and 1.3 again, we see that G contains a semiregular element. This completes the proof of Corollary 1.4.

4 Examples

Lemma 4.1 *Let $n \geq 3$ and let $\Gamma = W(2^n, 2)$ and label the vertices of Γ by $\{1, 1', 2, 2', \dots, 2^n, (2^n)'\}$ such that i and i' both have $\{i-1, (i-1)', i+1, (i+1)'\}$ as their neighbourhood (addition is computed modulo 2^n). Let σ, τ and α_i for $i \in \mathbb{Z}_{2^n}$ be permutations of $V(\Gamma)$ defined by*

$$\begin{aligned}\sigma &= (1\ 3\ 5\ \dots\ 2^n - 1)(0\ 2\ 4\ \dots\ 2^n - 2) \\ &\quad (1'\ 3'\ 5'\ \dots\ (2^n - 1)')(0'\ 2'\ 4'\ \dots\ (2^n - 2)'), \\ \tau &: i \mapsto -i + 4, \quad \tau : i' \mapsto (-i + 4)', \quad \alpha_i = (i\ i').\end{aligned}$$

These are clearly automorphisms of Γ . Let $x = \alpha_1\sigma$ and let $z = \alpha_0\alpha_2\alpha_3 \dots \alpha_{2^n-4}\alpha_{2^n-2}\alpha_{2^n-1}$, that is z is the product of all α_i s with i not congruent to 1 (mod 4). Let $G = \langle x, \tau, z \rangle$. Then G is an edge-regular group of automorphisms of Γ of order 2^{n+2} and every involution of G fixes a vertex.

Proof Clearly, $\langle \sigma, \tau \rangle \cong D_{2^{n-1}}$ and $|\tau| = |z| = 2$. Note that $|\sigma| = 2^{n-1}$ and $x^{2^{n-1}} = \alpha_1\alpha_3\alpha_5 \dots \alpha_{2^n-1}$ hence $|x| = 2^n$. Moreover,

$$\tau x \tau x = \tau \alpha_1 \sigma \tau \alpha_1 \sigma = \tau \alpha_1 \tau \sigma^{-1} \alpha_1 \sigma = \alpha_1^\tau \alpha_1^\sigma = \alpha_3 \alpha_3 = 1,$$

hence $\langle x, \tau \rangle \cong D_{2^n}$.

Note that $z^\tau = \alpha_0\alpha_1\alpha_2 \dots \alpha_{2^n-4}\alpha_{2^n-3}\alpha_{2^n-2}$, that is z^τ is the product of all α_i s with i not congruent to 3 (mod 4). It follows that $\langle z, z^\tau \rangle \cong C_2^2$. Note that $z^\sigma = z^\tau$. It follows that $\langle z, z^\tau \rangle$ is normal in G . Note that $zz^\tau = x^{2^{n-1}}$ and hence $\langle x, \tau \rangle \cap \langle z, z^\tau \rangle = \langle x^{2^{n-1}} \rangle$ and hence $|G| = \frac{|(x, \tau)||\langle z, z^\tau \rangle|}{|\langle x^{2^{n-1}} \rangle|} = 2^{n+2}$.

We now determine representatives for the conjugacy classes of involutions of G . Since $\langle z, z^\tau \rangle$ is normal in G and $\langle x, \tau \rangle \cong D_{2^n}$, it follows that every involution in G is conjugate to an element of one of the following three cosets: $\langle z, z^\tau \rangle$, $\langle z, z^\tau \rangle \tau$ and $\langle z, z^\tau \rangle x \tau$.

The involutions in $\langle z, z^\tau \rangle$ are z, z^τ and $x^{2^{n-1}}$, but z is conjugate to z^τ . The only involutions in $\langle z, z^\tau \rangle \tau$ are τ and $x^{2^{n-1}} \tau$, but τ is conjugate to $x^{2^{n-1}} \tau$ under $x^{2^{n-2}}$. The four

elements of $\langle z, z^\tau \rangle x\tau$ are involutions, but $x\tau$ and $x^{2^{n-1}}x\tau$ are conjugate under $x^{2^{n-2}}$. We have shown that every involution in G is conjugate to one of $z, x^{2^{n-1}}, \tau, x\tau, zx\tau$ or $z^\tau x\tau$.

We show that each of them fixes a vertex of Γ . Clearly, z fixes the vertex 1, while $x^{2^{n-1}}$ and τ both fix the vertex 2. It is not too hard to check that both $x\tau$ and z fix the vertex $2^{n-1} + 1$; hence so does $zx\tau$. Finally, $z^\tau x\tau$ fixes the vertex 1.

We now show that Γ is G -edge-transitive. There is a natural G -invariant partition of $V(\Gamma)$ consisting of the blocks $B_i = \{i, i'\}$ for $i \in \mathbb{Z}_{2^n}$. Clearly $\langle \sigma \rangle$ has two orbits on these blocks and x maps 1 to $3'$. It follows that G has at most two orbits on $V(\Gamma)$. Let u be the vertex 2 and v be the vertex $2^{n-1} + 1$. Note that u and v are in different G -orbits (and thus G has precisely 2 orbits on $V(\Gamma)$). Moreover, G_u contains τ and $x^{2^{n-1}}$, and hence $G_u^{\Gamma(u)}$ is transitive. Similarly, G_v contains $x\tau$ and z and hence $G_v^{\Gamma(v)}$ is transitive. Since G has two orbits on vertices, it follows that Γ is locally G -arc-transitive and hence G -edge-transitive by Lemma 2.1. Since Γ has 2^{n+2} edges, G is actually edge-regular. $\square$

Lemma 4.2 *Let $n \geq 2$, let $\Gamma = W(2^n, 2)$ and label the vertices of Γ by $\{1, 1', 2, 2', \dots, 2^n, (2^n)'\}$ such that i and i' both have $\{i-1, (i-1)', i+1, (i+1)'\}$ as their neighbourhood (addition is computed modulo 2^n). Let t, σ and τ be permutations of $V(\Gamma)$ defined by*

$$\begin{aligned} t &= (1 \ 1')(3 \ 3')(5 \ 5') \dots (2^n - 1 \ (2^n - 1)'), \\ \sigma &= (1 \ 2 \ 3 \ \dots \ 2^n)(1' \ 2' \ 3' \ \dots \ (2^n)'), \\ \tau &: i \mapsto -i + 3, \quad \tau: i' \mapsto (-i + 3)', \end{aligned}$$

again, with addition computed modulo 2^n . Clearly, $t, \sigma, \tau \in \text{Aut}(\Gamma)$.

Let $\Sigma = \mathbb{D}(\Gamma)$ and recall that $V(\Sigma) = E(\Gamma) \cup (V(\Gamma) \times \mathbb{Z}_2)$ and $e \in E(\Gamma)$ is adjacent to (v, i) in Σ if v is incident with e in Γ . For $v \in V(\Gamma)$, let α_v be the automorphism of Σ interchanging $(v, 0)$ and $(v, 1)$ and fixing all other vertices of Σ . Further, there is a natural embedding of $\text{Aut}(\Gamma)$ in $\text{Aut}(\Sigma)$ by letting $g \in \text{Aut}(\Gamma)$ map a vertex e in $E(\Gamma)$ to e^g and a vertex (v, i) in $V(\Gamma) \times \mathbb{Z}_2$ to (v^g, i) . Note that the group generated by $\text{Aut}(\Gamma)$ and $\{\alpha_v \mid v \in V(\Gamma)\}$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \text{Aut}(\Gamma)$.

Let $\omega = \alpha_1 \alpha_3 \dots \alpha_{2^n-1}$, let $x = \alpha_1 \alpha_{1'} \sigma$ and let $z = \omega t$. Let $G = \langle x, \tau, z \rangle$. Then G is an edge-transitive group of automorphisms of Σ of order 2^{n+5} such that every involution fixes a vertex.

Proof Note that $|\tau| = 2$, $|\sigma| = 2^n$ and $x^{2^n} = \alpha_1 \alpha_{1'} \alpha_2 \alpha_{2'} \dots \alpha_{2^n} \alpha_{(2^n)'}; hence $|x| = 2^{n+1}$. Note that $\sigma\tau$ is an involution fixing both 1 and $1'$; hence $\sigma\tau$ commutes with both α_1 and $\alpha_{1'}$ and $(x\tau)^2 = 1$. It follows that $\langle x, \tau \rangle \cong D_{2^{n+1}}$. Observe that $z^2 = \alpha_1 \alpha_{1'} \alpha_3 \alpha_{3'} \dots \alpha_{2^n-1} \alpha_{(2^n-1)'}; hence $|z| = 4$. Note also that $(z^\tau)^2 = \alpha_2 \alpha_{2'} \alpha_4 \alpha_{4'} \dots \alpha_{2^n} \alpha_{(2^n)'}; hence $x^{2^n} = z^2 (z^\tau)^2$.$$$

Note that t commutes with both t^τ and ω^τ . Since τ is an involution, it follows that t^τ commutes with ω . Thus

$$zz^\tau = \omega t \omega^\tau t^\tau = \omega \omega^\tau t^\tau t = \omega^\tau t^\tau \omega t = z^\tau z.$$

This implies that $\langle z, z^\tau \rangle \cong C_4^2$. Observe that $\alpha_1 \alpha_{1'}$ commutes with both t and t^τ and hence $\alpha_1 \alpha_{1'}$ commutes with both z and z^τ . In particular, $z^x = z^\sigma = \omega^\sigma t^\sigma = \omega^\tau t^\tau = z^\tau$ and $z^{\tau x} = z^{\tau\sigma} = z^{\sigma\tau^{-1}} = z^{\tau\tau^{-1}} = z$. It follows that $\langle z, z^\tau \rangle$ is normal in G . Clearly, $\langle x, \tau \rangle \cap \langle z, z^\tau \rangle = \langle x^{2^n} \rangle$ and hence $|G| = \frac{|\langle x, \tau \rangle| |\langle z, z^\tau \rangle|}{|\langle x^{2^n} \rangle|} = 2^{n+5}$.

We now determine representatives for the conjugacy classes of involutions of G . Since $\langle z, z^\tau \rangle$ is normal in G and $\langle x, \tau \rangle \cong D_{2^{n+1}}$, it follows that every involution in G is conjugate to an element of one of the following three cosets: $\langle z, z^\tau \rangle$, $\langle z, z^\tau \rangle \tau$ and $\langle z, z^\tau \rangle x \tau$.

Since $\langle z, z^\tau \rangle \cong C_4^2$, the only involutions in $\langle z, z^\tau \rangle$ are z^2 , $(z^\tau)^2$ and x^{2^n} , but z^2 is conjugate to $(z^\tau)^2$ under τ . The only involutions in $\langle z, z^\tau \rangle \tau$ are τ , $z(z^\tau)^3 \tau$, $z^3 z^\tau \tau$ and $x^{2^n} \tau$, but τ is conjugate to $x^{2^n} \tau$ under $x^{2^{n-1}}$, to $z(z^\tau)^3 \tau$ under z^τ and to $z^3 z^\tau \tau$ under z . Similarly, the only involutions in $\langle z, z^\tau \rangle x \tau$ are $x \tau$, $z^2 x \tau$, $(z^\tau)^2 x \tau$ and $x^{2^n} x \tau$, but $x \tau$ and $x^{2^n} x \tau$ are conjugate under $x^{2^{n-1}}$. We have shown that every involution in G is conjugate to one of z^2 , x^{2^n} , τ , $x \tau$, $z^2 x \tau$, or $(z^\tau)^2 x \tau$.

We show that each of them fixes a vertex of Σ . By definition, z^2 and x^{2^n} fix all the vertices of Σ corresponding to $E(\Gamma)$. Similarly, τ fixes the edge $\{1, 2\}$ of Γ and hence fixes the corresponding vertex of Σ . It is not hard to check that $x \tau$ fixes the vertex $((2^{n-1} + 1), 0)$ of Σ . Since $(z^\tau)^2$ also fixes $((2^{n-1} + 1), 0)$, so does $(z^\tau)^2 x \tau$. Finally, $z^2 x \tau$ fixes $(1, 0)$.

We now show that G is transitive on edges of Σ . Clearly, $\langle \sigma, t \rangle$ acts transitively on $V(\Gamma)$. Moreover, the group $\langle t^\tau, \sigma \tau \rangle$ fixes the vertex 1 and acts transitively on the set $\{2, 2', 2^n, (2^n)'\}$ of neighbours of 1 in Γ . Hence Γ is $\langle \sigma, t, \tau \rangle$ -arc-transitive. This implies that G is transitive on $E(\Gamma)$. Let $e = \{1, 2\}$ viewed as a vertex of Σ . The group $\langle \tau, x^{2^n} \rangle$ fixes e and acts transitively on the set $\{(1, 0), (1, 1), (2, 0), (2, 1)\}$ of neighbours of e in Σ . Since every edge of Σ has one endpoint in $E(\Gamma)$, G is transitive on $E(\Sigma)$. $\square$

Note that the examples in Lemma 4.2 are not the only ones that show that case (3) in Theorem 1.3 cannot be avoided. In fact, Λ does not even have to be a wreath graph. For example, if Λ is the graph called C4[16, 2] or C4[32, 3] in [14], then Λ has order 16 or 32, respectively, and a MAGMA [4] computation shows that $\mathbb{D}\Lambda$ admits many edge-transitive groups of automorphisms without a semiregular element.

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