Brauer algebras of type $I_2^n (n \geq 5)$

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Abstract We will present an algebra related to the Coxeter group of type I_2^n which can be taken as the twisted subalgebra in Brauer algebra of type A_{n-1} . Also, we will describe some properties of this algebra.

Keywords Diagram algebra · Coxeter group · Brauer algebra

1 Introduction

From studying the invariant theory for orthogonal groups, Brauer discovered algebras which are now called Brauer algebras of type A in [2]; Cohen et al. extended it to the definition of simply laced type in [6], the nodes of whose Dynkin diagrams are connected by simple bond. Mühlherr described how to get Coxeter group of type I_2^n by twisting Coxeter group of type A_{2n-1} in [15]. Here, we will apply a similar approach as Mühlherr on $\text{Br}(A_{n-1})$, to get an algebra $\text{Br}(I_2^n)$ called the Brauer algebra of type I_2^n .

In fact, Mühlherr's method can be considered as the generalization of obtaining Weyl groups of non-simply laced types from simply laced types ([3, 16]), such as C_n from A_{2n-1} , B_n from D_{n+1} , and F_4 from E_6 . These are motivated by considering the invariant subgroups under non-trivial automorphisms by action on their Dynkin diagrams. We have already utilized it to obtain Brauer algebras of type C_n ([5]), B_n ([7]), and F_4 ([14]). This paper can be regarded as a part of the project of finding Brauer algebras of non-simply laced types from simply laced types. In this way, we not only give some properties of these algebras but also construct diagram representations for them.

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We give the definition of $\text{Br}(\mathcal{I}_2^n)$ as follows. The cases n even and odd are different and so have different definitions. We arrive at these definitions when we compute relations of images of generators under ϕ in Theorem 1.6 for small values of n , and we generalize it for arbitrary n .

Definition 1.1 *The Brauer algebra of type $\mathcal{I}_2^{2m}$ for $m \in \mathbb{N}_{>2}$, denoted by $\text{Br}(\mathcal{I}_2^{2m})$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$ -algebra generated by r_0, r_1, e_0 and e_1 subject to the following relations and a set $\Theta \subset \mathbb{N}$ consisting of $\kappa_i, \eta_j, \xi_j, \theta_j$, where $i = 0, 1, j = 1, \dots, [m/2]$. Symbols $[r_0 r_1 \dots]_t$ and $[r_1 r_0 \dots]_t$ stand for words of length t with r_0 and r_1 alternated:*

$$r_i^2 = 1, \quad (1.1)$$

$$r_i e_i = e_i r_i = e_i, \quad (1.2)$$

$$e_i^2 = \delta^{\kappa_i} e_i, \quad (1.3)$$

$$e_1 e_0 e_1 = \delta e_1, \quad (1.4)$$

$$e_0[r_1 r_0 \dots]_{2m-1} = [r_1 r_0 \dots]_{2m-1} e_0, \quad (1.5)$$

$$e_1[r_0 r_1 \dots]_{2m-1} = e_1, \quad (1.6)$$

$$[r_0 r_1 \dots]_{2m-1} e_1 = e_1, \quad (1.7)$$

$$e_0[r_1 r_0 \dots]_{2k} e_1 = \delta^{\theta_k} e_0 e_1, \quad 0 \leq k \leq [m/2] \quad (1.8)$$

$$e_1[r_0 r_1 \dots]_{2k} e_0 = \delta^{\theta_k} e_1 e_0, \quad 0 \leq k \leq [m/2] \quad (1.9)$$

$$e_1[r_0 r_1 \dots]_{2k-1} e_1 = \delta^{\eta_k} e_1, \quad 0 \leq k \leq [m/2] \quad (1.10)$$

$$[r_1 r_0 \dots]_{2m} = [r_0 r_1 \dots]_{2m}. \quad (1.11)$$

When $2k \leq m$, let $l = \text{lcm}(k, m)$,

$$e_0[r_1 r_0 \dots]_{2k-1} e_0 = \delta^{\xi_k} [r_1 r_0 \dots]_{2m-1} e_0, \quad \frac{l}{m}, \frac{l}{k} \text{ odd}, \quad (1.12)$$

$$e_0[r_1 r_0 \dots]_{2k-1} e_0 = \delta^{\xi_k} e_0, \quad \frac{l}{m} \text{ even}, \quad (1.13)$$

$$e_0[r_1 r_0 \dots]_{2k-1} e_0 = \delta^{\xi_k} e_0 e_1 e_0, \quad \frac{l}{m} \text{ odd}, \frac{l}{k} \text{ even}. \quad (1.14)$$

The submonoid of the multiplicative monoid of $\text{Br}(\mathcal{I}_2^{2m})$ generated by $\delta^{\pm 1}, \{r_i\}_{i=0}^1, \{e_i\}_{i=0}^1$ is denoted by $\text{BrM}(\mathcal{I}_2^{2m})$. This is the monoid of monomials in $\text{Br}(\mathcal{I}_2^{2m})$.

Definition 1.2 *The Brauer algebra of type $\mathcal{I}_2^{2m-1}$ for $m \in \mathbb{N}_{>2}$, denoted by $\text{Br}(\mathcal{I}_2^{2m-1})$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$ -algebra generated by r_0, r_1, e_0 , and e_1 subject to the following relations and a set $\Theta \subset \mathbb{N}$ consisting of κ_i, ξ_j with $j = 1, \dots, m, i = 0, 1$,*

$\kappa_0 = \kappa_1$:

$$r_i^2 = 1, \quad (1.15)$$

$$r_i e_i = e_i r_i = e_i, \quad (1.16)$$

$$e_i^2 = \delta^{\kappa_i} e_i, \quad (1.17)$$

$$[r_0 r_1 \cdots]_{2m-2} e_0 = e_1 [r_0 r_1 \cdots]_{2m-2}, \quad (1.18)$$

$$e_0 [r_1 r_0 \cdots]_{2k-1} e_0 = \delta^{\xi_k} e_0, \quad 0 < k < m \quad (1.19)$$

$$[r_1 r_0 \cdots]_{2m-1} = [r_0 r_1 \cdots]_{2m-1}. \quad (1.20)$$

The submonoid of the multiplicative monoid of $\text{Br}(\mathbb{I}_2^{2m-1})$ generated by $\delta^{\pm 1}$, r_0 , r_1 , e_0 , e_1 is denoted by $\text{BrM}(\mathbb{I}_2^{2m-1})$. This is the monoid of monomials in $\text{Br}(\mathbb{I}_2^{2m-1})$.

Remark 1.3 Up to parameters, the Definition 1.2 defines the same algebra presented in [4], where Chen proved the rank by a geometric proof. Instead, here we prove this in an algebraic and combinatorial way. But we have different definitions for the even case because Chen has $e_0 e_1 = 0$ in his defining relations, our algebra have bigger rank than his.

The Coxeter group of type $\mathbb{I}_2^n$ is known to be the dihedral group of order $2n$ which is also known to be the group of symmetries of a regular n -gon. These are generated by the reflections about the vertices for n odd and n even these reflections plus the reflections about the centers of opposite faces.

In [15], the nontrivial diagram automorphisms on Dynkin diagrams were generalized as admissible partitions. We introduce the definition and a theorem from [15].

Definition 1.4 Let (W, S) be a Coxeter system of type M . If J is a spherical subset of S , we write w_J for the longest element of W_J , the parabolic subgroup of W generated by J . If $\mathbb{P}$ is a partition of S all of whose parts are spherical, we denote $C_W(\mathbb{P})$ the subgroup of W generated by all w_J for $J \in \mathbb{P}$. A partition $\mathbb{P}$ of S is called an *admissible partition* if, for each part $A \in \mathbb{P}$, the Coxeter diagram restricted to A is spherical and, for each element $w \in C_W(\mathbb{P})$, either $l(rw) < l(w)$ for all $r \in A$ or $l(rw) > l(w)$ for all $r \in A$.

Theorem 1.5 ([15]) *Let $\mathbb{P}$ be an admissible partition of S and write $S_{\mathbb{P}} = \{w_A \mid A \in \mathbb{P}\}$. Then, the pair $(C_W(\mathbb{P}), S_{\mathbb{P}})$ is a Coxeter system. Its type is the Coxeter diagram $M_{\mathbb{P}}$ on $S_{\mathbb{P}}$ whose A, B -entry is the order of $w_A w_B$ in W .*

We apply this theorem to obtain the Coxeter group of type $\mathbb{I}_2^n$ from the Coxeter group of type A_{n-1} . The Fig. 1 is the Dynkin diagrams of type A_{n-1} and type $\mathbb{I}_2^n$, and the partition to get $\mathbb{I}_2^6$ from A_5 . The main theorem of this paper can be stated as below.

Theorem 1.6 *For $n > 4$, there is an injective algebra homomorphism*

$$\phi : \text{Br}(\mathbb{I}_2^n) \longrightarrow \text{Br}(A_{n-1})$$

determined by $\phi(r_0) = \prod_{i \text{ even}}^{0 < i < n} R_i$, $\phi(r_1) = \prod_{i \text{ odd}}^{0 < i < n} R_i$, $\phi(e_0) = \prod_{i \text{ even}}^{0 < i < n} E_i$, and $\phi(e_1) = \prod_{i \text{ odd}}^{0 < i < n} E_i$ when each parameter in Θ takes special value in $\mathbb{N}$ to make

Fig. 1 I_2^n in A_5

ϕ an algebra homomorphism, where R_i and E_i are generators of $\text{Br}(A_{n-1})$ (see in Definition 3.1). Furthermore,

$$\text{rank}_{\mathbb{Z}[\delta^{\pm 1}]} \text{Br}(I_2^n) = \begin{cases} 2n + n^2, & \text{if } n \text{ is odd,} \\ 2n + \frac{3}{2}n^2, & \text{if } n \text{ is even.} \end{cases}$$

This paper is included as Chapter 5 in the author's PhD thesis [13].

2 An interesting elementary problem

The main purpose of this section is to prepare for the proof of Lemma 3.5 for relations (1.12), (1.13), and (1.14).

Suppose that $k, m \in \mathbb{N}$ such that $1 < 2k \leq m$. There is a box in the x, y plane $\mathbb{R}^2$ fixed by four lines $x = 1$, $x = 2m$, $y = 2k - \frac{1}{2}$, and $y = -\frac{1}{2}$. Imagine you have a particle, which starts to move from $(1, 2k - 1)$ with slope -1 ; when it touches the bottom (the top), it will be reflected with the bottom (the top) as a mirror; but when it touches the right (left) wall, it first goes down (up) 1 unit vertically, if it comes at the wall from the top (bottom), and continues its path with the wall as the mirror; it stops if it reaches the points $(1, 0)$, $(2m, 0)$, $(1, 2k - 1)$, or $(2m, 2k - 1)$. For different values of k, m , the problem is to decide at which point the particle stops. One example is Fig. 2, when $m = 5, k = 2$.

To solve the problem, we unfold its path by “penetrating” the walls, which means that when the particle touches the right wall for the first time, we change the vertical step into one move of slope -1 when coming from the top (slope 1 when coming from the bottom) with Euclid length $\sqrt{2}$, or, in other words, we do not change its moving at the wall, and we see that the path of the particle in the region between $x = 2m + 1$ and $x = 4m$ is just the reverse of the path of the particle when it goes from the right to the left for the first time. The algorithm can be similarly extended at the left wall to make the unfolded path look like the graph of a function of a single variable. It can be verified that when it passes the point with the x -coordinate being a multiple of $2m$, the

Fig. 2 Case for $m = 5, k = 2$

Fig. 3 The unfolded path for $m = 5, k = 2$

movement stops. It can be seen that before it stops, the path in $[2tm + 1, 2(t + 1)m]$ is just a copy of a particle path in the above box of:

- (i) the $\frac{t+2}{2}$ th path from the left wall to the right wall if t is even,
- (ii) or $\frac{t+1}{2}$ th path from the right wall to the the left wall if t is odd.

Therefore, the above trick is just that we draw the picture on folded paper, then we unfold this and see a simple picture in which the original problem can be tackled. Here is an example for $m = 5, k = 2$ in Fig. 3 being the unfolded case of Fig. 2.

Lemma 2.1 *Let $l = \text{lcm}(k, m)$. The particle stops in the unfolded path when it moves $2l - 1$ for its x -coordinate. Furthermore*

- (i) when $\frac{l}{m}$ is even, the particle stops at $(1, 0)$;
- (ii) when $\frac{l}{m}$ and $\frac{l}{k}$ are odd, the particle stops at $(2m, 0)$;
- (iii) when $\frac{l}{m}$ is odd and $\frac{l}{k}$ is even, the particle stops at $(2m, 2k - 1)$.

Proof By prolonging the path at the beginning and the ending, respectively, by half unit for x -coordinate to complete a period, we can consider the particle starting from

the top and stopping at the top or the bottom. By observing the unfolded path of the particle, each time it goes from the top ceiling to the bottom ground or from the bottom to the top, the x -coordinate is increased by $2k$, so the first conclusion follows naturally. The other two conclusions hold easily by basic number theory knowledge about congruence. Furthermore, elementary number theory tells us that $\frac{l}{m}$ and $\frac{l}{k}$ can not both be even, which implies that the particle never stops at $(1, 2k - 1)$. $\square$

3 The map ϕ inducing a homomorphism

In order to avoid confusion with the above generators, the symbols of [6] have been capitalized.

Definition 3.1 Let Q be a graph. The Brauer monoid $\text{BrM}(Q)$ is the monoid generated by the symbols R_i and E_i , for each node i of Q and δ, δ^{-1} subject to the following relation, where $\sim$ denotes adjacency between nodes of Q :

$$\delta\delta^{-1} = 1, \quad (3.1)$$

$$R_i^2 = 1, \quad (3.2)$$

$$R_i E_i = E_i R_i = E_i, \quad (3.3)$$

$$E_i^2 = \delta E_i, \quad (3.4)$$

$$R_i R_j = R_j R_i, \text{ for } i \approx j, \quad (3.5)$$

$$E_i R_j = R_j E_i, \text{ for } i \approx j, \quad (3.6)$$

$$E_i E_j = E_j E_i, \text{ for } i \approx j, \quad (3.7)$$

$$R_i R_j R_i = R_j R_i R_j, \text{ for } i \sim j, \quad (3.8)$$

$$R_j R_i E_j = E_i E_j, \text{ for } i \sim j, \quad (3.9)$$

$$R_i E_j R_i = R_j E_i R_j, \text{ for } i \sim j. \quad (3.10)$$

The Brauer algebra $\text{Br}(Q)$ is the the free $\mathbb{Z}$ -algebra for Brauer monoid $\text{BrM}(Q)$.

In [2], Brauer defines $\text{BrM}(A_t)$, there A_t is the type A Dynkin diagram, the path on $2t + 2$ vertices. He gives a diagrammatic description of the monoid, which consists of diagrams with parallel rows of $t + 1$ vertices (drawn at $(0, i)$ and $(1, i)$ for $i = 1, \dots, t + 1$) and $t + 1$ edges forming a perfect matching on the vertices. The multiplication of any two diagrams is formed by vertical concatenation where any closed loop is replaced by a factor of δ . The generators for $\text{BrM}(A_t)$, R_i and E_i , are depicted in Fig. 4. Henceforth, we identify $\text{Br}(A_t)$ with its diagrammatic version. It is a free algebra over $\mathbb{Z}[\delta^{\pm 1}]$ of rank $(t + 1)!!$, the product of the first $t + 1$ positive odd integers.

We consider the root system of the Coxeter group of type A_t , focussing on special collections of mutually orthogonal positive roots called admissible sets. Also, the notion of height for elements of the Brauer algebra $\text{Br}(A_t)$ is introduced and discussed.

Definition 3.2 Let $t \geq 1$. The root system of the Coxeter group $W(A_t)$ of type A_t is denoted by Φ . It is realized as $\Phi := \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq t + 1, i \neq j\}$ in the Euclidean space $\mathbb{R}^{t+1}$, where ϵ_i is the i th standard basis vector. Put $\alpha_i := \epsilon_i - \epsilon_{i+1}$.

Fig. 4 Diagrams of R_i , E_i

Then $\{\alpha_i\}_{i=1}^t$ is called the set of simple roots of Φ . Φ^+ denotes the set of positive roots in Φ with respect to these simple roots; that is, $\Phi^+ := \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq t+1\}$. The group $W(A_t)$ has a natural action on Φ as a group generated by reflections, which can be found in many books about Weyl groups or Coxeter groups, such as [3]. We consider the natural action of $W(A_t)$ on Φ^+ by negating the negative roots once these appear in this action.

We have seen that, up to powers of δ , the monomials of $\text{Br}(A_t)$ correspond to Brauer diagrams. In order to work with the tops and bottoms of Brauer diagrams, we introduce the following notion.

Definition 3.3 Let $\mathcal{A}$ denote the collection of all subsets of Φ^+ containing mutually orthogonal positive roots. Members of $\mathcal{A}$ are called *admissible sets*.

An admissible set B corresponds to a Brauer diagram top in the following way: for each $\beta \in B$, where $\beta = \epsilon_i - \epsilon_j$ for some $i, j \in \{1, \dots, t+1\}$, draw a horizontal strand in the corresponding Brauer diagram top from the dot $(i, 1)$ to the dot $(j, 1)$. All horizontal strands on the top are obtained this way, so there are precisely $|B|$ horizontal strands.

For any $\beta \in \Phi^+$ and $i \in \{1, \dots, t\}$, there exists a $w \in W(A_t)$ such that $\beta = w\alpha_i$. Then, $E_\beta := wE_iw^{-1}$ is well defined (see [6]). If $\beta, \gamma \in \Phi^+$ are mutually orthogonal,

then E_β and E_γ commute (see Lemma 4.3 of [6]). Hence, for $B \in \mathcal{A}$, we can define

$$\hat{E}_B := \delta^{-|B|} \prod_{\beta \in B} E_\beta.$$

This is an idempotent element of the Brauer monoid.

In [6, Definition 3.2], an action of the Brauer monoids of simply laced types (type A, D, E_6 , E_7 , E_8) on certain positive root sets are defined; here, we present the Brauer monoid $\text{BrM}(A_t)$ on the collection $\mathcal{A}$ as follows. The generators $\{R_i\}_{i=1}^t$ act by the natural action of Coxeter group elements on its root sets, where negative roots are negated so as to obtain positive roots, and the action of $\{E_i\}_{i=1}^t$ is defined as

$$E_i B := \begin{cases} B, & \text{if } \alpha_i \in B, \\ B \cup \{\alpha_i\}, & \text{if } \alpha_i \perp B, \\ R_\beta R_i B, & \text{if } \beta \in B - \alpha_i^\perp. \end{cases} \quad (3.11)$$

Alternatively, this action can be described as follows: complete the top corresponding to B into a Brauer diagram b , without increasing the number of horizontal strands at the top. Now aB is the top of the Brauer diagram ab . This action can provide a normal form for elements of $\text{BrM}(A_t)$, as proved in [6, Theorem 3.6].

Remark 3.4 From [6, Remark 5.7], there is an anti-involution on $\text{Br}(A_t)$ determined by

$$\gamma_1 \cdots \gamma_k \mapsto \gamma_k \cdots \gamma_1$$

on products of generators of $\text{Br}(A_n)$. We denote it by $x \mapsto x^{\text{op}}$.

Lemma 3.5 *The map ϕ defined on the generators in Theorem 1.6 induces a $\mathbb{Z}[\delta^{\pm 1}]$ -algebra homomorphism.*

Proof We illustrate the images of generators under ϕ in the Fig. 5. We deal first with the case n is odd. In Definition 1.2, the nontrivial relations to be verified are just (1.18), (1.19), and (1.20). By [15], the relation (1.20) holds for the inspection of the images of generators. The relation (1.19) follows by observing the diagrams in $\text{Br}(A_{2m-2})$ because the top and the bottom are fixed, both of which have just one point having a vertical strand. By the diagram, we see that the action of $\phi([r_1 r_0 r_1 \dots]_s)$ in $W(A_{2m-2})$ (which we identify with Sym_{2m-1}) on a number $1 \leq a \leq 2m-1$, for $s \leq 2m-2$, is given by

$$\phi([r_1 r_0 r_1 \dots]_s)(a) = \begin{cases} a+s, & a \text{ odd}, a+s \leq 2m-1, \\ a-s, & a \text{ even}, a-s \geq 1, \\ 4m-1-a-s, & a \text{ odd}, a+s > 2m-1, \\ 1+s-a, & a \text{ even}, a-s < 1. \end{cases} \quad (3.12)$$

Fig. 5 Images of generators under ϕ

Then, for $t > 0$,

$$\begin{aligned}\phi([r_1 r_0 r_1 \dots]_{2m-2})(2t) &= 2m - 1 - 2t, \\ \phi([r_1 r_0 r_1 \dots]_{2m-2})(2t + 1) &= 2m - 1 - 2t + 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\alpha_{2t} &= \alpha_{2m-1-2t}, \\ \phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\{\alpha_{2t}\}_{t=1}^{m-1} &= \{\alpha_{2t-1}\}_{t=1}^{m-1},\end{aligned}$$

which implies that $\phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\phi(e_0)\phi([r_1 r_0 r_1 \dots]_{2m-2}) = \phi(e_1)$ and (1.18) holds for the generator images under ϕ .

Now consider ϕ when $n = 2m > 5$ even. The fact that (1.1)–(1.4) and (1.8)–(1.11) still hold for the generator images under ϕ can be proved easily

by Brauer diagrams. As above we see that for $1 \leq a \leq 2m$, $s \leq 2m - 2$,

$$\phi([r_1 r_0 r_1 \dots]_s)(a) = \begin{cases} a + s, & a \text{ odd}, a + s \leq 2m - 1, \\ a - s, & a \text{ even}, a - s \geq 1, \\ 4m + 1 - a - s, & a \text{ odd}, a + s > 2m - 1, \\ 1 + s - a, & a \text{ even}, a - s < 1, \end{cases} \quad (3.13)$$

$$\phi([r_0 r_1 r_0 \dots]_s)(a) = \begin{cases} a + s, & a \text{ even}, a + s \leq 2m - 1, \\ a - s, & a \text{ odd}, a - s \geq 1, \\ 4m + 1 - a - s, & a \text{ even}, a + s > 2m - 1, \\ 1 + s - a, & a \text{ odd}, a - s < 1. \end{cases} \quad (3.14)$$

By diagram inspection, we have that

$$\begin{aligned} \phi([r_1 r_0 r_1 \dots]_{2m-1}^{-1})(\alpha_{2t}) &= \alpha_{2m-2t}, \\ \phi([r_1 r_0 r_1 \dots]_{2m-1}^{-1})(\{\alpha_{2t}\}_{t=1}^{m-1}) &= \{\alpha_{2t}\}_{t=1}^{m-1}, \end{aligned}$$

and hence $\phi([r_1 r_0 r_1 \dots]_{2m-1})\phi(e_0)\phi([r_1 r_0 r_1 \dots]_{2m-1}) = \phi(e_0)$; therefore, relation (1.5) holds for the images of the generators under ϕ . On the other hand,

$$\begin{aligned} \phi([r_0 r_1 r_0 \dots]_{2m-1}^{-1})(\alpha_{2t-1}) &= \alpha_{2m-2t+1}, \\ \phi([r_0 r_1 r_0 \dots]_{2m-1}^{-1})(\{\alpha_{2t-1}\}_{t=1}^m) &= \{\alpha_{2t}\}_{t=1}^{m-1}. \end{aligned}$$

Just observing Brauer diagrams, (1.6) and (1.7) hold for the images of the generators under ϕ .

As for (1.12)–(1.14), consider the top and the bottom of the diagram of the left under ϕ ; both the top and the bottom have horizontal the same strands among those points $\{(i, 0)\}_{i=2}^{2m-1} \cup \{(i, 1)\}_{i=2}^{2m-1}$ as $\phi(e_0)$. Except those $2m - 2$ strands in the top and in the bottom fixed for $\phi(e_0)$, there are still two strands of the left side under ϕ unknown. Those two strands are between the remaining four points, $(0, 1)$, $(0, 2m)$, $(1, 1)$, and $(1, 2m)$. If we find another end of the strand from $(1, 1)$, the other strand is fixed as a consequence. The strands starting from $(1, 1)$ in the images of under ϕ the right hand sides of (1.12)–(1.14) are ended at $(0, 2m)$, $(0, 1)$ and $(1, 2m)$, respectively. By observation, we can transform this equality problem to the elementary problem solved at the beginning of this section in the following way. Consider the paths of a particle starting from $(1, 1)$ in the diagram of the left-hand sides of the images under ϕ of (1.12)–(1.14) with the $m - 1$ horizontal strands at the top and the $m - 1$ horizontal strands at the bottom removed and transform the horizontal strands as in Fig. 6. We give an example for this in Fig. 7 for $\phi(e_0)\phi(r_1)\phi(r_0)\phi(r_1)\phi(e_0)$ in $\text{Br}(\mathbb{I}_2^8)$. By observation, Lemma 2.1 can be applied here, and gives that the three equalities hold under ϕ acting on both sides. $\square$

Fig. 6 Transformation of horizontal strands

Fig. 7 $\phi(e_0)\phi(r_1)\phi(r_0)\phi(r_1)\phi(e_0)$ for I_2^8

4 Normal forms for $\text{BrM}(I_2^n)$

Lemma 4.1 *The submonoid generated by r_0 and r_1 in $\text{BrM}(I_2^n)$ is isomorphic to $W(I_2^n)$.*

Proof The lemma follows from the natural homomorphisms chain below.

$$\mathbb{Z}[\delta^{\pm 1}](W(I_2^n)) \rightarrow \text{Br}(I_2^n) \rightarrow \text{Br}(I_2^n)/(e_0, e_1) \rightarrow \mathbb{Z}[\delta^{\pm 1}](W(I_2^n)).$$

The composition is the identity and so the lemma follows. $\square$

From now on, we do not distinguish the $W(I_2^n)$ in $\text{BrM}(I_2^n)$ and its image under ϕ . Analogous to Remark 3.4, we can also define an anti-involution $\text{Br}(I_2^n)$ denoted by $x \mapsto x^{\text{op}}$.

Proposition 4.2 *The natural anti-involution above induces an automorphism of the $\mathbb{Z}[\delta^{\pm 1}]$ -algebra $\text{Br}(I_2^n)$.*

Proof It suffices to check the defining relations given in Definition 1.1 and Definition 1.2 still hold under the anti-involution. An easy inspection shows that all relations involved in the definition are invariant under op , except for (1.6), (1.7), (1.8), (1.9), and (1.12). The relation obtained by applying the anti-involution to (1.12) holds due to (1.5). The equalities (1.6) and (1.8) are the op -duals of (1.7) and (1.9), respectively. Hence, our claim holds. $\square$

Proposition 4.3 Suppose that D_0^{2m-1} is the left coset of the subgroup generated by r_0 in $W(I_2^{2m-1})$. Up to some powers of δ , each element in $\text{BrM}(I_2^{2m-1})$ can be written as an element in $W(I_2^{2m-1})$ or ue_0v , where $u \in D_0^{2m-1}$ and $v \in (D_0^{2m-1})^{\text{op}}$.

Proof By (1.18), it follows that e_1 is conjugate to e_0 under $W(I_2^{2m-1})$; hence, we only need to prove that $D_0^{2m-1}e_0(D_0^{2m-1})^{\text{op}}$ is closed under multiplication by the generators e_0, r_0 , and r_1 up to some power of δ . By Proposition 4.2, and invariance of the set $D_0^{2m-1}e_0(D_0^{2m-1})^{\text{op}}$ under the natural involution, it suffices to prove that it is closed under left multiplication. For r_0 and r_1 , we just apply (1.16). For e_0 , it follows from (1.19). $\square$

Definition 4.4 Let Ψ be the root system of I_2^n , and Ψ^+ be the positive roots with respect to $\beta_0 = (\sqrt{2}, 0)$, $\beta_1 = (\sqrt{2} \cos \frac{(n-1)\pi}{n}, \sqrt{2} \sin \frac{(n-1)\pi}{n})$ in $\mathbb{R}^2$ which are roots corresponding to reflections r_0, r_1 , respectively, and r_0r_1 is a rotation with angle $\frac{2\pi}{n}$. Hence, $\Psi^+ = \{(\sqrt{2} \cos \frac{i\pi}{n}, \sqrt{2} \sin \frac{i\pi}{n})\}_{0 \leq i \leq n-1}$. As in Definition 3.2, we consider the natural action of $W(I_2^n)$ on Ψ^+ by negating the negative roots once these appear in this action.

Lemma 4.5 Let N_0 and N_1 be stabilizers of β_0 and β_1 in $W(I_2^n)$, respectively. Then, for any element $a \in N_i$, we have that $ae_ia^{-1} = e_i$, for $i = 0, 1$.

Proof When n is odd, $N_i = \langle r_i \rangle$ for $i = 0, 1$. Hence, the lemma holds because

$$r_i e_i r_i \stackrel{(1.16)}{=} e_i.$$

When $n = 2m$ even, we have that $N_0 = \langle r_0, [r_1 r_0 \cdots]_{2m-1} \rangle$ and $N_1 = \langle r_1, [r_0 r_1 \cdots]_{2m-1} \rangle$, and hence the lemma holds thanks to the following equalities:

$$\begin{aligned} r_i e_i r_i &\stackrel{(1.2)}{=} e_i, \\ [r_1 r_0 \cdots]_{2m-1} e_0 [r_1 r_0 \cdots]_{2m-1} &\stackrel{(1.5)}{=} e_0, \\ [r_0 r_1 \cdots]_{2m-1} e_1 [r_0 r_1 \cdots]_{2m-1} &\stackrel{(1.6)+(1.7)}{=} e_1. \end{aligned}$$

$\square$

Remark 4.6 As in [6], consider a positive root β and a node i of type I_2^n . If there exists $w \in W$ such that $w\beta_i = \beta$, then we can define the element e_β in $\text{BrM}(I_2^n)$ by

$$e_\beta = we_iw^{-1}.$$

The above lemma implies that e_β is well defined.

Lemma 4.7 Let D_i^{2m} be a left coset representatives for N_i in $W(I_2^{2m})$ for $i = 0, 1$, and $K_0 = \langle [r_1 r_0 \cdots]_{2m-1} \rangle \subset N_0$, $K_1 = \langle 1 \rangle \subset N_1$. Then, for any $r \in W(I_2^{2m})$, there exist $a \in D_i$ and $b \in K_i$, such that

$$re_i = ae_ib.$$

Proof It follows from (1.2), (1.5) and (1.7) by considering writing each element in $W(I_2^n)$ in forms of the coset of N_i in $W(I_2^n)$. $\square$

Lemma 4.8 *Up to some power of δ , each element in $\text{BrM}(I_2^m)$ can be written as*

- (i) $a \in W(I_2^m)$,
- (ii) $ue_i v w$, $u \in D_i^{2m}$, $v \in K_i$, $w \in (D_i^{2m})^{\text{op}}$ for $i = 0, 1$,
- (iii) $u' e_0 e_1 w'$, $u' \in D_0^{2m}$, $w' \in (D_1^{2m})^{\text{op}}$,
- (iv) $u'' e_1 e_0 w''$, $u'' \in D_1^{2m}$, $w'' \in (D_0^{2m})^{\text{op}}$, or
- (v) $u''' e_0 e_1 e_0 w'''$, $u''' \in D_0^{2m}$, $w''' \in (D_0^{2m})^{\text{op}}$.

Proof Let us first prove the claim that the monomial $e_0 r e_1$ can be written as $e_0 e_1$ for any $r \in W(I_2^m)$ up to some power of δ . In view of (1.2), we only need to consider the elements that can be written as $[r_1 r_0 \cdots]_{2k}$ with $2k \leq 2m$. Also, thanks to (1.8), when we can restrict ourselves to $m < 2k \leq 2m$, it follows from the below:

$$\begin{aligned} e_0[r_1 r_0 \cdots]_{2k} e_1 &\stackrel{(1.1)}{=} e_0[r_0 r_1 \cdots]_{2m-2k-1} [r_0 r_1 \cdots]_{2m-1} e_1, \\ &\stackrel{(1.7)}{=} e_0 r_0 [r_1 r_0 \cdots]_{2m-2k-2} e_1, \\ &\stackrel{(1.2)}{=} e_0 [r_1 r_0 \cdots]_{2m-2k-2} e_1, \\ &\stackrel{(1.8)}{=} \delta^{\theta_{m-k-1}} e_0 e_1. \end{aligned}$$

Using similar argument, it follows that for any $r \in W(I_2^m)$, the monomial $e_i r e_j$ can be written as one of e_0 , $[r_1 r_0 \cdots]_{2m-1} e_0$, e_1 , $e_0 e_1$, $e_1 e_0$, or $e_0 e_1 e_0$ up to some power of δ .

To prove the lemma, it remains to prove that those five kinds of normal forms are closed under multiplication by generators e_i . Thanks to Proposition 4.2, we only need to consider multiplication from the left. By the conclusion from the above paragraph, the lemma holds. $\square$

5 The rank of $\text{Im}\phi$

Because we apply the well-known Lagrange's Theorem many times in the section, we present it below.

Theorem 5.1 ([12]) *Suppose G is a group and X is a G -set with $x \in X$. Let H be the stabilizer subgroup of x in G . Then,*

$$\#G = \#Gx \cdot \#H,$$

where Gx is the orbit of x under G .

To prove Theorem 1.6, it suffices to prove that those rewritten forms in Proposition 4.3 and Lemma 4.8 are different diagrams in $\text{Br}(A_t)$ with $t = n - 1$. The problem can

be reduced to the counting of orbit sizes.

(I) If $n = 2m - 1$ odd, then

$$\#\phi(W(I_2^{2m-1}))(\{\alpha_{2t}\}_{t=1}^{m-1}) = 2m - 1.$$

(II) If $n = 2m$ even, then

$$\begin{aligned}\#\phi(W(I_2^{2m}))(\{\alpha_{2t}\}_{t=1}^{m-1}) &= m, \\ \#\phi(W(I_2^{2m}))(\{\alpha_{2t-1}\}_{t=1}^m) &= m, \\ \#\phi(W(I_2^{2m}))(\phi(e_0)\{\alpha_{2t-1}\}_{t=1}^m) &= m.\end{aligned}$$

and the last two orbits are different. Neither are the same as the first as the number of roots is different.

First, we consider $n = 2m - 1 \geq 5$.

By (3.12), when $0 < s < 2m - 1$, we see that $s + 1$ is not occupied in the horizontal strands of $\phi([r_1 r_0 r_1 \cdots]_s^{-1})(\{\alpha_{2t}\}_{t=1}^{m-1})$. Then, $\#\phi(W(I_2^{2m-1}))(\{\alpha_{2t}\}_{t=1}^{m-1})$ is at least $2m - 1$. But the subgroup $\langle r_0 \rangle$ stabilizes $\{\alpha_{2t}\}_{t=1}^{m-1}$; therefore, by Lagrange's Theorem, (I) holds.

When $n = 2m > 5$, we define

$$\begin{aligned}\alpha &= \sum_{i=1}^{2m-1} \alpha_i, \\ Y_0 &= \{\alpha_{2t}\}_{t=1}^{m-1}, \\ Y_1 &= \{\alpha_{2t-1}\}_{t=1}^m, \\ Y_2 &= \phi(e_0)Y_1 = Y_0 \cup \{\alpha\}.\end{aligned}$$

Now, we consider the case when $m = 2m' + 1$ is odd. Here, we denote by $h(\gamma)$ the height of $\gamma \in \Phi^+$ which means the sum of the coefficients of simple roots for γ written as the linear combination of simple roots. We find that when $0 \leq s \leq m - 2$,

$$\begin{aligned}\max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_s^{-1})(Y_0)\} &= h(\phi([r_1 r_0 \cdots]_s^{-1})(\alpha_{2m'})) = 2s + 1, \\ \max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_{2m'}^{-1})(Y_0)\} &= h(\phi([r_1 r_0 \cdots]_{2m'}^{-1})(\alpha_{2m'})) = 4m'.\end{aligned}$$

Then, it follows that $\#W(I_2^{4m'+2})(Y_0)$ is at least m . At the same time, $\langle r_0, [r_1 r_0 \cdots]_{2m-1} \rangle$ stabilizes Y_0 ; therefore, it follows from Lagrange's Theorem that $\#W(I_2^{4m'+2})(Y_0)$ is exactly m .

Similarly, we see that when $0 \leq s \leq m - 1$,

$$\max\{h(\gamma) \mid \gamma \in \phi([r_0 r_1 \cdots]_s^{-1})(Y_1)\} = h(\phi([r_0 r_1 \cdots]_s^{-1})(\alpha_m)) = 2s + 1.$$

Thus, $W(I_2^{4m'+2})(Y_1)$ has at least m elements. At the same time $\langle r_1, [r_0 r_1 \cdots]_{2m-1} \rangle$ stabilizes Y_1 , and so by Lagrange's Theorem, $\#W(I_2^{4m'+2})(Y_1) = m$.

When we consider $W(I_2^{4m'+2})(Y_2)$, we need some results from [5, Sect. 5]. We see that Y_2 consists of m' symmetric pairs and 1 symmetric root, where $\alpha \in \Phi^+$ is

called a symmetric root if $\alpha = \epsilon_j - \epsilon_i$ with $i + j = 4m' + 3$, and this numerical information is not changed under $W(I_2^{4m'+2}) \subset W(C_{2m'+1})$ (Weyl group of type $C_{2m'+1}$ in [5, Sect. 5]). It also follows from inspection from the diagram of R_i that the orbit $W(I_2^{4m'+2})(\{\alpha\})$ has at least m elements; hence, using the same argument as the above, we see that $\#W(I_2^{4m'+2})(Y_2)$ is also exactly m .

To prove that the orbits of Y_1 and Y_2 have no intersection, it suffices to verify that Y_2 is not in the orbit of Y_1 . By the above, we see that α only occurs in $Y = \phi([r_0 r_1 \cdots]_{2m'})(Y_1)$ in the orbit of Y_1 under $W(I_2^{2m})$. But $h(\phi([r_0 r_1 \cdots]_{2m'})(\alpha_{m-2})) = 4m' - 2 > 1$, which contradicts the heights of elements in $Y \setminus \{\alpha\}$. With (II) verified, we have proved Theorem 1.6 for $n \equiv 2 \pmod{4}$ and $n \geq 5$.

At last, consider the case when $n = 2m \geq 5$ and $m = 2m'$. The formula $\#W(I_2^{4m'})(Y_1) = \#W(I_2^{4m'})(Y_0) = m$ can be proved by the same argument as the above.

From [5, Sect. 5], we see that Y_1 has m' pairs of symmetric roots and no symmetric root, and Y_2 has $m' - 1$ pairs and 2 symmetric roots α and $\alpha_{2m'}$. Hence, the $W(I_2^{2m})$ -orbits of Y_1 and Y_2 have no intersection.

When $0 \leq s < 2m' - 3$, we have

$$\begin{aligned} & \max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_{s+1}^{-1})(Y_2 \setminus \{\alpha_{2m'}, \alpha\})\} \\ & > \max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_s^{-1})(Y_2 \setminus \{\alpha_{2m'}, \alpha\})\}, \end{aligned}$$

so the orbit of Y_2 has at least $2m' - 2$ elements. Therefore, the cardinality of the stabilizer in $W(I_2^m)$ is smaller than $\frac{8m'}{2m'-2}$. If $m' > 3$, then $\frac{8m'}{2m'-2} < 6$, but the group $\langle r_0, [r_1 r_0 \cdots]_{4m'-1} \rangle$ stabilizes Y_2 ; hence, the subgroup will be the full stabilizer. By checking when $m' = 2, 3$, finally, we see that $\#W(I_2^{4m'})(Y_1) = 2m' = m$. With (II) verified, we have proved the main theorem for $n \equiv 0 \pmod{4}$, and $n \geq 5$. Therefore, we obtain the rank claimed in Theorem 1.6 with Proposition 4.3 and Lemma 4.8. So, the Theorem 1.6 is proved.

We say an algebra is cellular; it implies that we have some ordered relations and pairings on ideas, and we call the algebra cellularly stratified which means it is cellular and has an identity element on each ideal of different levels, and the precise definitions can be found in [9, 10], and [11].

Now, we have the following decomposition of $\text{Br}(I_2^n)$ as a $\mathbb{Z}[\delta^{\pm 1}]$ -module.

$$\begin{aligned} \text{Br}(I_2^n) &= \text{Br}(I_2^n)/(e_0) \oplus (e_0), \quad 2 \nmid n, \\ \text{Br}(I_2^n) &= \text{Br}(I_2^n)/(e_0) \oplus (e_0)/(e_0 e_1 e_0) \oplus (e_0 e_1 e_0), \quad 2 \mid n. \end{aligned}$$

If we read it from diagram version of these algebras, the decomposition is indicating the number of horizontal stands appearing in the diagrams of those monomials. This decomposition gives us better understanding of the ideal structures on those two algebras.

Applying the similar arguments about cellularity in [1, 5, 7], and [14]; therefore, the theorem below can be obtained.

Theorem 5.2 *If R is a field such that the group ring $R[W(I_2^n)]$ is a cellular algebra, then the algebra $\text{Br}(I_2^n) \otimes R$ is a cellularly stratified algebra.*

Remark 5.3 For the hypothesis of the Theorem 5.2, with the method in [8], we conjecture that a sufficient condition for $R[W(I_2^n)]$ is a cellular algebra is that the characteristic of R does not divide n .

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