Positivity and tameness in rank 2 cluster algebras

Kyungyong Lee · Li Li · Andrei Zelevinsky

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Abstract We study the relationship between the positivity property in a rank 2 cluster algebra, and the property of such an algebra to be tame. More precisely, we show that a rank 2 cluster algebra has a basis of indecomposable positive elements if and only if it is of finite or affine type. This statement disagrees with a conjecture by Fock and Goncharov.

Keywords Cluster algebras · Positivity · Tameness

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1 Introduction and main results

This note continues the study of the positivity structure for coefficient-free rank 2 cluster algebras initiated in [8–10]. These algebras can be quickly defined as follows: they form a 2-parametric family depending on a pair of positive integers (b, c) , and the cluster algebra $\mathcal{A}(b, c)$ is the subring of the *ambient field* $\mathcal{F} = \mathbb{Q}(x_1, x_2)$ generated by the *cluster variables* x_m for all $m \in \mathbb{Z}$, where the (two-sided) sequence of cluster

K. Lee (✉)

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
e-mail: klee@math.wayne.edu

L. Li

Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA
e-mail: li2345@oakland.edu

A. Zelevinsky

Department of Mathematics, Northeastern University, Boston, MA 02115, USA
e-mail: andrei@neu.edu

variables is given recursively by the relations

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & \text{for } m \text{ odd;} \\ x_m^c + 1 & \text{for } m \text{ even.} \end{cases} \quad (1.1)$$

Recall also that the sets $\{x_m, x_{m+1}\}$ for $m \in \mathbb{Z}$ are called *clusters*, and the ambient field $\mathcal{F}$ is naturally identified with $\mathbb{Q}(x_m, x_{m+1})$ for any $m \in \mathbb{Z}$.

Despite such an elementary definition, we find the structure theory of these algebras rather deep, and some of the natural questions surprisingly difficult. Here is a fundamental result, which is a special case of the Laurent phenomenon discovered and proved in [2, 5, 6]: every cluster variable is not just a rational function in the two elements of any given cluster but a Laurent polynomial with integer coefficients. The following stronger result is a special case of the results in [2]:

$$\mathcal{A} = \bigcap_{m \in \mathbb{Z}} \mathbb{Z}[x_m^{\pm 1}, x_{m+1}^{\pm 1}], \quad (1.2)$$

where $\mathbb{Z}[x_m^{\pm 1}, x_{m+1}^{\pm 1}]$ denotes the ring of Laurent polynomials with integer coefficients in x_m and x_{m+1} .

The main focus in our study of the algebras $\mathcal{A}(b, c)$ is *positivity*. Recall that a non-zero element $x \in \mathcal{A}(b, c)$ is *positive at a cluster* $\{x_m, x_{m+1}\}$ if all the coefficients in the expansion of x as a Laurent polynomial in x_m and x_{m+1} are positive. We say that $x \in \mathcal{A}(b, c)$ is *positive* if it is positive at all the clusters.

Following [10], we introduce the following important definition.

Definition 1.1 A positive element $x \in \mathcal{A}(b, c)$ is *indecomposable* if it cannot be expressed as the sum of two positive elements.

Recall that $\mathcal{A}(b, c)$ is of *finite* (resp. *affine*) type if $bc \leq 3$ (resp. $bc = 4$). It is also common to refer to the case $bc \leq 4$ as *tame*, and that of $bc > 4$ as *wild* (this terminology comes from the theory of quiver representations). One of the main results of [10] is the following: if $\mathcal{A}(b, c)$ is tame then indecomposable positive elements form a $\mathbb{Z}$ -basis in $\mathcal{A}(b, c)$.

Motivated by this result (among other considerations) it was conjectured in [4, Conjecture 4.1] that indecomposable positive elements form a $\mathbb{Z}$ -basis in any cluster algebra. However, already the authors of [10] suspected that this is not true, and this suspicion was detailed and stated explicitly in [8]. Here we finally settle this question by proving the following.

Theorem 1.2 *The set of indecomposable positive elements forms a $\mathbb{Z}$ -basis for $\mathcal{A}(b, c)$ if and only if $bc \leq 4$ i.e., $\mathcal{A}(b, c)$ is tame.*

Recall that in [8] for each (b, c) a family $\{x[a_1, a_2] : (a_1, a_2) \in \mathbb{Z}^2\}$ of *greedy elements* was introduced in $\mathcal{A}(b, c)$ which include all cluster variables, and it was proved among other things that they are indecomposable positive, and that they form a $\mathbb{Z}$ -basis in $\mathcal{A}(b, c)$. Taking this into account, it is easy to see that the following conditions on (b, c) are equivalent:

- (1) Indecomposable positive elements do not form a $\mathbb{Z}$ -basis for $\mathcal{A}(b, c)$.
- (2) Indecomposable positive elements in $\mathcal{A}(b, c)$ are linearly dependent.
- (3) There exists a non-greedy indecomposable positive element in $\mathcal{A}(b, c)$.
- (4) There exists a positive element $p \in \mathcal{A}(b, c)$ whose expansion in the basis of greedy elements has at least one negative coefficient.

For instance, to deduce (3) from (4) take any expansion of p into the sum of indecomposable positive elements, and note that it is *different* from the expansion of p in the basis of greedy elements. Hence at least one of indecomposable positive components of p must be non-greedy.

We see that to prove Theorem 1.2 it suffices to show that in the wild case there always exists an element p satisfying (4). We exhibit such an element explicitly as follows.

Theorem 1.3 *Suppose that $bc > 4$, i.e., $\mathcal{A}(b, c)$ is wild. Define an element $p \in \mathcal{A}(b, c)$ as follows:*

$$p = \begin{cases} x[bc - b + 1, c + 1] + x[b + 1, bc - c + 1] - x[1, 1] & \text{if } \min(b, c) > 1; \\ x[b + 2, 3] + x[b + 2, b - 1] - x[2, 1] & \text{if } c = 1; \\ x[3, c + 2] + x[c - 1, c + 2] - x[1, 2] & \text{if } b = 1. \end{cases} \quad (1.3)$$

Then p is positive, hence satisfies condition (4) above.

To show that the element p given by (1.3) is positive, we use the group of automorphisms W of $\mathcal{A}(b, c)$ introduced in [8]. By the definition, W is generated by the involutions σ_ℓ for $\ell \in \mathbb{Z}$, where σ_ℓ acts on cluster variables by a permutation $\sigma_\ell(x_m) = x_{2\ell-m}$. It is easy to see that W is a dihedral group generated by σ_1 and σ_2 (this group is finite if $\mathcal{A}$ is of finite type, and infinite otherwise). As shown in [8, Proposition 1.8], the set of greedy elements is W -invariant, and the automorphisms σ_1 and σ_2 act on greedy elements as follows:

$$\sigma_1(x[a_1, a_2]) = x[a_1, c[a_1]_+ - a_2], \quad \sigma_2(x[a_1, a_2]) = x[b[a_2]_+ - a_1, a_2] \quad (1.4)$$

for all $(a_1, a_2) \in \mathbb{Z}^2$, where we use the standard notation $[a]_+ = \max(a, 0)$.

Remark 1.4 As [3] pointed out, if $b \neq c$, the group W is equal to the group $\text{Aut } A$ of cluster automorphisms introduced in [1], otherwise W is a subgroup of $\text{Aut } A$ of index 2 (since for each $\ell \in \mathbb{Z}$, there is an automorphism that sends x_m to $x_{2\ell+1-m}$).

Clearly, W acts transitively on the set of all clusters of $\mathcal{A}(b, c)$. Thus, the positivity of p is equivalent to the property that $\sigma(p)$ is positive at the initial cluster $\{x_1, x_2\}$ for every $\sigma \in W$.

We identify $\mathbb{Z}^2$ with the root lattice associated with the (generalized) Cartan matrix

$$A = A(b, c) = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix} \quad (1.5)$$

using an unorthodox convention that the simple roots α_1 and α_2 are identified with $(0, 1)$ and $(1, 0)$, respectively. The Weyl group $W(A)$ is a group of linear transforma-

tions of $\mathbb{Z}^2$ generated by two *simple reflections* s_1 and s_2 whose action in $\mathbb{Z}^2$ is given by

$$s_1 = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}. \quad (1.6)$$

Comparing this with (1.4) we see that s_1 agrees with σ_1 , and s_2 agrees with σ_2 on $\mathbb{Z}_{\geq 0}^2$.

It is well known (see e.g., [7]) that the difference between tame and wild cases manifests itself in the appearance and behavior of *imaginary roots*. According to [7], the set Φ_+^{im} of *positive imaginary roots* can be defined as follows:

$$\Phi_+^{\text{im}} = \{(a_1, a_2) \in (\mathbb{Z}_{>0})^2 : Q(a_1, a_2) \leq 0\} \quad (1.7)$$

where Q is the $W(A)$ -invariant quadratic form on $\mathbb{Z}^2$ given by

$$Q(a_1, a_2) = ca_1^2 - bca_1a_2 + ba_2^2. \quad (1.8)$$

It is known (and easy to check) that

- in the finite type $bc < 4$ the form Q is positive definite, and so $\Phi_+^{\text{im}} = \emptyset$;
- in the affine type $bc = 4$, we have $Q(a_1, a_2) \geq 0$ on $\mathbb{Z}^2$, and $\Phi_+^{\text{im}} = \{(a_1, a_2) \in \mathbb{Z}_{>0} : Q(a_1, a_2) = 0\}$ is the set of integer positive multiples of the *minimal imaginary root*;
- in the wild case $bc > 4$, the form Q does not vanish on $\mathbb{Z}^2$, and we have

$$\begin{aligned} \Phi_+^{\text{im}} &= \{(a_1, a_2) \in (\mathbb{Z}_{>0})^2 : Q(a_1, a_2) < 0\} \\ &= \left\{ (a_1, a_2) \in (\mathbb{Z}_{>0})^2 : \frac{bc - \sqrt{bc(bc-4)}}{2b} < \frac{a_2}{a_1} < \frac{bc + \sqrt{bc(bc-4)}}{2b} \right\}. \end{aligned}$$

In this note our main interest is in the wild case. In this case we can and will identify the group W of automorphisms of $\mathcal{A}(b, c)$ generated by σ_1 and σ_2 with the Weyl group $W(A)$ by identifying σ_1 with s_1 and σ_2 with s_2 . The above facts imply that Φ_+^{im} is $W(A)$ -invariant, and we have the following useful property.

Proposition 1.5 *In the wild case $bc > 4$, if $\sigma \in W$ and $w \in W(A)$ are identified with each other and $(a_1, a_2) \in \Phi_+^{\text{im}}$, then we have $\sigma(x[a_1, a_2]) = x[w(a_1, a_2)]$.*

Returning to Theorem 1.3, it is easy to check that all the elements $x[a_1, a_2]$ appearing in the right-hand side of (1.3) correspond to positive imaginary roots (see Lemma 4.2). Thus to prove Theorems 1.2 and 1.3 it suffices to establish the following key lemma.

Lemma 1.6 *In the setup of Theorem 1.3, for every $w \in W$, the element obtained from p by replacing each element $x[a_1, a_2]$ in the right-hand side of (1.3) with $x[w(a_1, a_2)]$ is positive at the initial cluster $\{x_1, x_2\}$.*

The proof of Lemma 1.6 is carried out in Sect. 4. All the tools in our proof are already developed in [8]. The two most important ingredients are as follows:

- A combinatorial definition of greedy elements in terms of *compatible pairs* (to be recalled later). The combinatorics of compatible pairs plays a crucial part in our argument.
- A precise description of the set of Laurent monomials that appear in the expansion of $x[a_1, a_2]$ in terms of the initial cluster $\{x_1, x_2\}$, for an arbitrary positive imaginary root (a_1, a_2) . Specifically, we show that the upper bound for this set given in [8, Proposition 4.1, Case 6] is exact.

2 Compatibility and greedy elements in rank 2 cluster algebras

In this section we recall some definitions and results from [8].

Let (a_1, a_2) be a pair of non-negative integers. A *Dyck path* of type $a_1 \times a_2$ is a lattice path from $(0, 0)$ to (a_1, a_2) that never goes above the main diagonal joining $(0, 0)$ and (a_1, a_2) . Among the Dyck paths of a given type $a_1 \times a_2$, there is a (unique) *maximal* one denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. It is defined by the property that any lattice point strictly above $\mathcal{D}$ is also strictly above the main diagonal.

Let $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. Let $\mathcal{D}_1 = \{u_1, \dots, u_{a_1}\}$ be the set of horizontal edges of $\mathcal{D}$ indexed from left to right, and $\mathcal{D}_2 = \{v_1, \dots, v_{a_2}\}$ be the set of vertical edges of $\mathcal{D}$ indexed from bottom to top.

Given any points A and B on $\mathcal{D}$, let AB be the subpath starting from A , and going in the Northeast direction until it reaches B [(if we reach (a_1, a_2) first, we continue from $(0, 0)$]. By convention, if $A = B$, then AA is the subpath that starts from A , then passes (a_1, a_2) and ends at A . If we represent a subpath of $\mathcal{D}$ by its set of edges, then for $A = (i, j)$ and $B = (i', j')$, we have

$$AB = \begin{cases} \{u_k, v_\ell : i < k \leq i', j < \ell \leq j'\}, & \text{if } B \text{ is to the Northeast of } A; \\ \mathcal{D} - \{u_k, v_\ell : i' < k \leq i, j' < \ell \leq j\}, & \text{otherwise.} \end{cases}$$

We denote by $(AB)_1$ the set of horizontal edges in AB , and by $(AB)_2$ the set of vertical edges in AB . Also, let AB° denote the set of lattice points on the subpath AB excluding the endpoints A and B (here $(0, 0)$ and (a_1, a_2) are regarded as the same point).

Definition 2.1 [8] For $S_1 \subseteq \mathcal{D}_1, S_2 \subseteq \mathcal{D}_2$, we say that the pair (S_1, S_2) is *compatible* if for every $u \in S_1$ and $v \in S_2$, denoting by E the left endpoint of u and F the upper endpoint of v , there exists a lattice point $A \in EF^\circ$ such that

$$|(AF)_1| = b|(AF)_2 \cap S_2| \text{ or } |(EA)_2| = c|(EA)_1 \cap S_1|. \quad (2.1)$$

One of the main results of [8] is the following combinatorial expression for greedy elements.

Theorem 2.2 [8] For every $(a_1, a_2) \in \mathbb{Z}_{\geq 0}^2$, the greedy element $x[a_1, a_2] \in \mathcal{A}(b, c)$ at (a_1, a_2) is given by

$$x[a_1, a_2] = x_1^{-a_1} x_2^{-a_2} \sum_{(S_1, S_2)} x_1^{b|S_2|} x_2^{c|S_1|}, \quad (2.2)$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{a_1 \times a_2}$.

For the purposes of this paper we can view (2.2) as a *definition* of greedy elements. Note that when (a_1, a_2) is the denominator of a cluster variable, the formula is (after some interpretation) equivalent to the expansion formula in [9, Theorem 9]; we plan to give more details on this interpretation elsewhere.

3 Extremal pairs and their compatibility

In this section we define extremal pairs and study their compatibility condition. As a consequence, we give a precise description of the set of Laurent monomials that appear in the expansion of $x[a_1, a_2]$ in terms of the initial cluster $\{x_1, x_2\}$, for an arbitrary positive imaginary root (a_1, a_2) .

Definition 3.1 Let $(a_1, a_2) \in \mathbb{Z}_{\geq 0}^2$, and $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. Let s_1 and s_2 be two integers such that $0 \leq s_1 \leq a_1$ and $0 \leq s_2 \leq a_2$. If $S_1 = \{u_1, \dots, u_{s_1}\}$ is the set of the first s_1 horizontal edges in $\mathcal{D}$ and $S_2 = \{v_{a_2-s_2+1}, \dots, v_{a_2}\}$ is the set of the last s_2 vertical edges in $\mathcal{D}$, then we call (S_1, S_2) the *extremal pair* (in $\mathcal{D}$) of size $(s_1; s_2)$.

As in [8], we denote by $c(p, q)$ the coefficients of $x[a_1, a_2]$:

$$x[a_1, a_2] = x_1^{-a_1} x_2^{-a_2} \sum_{p, q \geq 0} c(p, q) x_1^{bp} x_2^{cq},$$

and define the pointed support of $x[a_1, a_2]$ to be

$$PS[a_1, a_2] = \{(p, q) \in \mathbb{Z}_{\geq 0}^2 : c(p, q) \neq 0\}.$$

Let $P = P[a_1, a_2] \in \mathbb{R}^2$ be the region bounded by the broken line

$$(0, 0), (a_2, 0), (a_1/b, a_2/c), (0, a_1), (0, 0),$$

with the convention that P includes sides $[(0, 0), (a_2, 0)]$ and $[(0, a_1), (0, 0)]$ but excludes the rest of the boundary. The following lemma gives a simple geometric characterization of positive imaginary roots.

Lemma 3.2 A lattice point $(a_1, a_2) \in \mathbb{Z}_{>0}^2$ is a positive imaginary root if and only if $a_1 p + a_2 q \leq a_1 a_2$ for every $(p, q) \in P$; in other words, P is contained in the triangle with vertices $(0, 0)$, $(a_2, 0)$, and $(0, a_1)$.

Proof The latter condition is equivalent to the condition that the vertex $(a_1/b, a_2/c)$ lies inside or on the boundary of the triangle, which is equivalent to the inequality $a_1 \cdot a_1/b + a_2 \cdot a_2/c \leq a_1 a_2$. This is in turn equivalent to $Q(a_1, a_2) \leq 0$, i.e., to the condition that (a_1, a_2) is a positive imaginary root. $\square$

Lemma 3.3 Suppose (a_1, a_2) is a positive imaginary root, and (p, q) is a pair of positive integers such that $q \leq a_1$ and $p \leq a_2$. Let (S_1, S_2) be the extremal pair of size $(q; p)$ in $\mathcal{D}$. Then the following conditions are equivalent:

- (1) every $u \in S_1$ precedes every $v \in S_2$;
- (2) $a_1 p + a_2 q < a_1 a_2 + a_1 + a_2$.

Proof The condition (1) is equivalent to the condition that the edge u_q is not higher than the lower endpoint of v_{a_2-p+1} ; that is, $\lfloor (q-1)a_2/a_1 \rfloor \leq a_2 - p$. This inequality is equivalent to $(q-1)a_2/a_1 - 1 < a_2 - p$, thus is equivalent to (2). $\square$

Recall that in [8, Proposition 4.1 (6)] we showed that $PS[a_1, a_2]$ is contained in P . Our goal in the rest of this section is to prove the following strengthening of this result.

Proposition 3.4 Assume that (a_1, a_2) is a positive imaginary root. If (p, q) is a lattice point in P and both p and q are positive, then the extremal pair (S_1, S_2) of size $(q; p)$ is compatible.

Corollary 3.5 Assume that (a_1, a_2) is a positive imaginary root. The set $PS[a_1, a_2]$ is the set of all lattice points in the region P .

Proof Since $PS[a_1, a_2]$ is contained in P , we only need to show that for every lattice point (p, q) in P , the extremal pair (S_1, S_2) of size $(q; p)$ is compatible. If $p = 0$ or $q = 0$, the conclusion is immediate. Other cases follow from Proposition 3.4. $\square$

The following lemma plays a key role in our proof of Proposition 3.4.

Lemma 3.6 Suppose (a_1, a_2) is a positive imaginary root, and (p, q) a pair of positive integers satisfying $a_1 p + a_2 q < a_1 a_2 + a_1 + a_2$ (hence $q \leq a_1$ and $p \leq a_2$). Let (S_1, S_2) be the extremal pair of size $(q; p)$ in $\mathcal{D}$. Then (S_1, S_2) is compatible if every horizontal edge $u \in S_1$ with the left endpoint E and every vertical edge $v \in S_2$ with the top endpoint F satisfy at least one of the following inequalities:

$$|(EF)_1| > b|(EF)_2 \cap S_2|, \quad (3.1)$$

$$|(EF)_2| > c|(EF)_1 \cap S_1|. \quad (3.2)$$

Equivalently, the extremal pair (S_1, S_2) is compatible if for every $1 \leq p' \leq p$, $1 \leq q' \leq q$, at least one of the following inequalities holds:

$$(ba_2 - a_1)(p' - 1) - a_2(q' - 1) > (bp - a_1)a_2, \quad (3.3)$$

$$(ca_1 - a_2)(q' - 1) - a_1(p' - 1) > (cq - a_2)a_1. \quad (3.4)$$

Proof Define $f(AB) = b|(AB)_2 \cap S_2| - |(AB)_1|$. Without loss of generality, we assume (3.1), that is, $f(EF) < 0$. As A moves through the lattice points along $\mathcal{D}$ from E to the lower endpoint of v , $f(AF)$ either decreases by b , stays constant, or increases by 1 at each step (here we use implicitly the fact that u precedes v , which follows from Lemma 3.3); moreover, it starts with the negative value $f(EF)$

and ends at the positive value b . Thus, there exists $A \in EF^\circ$ such that $f(AF) = 0$. Since the same argument works for every u and v , we conclude that (S_1, S_2) is compatible.

Next, we show that, for $u = u_{q'}$ and $v = v_{a_2 - p' + 1}$, (3.1) is equivalent to (3.3). Indeed, Lemma 3.3 implies that u precedes v ; thus

$$\begin{aligned} (3.1) &\Leftrightarrow (|(OO)_1| - |(FO)_1|) - |(OE)_1| > b(p - p' + 1) \\ &\Leftrightarrow (a_1 - \lfloor a_1(p' - 1)/a_2 \rfloor) - (q' - 1) > b(p - p' + 1) \\ &\Leftrightarrow a_1 - q' - b(p - p' + 1) \geq \lfloor a_1(p' - 1)/a_2 \rfloor \\ &\Leftrightarrow a_1 - q' - b(p - p' + 1) > a_1(p' - 1)/a_2 - 1 \Leftrightarrow (3.3), \end{aligned}$$

where $O = (0, 0)$.

The equivalence of (3.2) and (3.4) is proved similarly. $\square$

Proof of Proposition 3.4 It is then easy to check that (p, q) lies in P if and only if at least one of the following two conditions hold:

$$a_1 \geq bp \text{ and } (ca_1 - a_2)(a_1 - bp) > (cq - a_2)a_1, \quad (3.5)$$

$$a_2 \geq cq \text{ and } (ba_2 - a_1)(a_2 - cq) > (bp - a_1)a_2. \quad (3.6)$$

Without loss of generality we assume (3.5) holds.

Thanks to Lemma 3.6, to show the compatibility of (S_1, S_2) , it suffices to prove

$$R \subset H_1 \cup H_2,$$

where

$$\begin{aligned} R &= \{(p', q') \in \mathbb{Z}^2 \mid 1 \leq p' \leq p, 1 \leq q' \leq q\}, \\ H_1 &= \{(p', q') \in \mathbb{R}^2 \mid (ba_2 - a_1)(p' - 1) - a_2(q' - 1) > (bp - a_1)a_2\}, \\ H_2 &= \{(p', q') \in \mathbb{R}^2 \mid (ca_1 - a_2)(q' - 1) - a_1(p' - 1) > (cq - a_2)a_1\}. \end{aligned}$$

Note that H_1 is the half plane below the line passing through $(1, a_1 - bp + 1)$ with slope $m_1 = (ba_2 - a_1)/a_2 > 0$, and H_2 is a half plane above a line with slope $m_2 = a_1/(ca_1 - a_2) > 0$. Also note that $m_1 \geq m_2$ because $Q(a_1, a_2) \leq 0$. Thus to show that $R \subset H_1 \cup H_2$, it suffices to show that $(1, a_1 - bp + 1) \in H_2$ (as illustrated in Fig. 1). But this is exactly the statement of (3.5). $\square$

4 Proofs of main results

In this section we prove Theorems 1.2 and 1.3. As discussed in Section 1, it is enough to prove Lemma 1.6, which will be our goal.

Due to obvious symmetry, we can and will assume in the rest of this Section that

$$\min(b, c) = c. \quad (4.1)$$

Fig. 1 This figure shows that R is contained in the union of the two open half planes H_1 and H_2 if the point $(1, a_1 - bp + 1)$ is in H_2

In particular, we can disregard the last case in (1.3). Now note that since both s_1 and s_2 are involutions ((see (1.6)), each element of W is one of the following (for $k \geq 0$):

$$w(1; k) = \underbrace{s_{\langle k \rangle} \cdots s_1 s_2 s_1}_{k \text{ factors}}, \quad w(2; k) = \underbrace{s_{\langle k+1 \rangle} \cdots s_2 s_1 s_2}_{k \text{ factors}};$$

Here we use the convention $w(1; 0) = w(2; 0) = e$, the identity element of W , and $\langle k \rangle = 1$ if k is odd, or 2 if k is even.

Let

$$r_k = \begin{cases} b & \text{for } k \text{ odd,} \\ c & \text{for } k \text{ even.} \end{cases}$$

Definition 4.1 For $(a_1, a_2) \in \mathbb{Z}^2$, we define a sequence $(a(k))_{k \geq -1}$ with initial data (a_1, a_2) as follows: let $a(-1) = a_2$, $a(0) = a_1$, and for $k > 0$ recursively define

$$a(k) = r_{k-1}a(k-1) - a(k-2). \quad (4.2)$$

Using this recursion, we define sequences $(\alpha(k))$, $(\beta(k))$, $(\gamma(k))$ with initial data given in the table:

	$(\alpha(0), \alpha(-1))$	$(\beta(0), \beta(-1))$	$(\gamma(0), \gamma(-1))$
if $\min(b, c) > 1$	$(1, 1)$	$((c-1)b+1, c+1)$	$(b+1, (b-1)c+1)$
if $c = 1$	$(2, 1)$	$(b+2, 3)$	$(b+2, b-1)$

The next step is to show that all components of the element p in (1.3) correspond to positive imaginary roots. Thus we show the following.

Lemma 4.2 Suppose we are in the wild case, that is, $bc > 4$.

- (1) If $\min(b, c) = c > 1$ then each of the vectors $(1, 1)$, $(b+1, bc-c+1)$, and $(bc-b+1, c+1)$ is a positive imaginary root (recall that this means that the form Q given by (1.8) takes negative values at all these vectors).

- (2) If $c = 1$ and $b > 4$, then each of the vectors $(2, 1)$, $(b + 2, 3)$, and $(b + 2, b - 1)$ is a positive imaginary root.

Proof (1) A direct computation shows $Q(1, 1) = b + c - bc$, $Q(b + 1, bc - c + 1) = Q(bc - b + 1, c + 1) = (2bc + 1)(b + c - bc)$. Note that $b + c - bc = (2 - c)c + (b - c)(1 - c)$ is a sum of two non-positive terms. Furthermore, we see that the equality $b + c - bc = 0$ is achieved only when $b = c = 2$, which is not a wild case. So in the wild case we have a strict equality $b + c - bc < 0$, and therefore $Q(1, 1) < 0$, $Q(b + 1, bc - c + 1) < 0$.

- (2) A direct computation shows $Q(2, 1) = 4 - b < 0$, $Q(b + 2, 3) = Q(b + 2, b - 1) = (2b + 1)(4 - b) < 0$. $\square$

Proposition 4.3 Using notation in Definition 4.1, for any $(a_1, a_2) \in \mathbb{Z}^2$ and any $k \geq 0$, we have

$$w(1; k)(a_1, a_2) = \begin{cases} (a(k - 1), a(k)), & \text{if } k \text{ is odd;} \\ (a(k), a(k - 1)), & \text{if } k \text{ is even.} \end{cases}$$

As a consequence, for p defined in (1.3),

$$w(1; k)(p) = \begin{cases} x[\beta(k - 1), \beta(k)] + x[\gamma(k - 1), \gamma(k)] - x[\alpha(k - 1), \alpha(k)], & \text{if } k \text{ is odd;} \\ x[\beta(k), \beta(k - 1)] + x[\gamma(k), \gamma(k - 1)] - x[\alpha(k), \alpha(k - 1)], & \text{if } k \text{ is even.} \end{cases}$$

Proof Straightforward induction on k . $\square$

In order to treat $w(1; k)(p)$ uniformly, we denote by $x_{b \leftrightarrow c}[a(k), a(k - 1)]$ the greedy element in $\mathcal{A}(r_{k-1}, r_k)$ pointed at $(a(k), a(k - 1))$ for every k . For convenience, we extend the sequences $(\alpha(k))$, $(\beta(k))$, and $(\gamma(k))$ to all $k < -1$ using the relation (4.2).

Proof of Lemma 1.6 We only show that $w(1; r)(p)$ is positive at $\{x_1, x_2\}$ (recall that p is defined in (1.3)), since the treatment of $w(2; r)(p)$ is completely similar and will be left to the reader.

Thanks to Proposition 4.3, it suffices to prove that the Laurent polynomial

$$p_k = x_{b \leftrightarrow c}[\beta(k), \beta(k - 1)] + x_{b \leftrightarrow c}[\gamma(k), \gamma(k - 1)] - x_{b \leftrightarrow c}[\alpha(k), \alpha(k - 1)] \in \mathcal{A}(r_{k-1}, r_k)$$

is positive at $\{x_1, x_2\}$ for every $k \geq 0$. Since p_k is the sum of the following two Laurent polynomials

$$x_{b \leftrightarrow c}[\gamma(k), \gamma(k - 1)] - x_1^{\alpha(k-2)} x_2^{-\alpha(k-1)}, \quad (4.3)$$

$$x_{b \leftrightarrow c}[\beta(k), \beta(k - 1)] + x_1^{\alpha(k-2)} x_2^{-\alpha(k-1)} - x_{b \leftrightarrow c}[\alpha(k), \alpha(k - 1)], \quad (4.4)$$

it suffices to show that (4.3) and (4.4) are positive at $\{x_1, x_2\}$. They are proved separately in Lemmas 4.4 and 4.5. $\square$

Lemma 4.4 For every $k \geq 0$, (4.3) is positive at $\{x_1, x_2\}$.

Lemma 4.5 For every $k \geq 0$, (4.4) is positive at $\{x_1, x_2\}$.

In order to prove the above two lemmas, we claim some identities among sequences $(\alpha(k))$, $(\beta(k))$ and $(\gamma(k))$.

Lemma 4.6 For every non-negative integer k , we have $\alpha(k) < \beta(k)$. For every integer k ,

$$2\alpha(k-2) + \alpha(k) = \beta(k-2), \quad (4.5)$$

$$\alpha(k-1) + r_k \alpha(k) = \beta(k-1), \quad (4.6)$$

$$2r_k \alpha(k) - \alpha(k-1) = \gamma(k+1), \quad (4.7)$$

$$\alpha(k) + r_k \alpha(k-1) = \gamma(k), \quad (4.8)$$

and

$$\begin{aligned} \alpha(k-1)\beta(k) - \alpha(k)\beta(k-1) &= \alpha(k)\gamma(k-1) - \alpha(k-1)\gamma(k) \\ &= r_{k-1}\alpha(k-1)\alpha(k+1) - r_k\alpha(k)^2 \\ &= bc\alpha(k-1)\alpha(k) - r_{k-1}\alpha(k-1)^2 - r_k\alpha(k)^2 = \delta(b, c) > 0, \end{aligned} \quad (4.9)$$

where

$$\delta(b, c) = \begin{cases} bc - b - c & \text{if } \min(b, c) > 1; \\ b - 4 & \text{if } c = 1. \end{cases}$$

Proof All identities can be easily proved by induction. To prove $\alpha(k) < \beta(k)$, it suffices to show that $(\beta(0) - \alpha(0), \beta(-1) - \alpha(-1))$ is a positive imaginary root. This is true because in the case $\min(b, c) > 1$ we have $Q((c-1)b, c) = bc(b+c-bc) < 0$ and in the case $c = 1, b > 4$ we have $Q(b, 2) = b(4-b) < 0$. $\square$

We denote by $PS_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)]$ the pointed support of $x_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)] \in \mathcal{A}(r_{k-1}, r_k)$. Define the map $\varphi_{\alpha(k), \alpha(k-1)}$ by sending (p, q) to $(-\alpha(k) + r_{k-1}p, -\alpha(k-1) + r_kq)$. Similar notation applies with α being replaced by β and γ . Now we are ready to prove Lemma 4.4.

Proof of Lemma 4.4 The positivity of (4.3) is equivalent to saying that the support of $x_{b \leftrightarrow c}[\gamma(k), \gamma(k-1)]$ contains the point $(\alpha(k-2), -\alpha(k-1))$, which follows immediately from Proposition 3.4 and Lemma 4.6. Indeed, we need to show that $(p, q) := \varphi_{\gamma(k), \gamma(k-1)}^{-1}(\alpha(k-2), -\alpha(k-1))$ lies in the region $PS_{b \leftrightarrow c}[\gamma(k), \gamma(k-1)]$. It is easy to check that

$$p = (\alpha(k-2) + \gamma(k))/r_{k-1} \stackrel{(4.7)}{=} 2\alpha(k-1) > \gamma(k)/r_{k-1},$$

$$q = (\gamma(k-1) - \alpha(k-1))/r_k \stackrel{(4.8)}{=} \alpha(k-2) > 0.$$

So we only need to show that (p, q) is below the line passing through $(\gamma(k)/r_{k-1}, \gamma(k-1)/r_k)$ and $(\gamma(k-1), 0)$, which is equivalent to the statement that the three points $(\gamma(k-1), 0)$, $(\gamma(k)/r_{k-1}, \gamma(k-1)/r_k)$, and (p, q) are in counter-clockwise order. Therefore, it follows from

$$\begin{aligned} \begin{vmatrix} 1 & \gamma(k-1) & 0 \\ 1 & \gamma(k)/r_{k-1} & \gamma(k-1)/r_k \\ 1 & p & q \end{vmatrix} &= \frac{1}{r_{k-1}r_k} \begin{vmatrix} 1 & r_{k-1}\gamma(k-1) & 0 \\ 1 & \gamma(k) & \gamma(k-1) \\ 1 & \alpha(k-2) + \gamma(k) & \gamma(k-1) - \alpha(k-1) \end{vmatrix} \\ &= \frac{1}{bc} \begin{vmatrix} 0 & \gamma(k-2) & -\gamma(k-1) \\ 1 & \gamma(k) & \gamma(k-1) \\ 0 & \alpha(k-2) & -\alpha(k-1) \end{vmatrix} = \frac{1}{bc} \begin{vmatrix} \gamma(k-2) & \gamma(k-1) \\ \alpha(k-2) & \alpha(k-1) \end{vmatrix} \stackrel{(4.9)}{=} \frac{\delta(b, c)}{bc} > 0. \end{aligned}$$

□

Lemma 4.5 can be deduced from the following lemma.

Lemma 4.7 *Let u_i and v_i be edges in $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$; and u'_i and v'_i be edges in $\mathcal{D}^{\beta(k) \times \beta(k-1)}$. The map*

$$\mu : \left\{ \begin{array}{l} \text{Compatible pairs} \\ \text{in } \mathcal{D}^{\alpha(k) \times \alpha(k-1)} \end{array} \right\} \setminus \{(\emptyset, \mathcal{D}_2^{\alpha(k) \times \alpha(k-1)})\} \rightarrow \left\{ \begin{array}{l} \text{Compatible pairs} \\ \text{in } \mathcal{D}^{\beta(k) \times \beta(k-1)} \end{array} \right\}$$

defined by $\mu(S_1, S_2) = (S'_1, S'_2)$, where

$$\begin{aligned} S'_1 &= \{u'_1, \dots, u'_{\alpha(k)}\} \cup \{u'_{i+\alpha(k)} | u_i \in S_1\}, \\ S'_2 &= \{v'_{i+\alpha(k-1)} | v_i \in S_2\} \cup \{v'_{2\alpha(k-1)+1}, \dots, v'_{\beta(k-1)}\}. \end{aligned}$$

is a well-defined injective map satisfying

$$\begin{aligned} |S'_1| &= |S_1| + (\beta(k-1) - \alpha(k-1))/r_k = |S_1| + \alpha(k), \\ |S'_2| &= |S_2| + (\beta(k) - \alpha(k))/r_{k-1} = |S_2| + \alpha(k+1). \end{aligned}$$

Indeed, assume that Lemma 4.7 is true. Since the pair (S_1, S_2) contributes a Laurent monomial $x_1^{-\alpha(k)+r_{k-1}p} x_2^{-\alpha(k-1)+r_kq}$ to $x_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)]$, while (S'_1, S'_2) contributes the same Laurent monomial to $x_{b \leftrightarrow c}[\beta(k), \beta(k-1)]$ because

$$x_1^{-\beta(k)+r_{k-1}(p+\alpha(k+1))} x_2^{-\beta(k-1)+r_k(q+\alpha(k))} \stackrel{(4.6)}{=} x_1^{-\alpha(k)+r_{k-1}p} x_2^{-\alpha(k-1)+r_kq},$$

every term in $x_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)] - x_1^{\alpha(k-2)} x_2^{-\alpha(k-1)}$ is cancelled out by a term in $x_{b \leftrightarrow c}[\beta(k), \beta(k-1)]$, which implies Lemma 4.5.

The rest of the paper is devoted to the proof of Lemma 4.7. For $P \subseteq \mathbb{Z}^2$ and $(d_1, d_2) \in \mathbb{Z}^2$, we denote

$$P - (d_1, d_2) = \{(i - d_1, j - d_2) | (i, j) \in P\}.$$

Fig. 2 Here $PS_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)]$ is the set of lattice points bounded by $OKLMO$, and $PS_{b \leftrightarrow c}[\beta(k), \beta(k-1)] - (\alpha(k+1), \alpha(k))$ is the set of lattice points bounded by $O'K'L'M'O'$

Lemma 4.8 *Let $k \geq 0$. Then*

$$PS_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)] \setminus \{(\alpha(k-1), 0)\} \subseteq PS_{b \leftrightarrow c}[\beta(k), \beta(k-1)] - (\alpha(k+1), \alpha(k)).$$

Proof Proposition 3.4 asserts that $PS_{b \leftrightarrow c}[\alpha(k), \alpha(k-1)]$ is the set of lattice points in the region bounded by the broken line

$$\begin{aligned} O &= (0, 0), \quad K = (\alpha(k-1), 0), \quad L = (\alpha(k)/r_{k-1}, \alpha(k-1)/r_k), \\ M &= (0, \alpha(k)), \quad (0, 0). \end{aligned}$$

$PS_{b \leftrightarrow c}[\beta(k), \beta(k-1)] - (\alpha(k+1), \alpha(k))$ is the set of lattice points in the region bounded by the broken line $O'K'L'M'O'$ with

$$\begin{aligned} O' &= (-\alpha(k+1), -\alpha(k)), \\ K' &= (-\alpha(k+1) + \beta(k-1), -\alpha(k)) \stackrel{(4.5)}{=} (2\alpha(k-1), -\alpha(k)), \\ L' &= (-\alpha(k+1) + \beta(k)/r_{k-1}, -\alpha(k) + \beta(k-1)/r_k) \stackrel{(4.6)}{=} L, \\ M' &= (-\alpha(k+1), -\alpha(k) + \beta(k)) \stackrel{(4.6)}{=} (-\alpha(k+1), r_{k+1}\alpha(k+1)). \end{aligned}$$

Note that O' , K' , L' , and M' are in the third, fourth, first, and second quadrant, respectively, as shown in Fig. 2.

First, note that O obviously lies strictly inside the region bounded by $O'K'L'M'O'$.

Next, we claim that M also lies strictly inside the same region. Indeed, it suffices to show that $L'M'M$ is anti-clockwise oriented, or equivalently the following determinant

$$\begin{vmatrix} 1 & \alpha(k)/r_{k-1} & \alpha(k-1)/r_k \\ 1 & -\alpha(k+1) & r_{k+1}\alpha_{k+1} \\ 1 & 0 & \alpha(k) \end{vmatrix} \stackrel{(4.2)}{=} \frac{1}{bc} (r_{k-1}\alpha(k-1)\alpha(k+1) - r_k\alpha(k)^2) \stackrel{(4.9)}{=} \frac{\delta(b, c)}{bc}$$

is positive, which is obviously the case.

Finally, the area of the triangle KLK' is

$$\frac{1}{2} \begin{vmatrix} 1 & \alpha(k-1) & 0 \\ 1 & \alpha(k)/r_{k-1} & \alpha(k-1)/r_k \\ 1 & 2\alpha(k-1) & -\alpha(k) \end{vmatrix} \stackrel{(4.2)}{=} \frac{1}{2bc} (bc\alpha(k-1)\alpha(k) - r_{k-1}\alpha(k-1)^2 - r_k\alpha(k)^2) \stackrel{(4.9)}{=} \frac{\delta(b, c)}{2bc}.$$

Thus KLK' is anti-clockwise oriented. Moreover, since the area is less than $1/2$, Pick's theorem asserts that there are no lattice points P other than K and K' that lie inside or on the boundary of the triangle KLK' (otherwise the triangle KPK' , which is contained in the triangle KLK' , would have area at least $1/2$). Therefore, every lattice point other than K in the region bounded by $OKLMO$ must be in the region bounded by $O'K'L'M'O'$. $\square$

Lemma 4.9 *Let $k, p, q \geq 0$, $p < \alpha(k-1)$. If the extremal pair (S_1, S_2) of size $(q; p)$ is compatible in $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$, then the extremal pair (S'_1, S'_2) of size $(q + \alpha(k); p + \alpha(k+1))$ is compatible in $\mathcal{D}^{\beta(k) \times \beta(k-1)}$.*

Proof It follows immediately from Proposition 3.4 and Lemma 4.8. $\square$

For any horizontal or vertical edge u , we denote

$$\begin{aligned} E_u &= \text{the left/lower endpoint of } u, \\ F_u &= \text{the right/upper endpoint of } u. \end{aligned}$$

To compare compatible pairs in $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$ with compatible pairs in $\mathcal{D}^{\beta(k) \times \beta(k-1)}$, we need the following crucial Lemma 4.10 which roughly says that a certain subpath of the Dyck path $\mathcal{D}^{\beta(k) \times \beta(k-1)}$ is almost identical with $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$.

We introduce some notation (see Fig. 3). Let $B = (\alpha(k), \alpha(k-1))$, $C = (2\alpha(k), 2\alpha(k-1))$, $B' = E_{u'_{\alpha(k)+1}}$, $C' = F_{v_{2\alpha(k-1)}}$ (so C' is of the same height as C), $G' = B + (1, 0) = (\alpha(k) + 1, \alpha(k-1))$, $H' = C - (0, 1) = (2\alpha(k), 2\alpha(k-1) - 1)$,

Fig. 3 The subpath of $\mathcal{D}^{\beta(k) \times \beta(k-1)}$ from G' to H' is almost identical with $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$

T be the intersection of the line CH' with the diagonal $\overline{O'P'}$ with $O' = (0, 0)$, and $P' = (\beta(k), \beta(k - 1))$. Let $BG'H'C$ be the path obtained by taking the union of the edge BG' , the subpath $G'H'$, and the edge $H'C$. Moreover, we denote B by B_k (and we use the subscript k in a similar fashion for other letters) if we need to specify the dependence on k .

Lemma 4.10 (1) *The points B, C, B', G' , and H' are above/below the diagonal $\overline{O'P'}$ as follows:*

- *The points B and C are above $\overline{O'P'}$ for all $k \geq 0$;*
- *B' has coordinates $(\alpha(k), \alpha(k - 1) - 1)$ and is below $\overline{O'P'}$ for all $k \geq 0$;*
- *G' is below $\overline{O'P'}$ for all $k \geq 1$;*
- *H' is below $\overline{O'P'}$ for $k \geq 1$ in the case $\min(b, c) > 1$, and for $k \geq 2$ in the case $c = 1$.*

(2) *The path $BG'H'C$ has the same shape as $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$ for $k \geq 1$ in the case $\min(b, c) > 1$, and for $k \geq 2$ in the case $c \geq 1$.*

Proof (1) It reduces to the following facts which are easy to check:

$\beta(k) > \delta(b, c)$ for $k \geq 0$ in both cases $\min(b, c) > 1$ and $c = 1$;

$\beta(k) > 2\delta(b, c)$ for $k \geq 1$ in the case $\min(b, c) > 1$, or for $k \geq 2$ in the case $c = 1$.

Indeed, the statement that B and C are above $\overline{O'P'}$ is equivalent to the inequality $\alpha(k-1)/\alpha(k) > \beta(k-1)/\beta(k)$, which is equivalent to $\delta(b, c) > 0$; the statement that B' has coordinates $(\alpha(k), \alpha(k-1) - 1)$ and is below $\overline{O'P'}$ follows from the inequality $(\alpha(k-1) - 1)/\alpha(k) < \beta(k-1)/\beta(k)$, which is equivalent to $\beta(k) > \delta(b, c)$; the statement that G' is below $\overline{O'P'}$ is equivalent to $\alpha(k-1)/(\alpha(k)+1) < \beta(k-1)/\beta(k)$, which is equivalent to $\beta(k-1) > \delta(b, c)$; the statement that H' is below $\overline{O'P'}$ is equivalent to $(2\alpha(k-1) - 1)/(2\alpha(k)) < \beta(k-1)/\beta(k)$, which is equivalent to $\beta(k) > 2\delta(b, c)$.

(2) Based on (1), it suffices to show that there are no lattice points that lie strictly inside the triangle $O'_k T_k C_k$ for all $k \geq 1$ in case $\min(b, c) > 1$, and all $k \geq 2$ in case $c = 1$. Aiming at the contradiction, assume that (i_k, j_k) is such a point. In other words, $\beta(k-1)/\beta(k) < j_k/i_k < \alpha(k-1)/\alpha(k)$, and $0 < i_k < 2\alpha(k)$. If $k > 0$ in case $\min(b, c) > 1$, or $k > 1$ in case $c = 1$, we would have that the lattice point $(i_{k-1}, j_{k-1}) = (j_k, r_k j_k - i_k)$ is strictly inside the triangle $O'_{k-1} T_{k-1} C_{k-1}$. Indeed,

$$0 < i_{k-1} = j_k < i_k \alpha(k-1)/\alpha(k) < 2\alpha(k) \alpha(k-1)/\alpha(k) = 2\alpha(k-1),$$

and

$$r_k - \beta(k)/\beta(k-1) < r_k - i_k/j_k < r_k - \alpha(k)/\alpha(k-1)$$

implies

$$\beta(k-2)/\beta(k-1) < j_{k-1}/i_{k-1} < \alpha(k-2)/\alpha(k-1).$$

By descending induction on k , we would conclude that

- (i) case $\min(b, c) > 1$: there is a lattice point strictly inside the triangle $O'_0 T_0 C_0$, where $O'_0 = (0, 0)$, $T_0 = (2, 2\beta(0)/\beta(1))$, $C_0 = (2, 2)$, which is absurd;

- (ii) case $c = 1$: there is a lattice point strictly inside the triangle $O'_1 T_1 C_1$, where $O'_1 = (0, 0)$, $T_1 = (2, 2(b+2)/(b-1))$, and $C_1 = (2, 4)$, which is also absurd. $\square$

Lemma 4.11 *Given a subpath EF of $\mathcal{D}^{\alpha_k \times \alpha_{k-1}}$ or $\mathcal{D}^{\beta_k \times \beta_{k-1}}$ and a pair (S_1, S_2) , we define*

$$\begin{aligned} f_{S_2}(EF) &= r_{k-1}|(EF)_2 \cap S_2| - |(EF)_1|, \\ g_{S_1}(EF) &= r_k|(EF)_1 \cap S_1| - |(EF)_2|. \end{aligned}$$

Assuming $u \in S_1$ and $v \in S_2$, the following are equivalent:

- there exists $A \in (E_u F_v)^\circ$ such that (2.1) holds, in other words, $f_{S_2}(AF_v) = 0$ or $g_{S_1}(E_u A) = 0$;
- there exists $A \in (E_u F_v)^\circ$ such that $f_{S_2}(AF_v) \leq 0$ or $g_{S_1}(E_u A) \leq 0$.

Proof It is straightforward as in the proof of Lemma 3.6. $\square$

Proof of Lemma 4.7. It is easy to check that $|S'_1| = |S_1| + \alpha(k)$ and $|S'_2| = |S_2| + \alpha(k+1)$, and μ is obviously injective if it is well defined. Thus, we are left to show that (S'_1, S'_2) is compatible.

If $k = 0$, or $k = 1$ for the case $c = 1$, then either $\alpha(k)$ or $\alpha(k-1)$ is equal to 1, thus any compatible pair (S_1, S_2) in $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$ satisfies either $S_1 = \emptyset$ or $S_2 = \emptyset$. We immediately conclude that (S'_1, S'_2) are compatible using Lemma 4.9.

For the rest of the proof we assume $k \geq 1$ in case $\min(b, c) > 1$ or $k \geq 2$ in case $c = 1$. Then Lemma 4.10 (2) can apply and we shall use the notation from there.

We will prove by contradiction by assuming that (S'_1, S'_2) is not compatible. Then by Lemma 4.11, there exist $u' \in S'_1$ and $v' \in S'_2$ such that $f_{S'_2}(AF_{v'}) > 0$ and $g_{S'_1}(E_{u'} A) > 0$ for every $A \in (E_{u'} F_{v'})^\circ$.

There are four cases to be considered.

Case 1: Both u' and v' are on the subpath $B'C'$. Thus, we can denote $u' = u'_{i+\alpha(k)}$, $v' = v'_{j+\alpha(k-1)}$ for some $1 \leq i \leq \alpha(k) + \lceil 2\delta(b, c)/\beta(k-1) \rceil$, $0 \leq j \leq \alpha(k-1)$.

Subcase 1-1: u' precedes v' . Since (S_1, S_2) is compatible, without loss of generality we assume the existence of $A \in (E_{u_i} F_{v_j})^\circ$ such that

$$f_{S_2}(AF_{v_j}) = 0.$$

Identify $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$ with $BG'H'C$ thanks to Lemma 4.10 (2). Since $|(AF_{v'})_1| \geq |(AF_{v_j})_1|$ and $|(AF_{v'})_2 \cap S'_2| = |(AF_{v_j})_2 \cap S_2|$, we get a contradiction

$$f_{S'_2}(AF_{v'}) \leq f_{S_2}(AF_{v_j}) = 0.$$

Subcase 1-2: u' is after v' . Then take $A = P' \in (E_{u'} F_{v'})^\circ$,

$$\begin{aligned} g_{S'_1}(E_{u'} A) &= r_k|(E_{u'} P')_1 \cap S'_1| - |(E_{u'} P')_2| \\ &\leq r_k(\alpha(k) - i + 1) - \left(\beta(k-1) - \left\lfloor \frac{(\alpha(k) + i - 1)\beta(k-1)}{\beta(k)} \right\rfloor \right) \end{aligned}$$

$$\begin{aligned} &\leq r_k(\alpha(k) - i + 1) - \beta(k - 1) + \frac{(\alpha(k) + i - 1)\beta(k - 1)}{\beta(k)} \\ &= \frac{(i - 1)(\beta(k - 1) - r_k\beta(k)) - \beta(k)(\beta(k - 1) - r_k\alpha(k)) + \alpha(k)\beta(k - 1)}{\beta(k)}. \end{aligned}$$

Since $\beta(k - 1) - r_k\beta(k) \stackrel{(4.2)}{=} -\beta(k + 1) < 0$ and $\beta(k - 1) - r_k\alpha(k) \stackrel{(4.6)}{=} \alpha(k - 1)$, we again get a contradiction

$$g_{S'_1}(E_{u'}A) \leq \frac{-\alpha(k - 1)\beta(k) + \alpha(k)\beta(k - 1)}{\beta(k)} \stackrel{(4.9)}{=} -\frac{\delta(b, c)}{\beta(k)} < 0.$$

Case 2: u' is on $B'C'$ but v' is not. Assume $u' = u'_{i+\alpha(k)}$ and $v' = v'_j$, where $1 \leq i \leq \alpha(k)$, $2\alpha(k - 1) + 1 \leq j \leq \beta(k - 1)$. Then we consider the extremal pair $(\tilde{S}'_1, \tilde{S}'_2)$ on $\mathcal{D}^{\beta(k) \times \beta(k-1)}$ of size

$$(|(F_{u'}C')_1 \cap S'_1| + i + \alpha(k); |(F_{u'}C')_2 \cap S'_2| + \beta(k - 1) - 2\alpha(k - 1)).$$

(In other words, $(\tilde{S}'_1, \tilde{S}'_2)$ is obtained from (S'_1, S'_2) by moving forward all edges in $(F_{u'}C')_1 \cap S'_1$ so that they are immediately after $F_{u'}$, moving backward all edges in $(F_{u'}C')_2 \cap S'_2$ so that they are immediately preceding C' , and adding all horizontal edges on $B'E_{u'}$ to S'_1 and removing all vertical edges on $B'E_{u'}$ from $\tilde{S}'_2$.)

We claim that $(\tilde{S}'_1, \tilde{S}'_2)$ is not compatible. Indeed, for any $A \in (E_{u'}F_{v'})^\circ$, we have

$$|(E_{u'}A)_1 \cap \tilde{S}'_1| \geq |(E_{u'}A)_1 \cap S'_1|, \quad |(AF_{v'})_2 \cap \tilde{S}'_2| \geq |(AF_{v'})_2 \cap S'_2|,$$

thus

$$g_{\tilde{S}'_1}(E_{u'}A) \geq g_{S'_1}(E_{u'}A) > 0, \quad f_{\tilde{S}'_2}(AF_{v'}) \geq f_{S'_2}(AF_{v'}) > 0.$$

Similarly, we consider the extremal pair $(\tilde{S}_1, \tilde{S}_2)$ on $\mathcal{D}^{\alpha(k) \times \alpha(k-1)}$ of size

$$(|(F_{u_i}C)_1 \cap S_1| + i; |(F_{u_i}C)_2 \cap S_2|).$$

(In other words, $(\tilde{S}_1, \tilde{S}_2)$ is obtained from (S_1, S_2) by moving forward all horizontal edges after u_i in S_1 so that they become immediately after u_i , moving backward all horizontal edges after u_i in S_2 to the northeast corner of the Dyck path, and adding all horizontal edges in front of u_i to S_1 and removing all vertical edges in front of u_i in S_2 .)

We claim that $(\tilde{S}_1, \tilde{S}_2)$ is compatible. To see this, let

$$\begin{aligned} U_1 &= (BF_{u_i})_1 \cap S_1, & U_2 &= (F_{u_i}C)_1 \cap S_1 = S_1 \setminus U_1, & W_1 &= \{u_1, \dots, u_i\} \cup U_2, \\ V_1 &= (BF_{u_i})_2 \cap S_2, & V_2 &= (F_{u_i}C)_2 \cap S_2 = S_2 \setminus V_1. \end{aligned}$$

We shall show that (W_1, V_2) is compatible, i.e., every $u_j \in W_1$ and $v_\ell \in V_2$ is separated by (W_1, V_2) . First consider the case $j \leq i$. Since (S_1, S_2) is compatible, there exists

$A \in (E_{u_i} F_{v_\ell})^\circ$ with either $f_{S_2}(AF_{v_\ell}) = 0$ or $g_{S_1}(E_{u_i} A) = 0$. On the other hand, $g_{S_1}(E_{u_i} A) \geq g_{S'_1}(E_{u'} A) > 0$. So $f_{V_2}(AF_{v_\ell}) = f_{S_2}(AF_{v_\ell}) = 0$. Next, consider the case $j > i$. Since (W_1, V_2) and (S_1, S_2) coincide on the subpath $E_{u_j} F_{v_\ell}$, a lattice point A that satisfies (2.1) for (S_1, S_2) also satisfies (2.1) for (W_1, V_2) . Since $|W_1| = |\tilde{S}_1|$ and $|V_2| = \tilde{S}_2$, Proposition 3.4 implies that $(\tilde{S}_1, \tilde{S}_2)$ is compatible.

Thus, we have a compatible extremal pair $(\tilde{S}_1, \tilde{S}_2)$ and an incompatible extremal pair $(\tilde{S}'_1, \tilde{S}'_2)$ which contradict Lemma 4.9.

Case 3: v' is on $B'C'$ but u' is not. The proof of this case is similar to Case 2.

Case 4: Neither u' nor v' is on $B'C'$. We let $(\tilde{S}'_1, \tilde{S}'_2)$ be the extremal pair of size $(|\tilde{S}'_1|; |\tilde{S}'_2|)$ (which is obtained from (S'_1, S'_2) by moving all horizontal edges in $B'C' \cap S'_1$ forward so that they are immediately after B' , and move all vertical edges in $B'C' \cap S'_2$ backward so that they are immediately in front of C'). Then $(\tilde{S}'_1, \tilde{S}'_2)$ is not compatible because for any $A \in (E_{u'} F_{v'})^\circ$,

$$\begin{aligned} f_{\tilde{S}'_2}(AF_{v'}) &\geq f_{S'_2}(AF_{v'}) > 0, \\ g_{\tilde{S}'_1}(E_{u'} A) &\geq g_{S'_1}(E_{u'} A) > 0. \end{aligned}$$

Similarly define $(\tilde{S}_1, \tilde{S}_2)$ to be the extremal pair of size $(|\tilde{S}_1|; |\tilde{S}_2|)$. Then by Proposition 3.4, $(\tilde{S}_1, \tilde{S}_2)$ is compatible. This again contradicts Lemma 4.9.

Since in all the four cases we get contradictions, (S'_1, S'_2) must be compatible. This completes the proof of Lemma 4.7. $\square$

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