On intransitive graph-restrictive permutation groups

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Received: 5 December 2012 / Accepted: 5 October 2013 / Published online: 26 October 2013
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Abstract Let Γ be a finite connected G -vertex-transitive graph and let v be a vertex of Γ . If the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is permutation isomorphic to L , then (Γ, G) is said to be *locally L* . A permutation group L is *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally L pair (Γ, G) and a vertex v of Γ , the inequality $|G_v| \leq c(L)$ holds. We show that an intransitive group is graph-restrictive if and only if it is semiregular.

Keywords Vertex-transitive · Graph-restrictive · Semiregular

1 Introduction

A graph Γ is said to be *G -vertex-transitive* if G is a subgroup of $\text{Aut}(\Gamma)$ acting transitively on the vertex set of Γ . Let Γ be a finite, connected, simple G -vertex-transitive graph and let v be a vertex of Γ . If the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is permutation isomorphic to L , then (Γ, G) is said to be *locally L* . Note that, up to permutation isomorphism, L does not depend on the choice of v , and, moreover, the degree of L is equal to the valency of Γ . In [6, p. 499], the second author introduced the following definition.

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Definition 1.1 A permutation group L is graph-restrictive if there exists a constant $c(L)$ such that, for every locally L pair (Γ, G) and for every vertex v of Γ , the inequality $|G_v| \leq c(L)$ holds.

To be precise, Definition 1.1 is a generalisation of the definition from [6], where the group L is assumed to be transitive. The problem of determining which transitive permutation groups are graph-restrictive was also proposed in [6]. A survey of the state of this problem can be found in [3], where it was conjectured ([3, Conjecture 3]) that a transitive permutation group is graph-restrictive if and only if it is semiprimitive. (A permutation group is said to be *semiregular* if each of its point-stabilisers is trivial and *semiprimitive* if each of its normal subgroups is either transitive or semiregular.)

Having removed the requirement of transitivity from the definition of graph-restrictive, it is then natural to try to determine which intransitive permutation groups are graph-restrictive. The main result of this note is a complete solution to this problem (which we did not expect, given the abundance and relative lack of structure of intransitive groups).

Theorem 1.2 An intransitive and graph-restrictive permutation group is semiregular.

It is easily seen that a semiregular permutation group is graph-restrictive. Indeed, if L is a semiregular permutation group of degree d and (Γ, G) is locally L , then for every arc vw of Γ the group G_{vw} fixes the neighbourhood $\Gamma(v)$ pointwise. Since Γ is connected, it follows that $G_{vw} = 1$ and hence $|G_v| \leq |\Gamma(v)| = d$ and L is graph-restrictive. Thus Theorem 1.2 provides a characterisation of intransitive graph-restrictive groups.

Corollary 1.3 An intransitive permutation group is graph-restrictive if and only if it is semiregular.

Note that an intransitive permutation group is semiregular if and only if it is semiprimitive. In particular, Corollary 1.3 completely settles the intransitive version of [3, Conjecture 3], giving remarkable new evidence towards its veracity.

2 Proof of Theorem 1.2

For the remainder of this paper, let L be a permutation group on a finite set Ω which is neither transitive nor semiregular. We show that L is not graph-restrictive, from which Theorem 1.2 follows.

2.1 The construction

Let $\omega_1, \dots, \omega_k \in \Omega$ be a set of representatives of the orbits of L on Ω . Since L is not transitive, $k \geq 2$ and, since L is not semiregular, we may assume without loss of

Fig. 1 Subgroup lattice for Sect. 2.1

generality that $L_{\omega_1} \neq 1$. Let $n \geq 2$ be an integer and let b_1 be the automorphism of $L_{\omega_1} \times L_{\omega_1}^n = L_{\omega_1}^{n+1}$ defined by

$$(x_0, x_1, \dots, x_{n-1}, x_n)^{b_1} = (x_n, x_{n-1}, \dots, x_1, x_0),$$

for each $(x_0, \dots, x_n) \in L_{\omega_1}^{n+1}$. Similarly, let b_2 be the automorphism of $L_{\omega_1}^n$ defined by

$$(x_1, x_2, \dots, x_{n-1}, x_n)^{b_2} = (x_n, x_{n-1}, \dots, x_2, x_1),$$

for each $(x_1, \dots, x_n) \in L_{\omega_1}^n$. Clearly, b_1 and b_2 are involutions, that is, $b_1^2 = 1$ and $b_2^2 = 1$. Now, let $\langle b_3 \rangle, \dots, \langle b_k \rangle$ be cyclic groups of order 2 and consider the following abstract groups:

$$A := L \times L_{\omega_1}^n,$$

$$B_1 := (L_{\omega_1} \times L_{\omega_1}^n) \rtimes \langle b_1 \rangle,$$

$$B_2 := L_{\omega_2} \times (L_{\omega_1}^n \rtimes \langle b_2 \rangle),$$

$$B_i := L_{\omega_i} \times L_{\omega_1}^n \rtimes \langle b_i \rangle, \quad \text{for } i \in \{3, \dots, k\},$$

$$C_i := L_{\omega_i} \times L_{\omega_1}^n, \quad \text{for } i \in \{1, \dots, k\},$$

where $b_1, \dots, b_k \notin A$. For every $i \in \{1, \dots, k\}$, there is an obvious embedding of C_i in both A and B_i . Hence, in what follows, we regard C_i as a subgroup of both A and B_i . Note that, for each $i \in \{1, \dots, k\}$, we have $A \cap B_i = C_i$, $|B_i : C_i| = 2$ and $|A : C_i| = |L : L_{\omega_i}|$ (see Fig. 1).

Lemma 2.1 *The core of $C_1 \cap \dots \cap C_k$ in A is $1 \times L_{\omega_1}^n$.*

Proof Let K be the core of $C_1 \cap \dots \cap C_k$ in A . Then

$$K = \bigcap_{a \in A} (C_1 \cap \dots \cap C_k)^a = \bigcap_{a \in A} ((L_{\omega_1} \cap \dots \cap L_{\omega_k}) \times L_{\omega_1}^n)^a.$$

Recall that L is a permutation group on Ω and that $\omega_1, \dots, \omega_k$ are representatives of the orbits of L on Ω . We thus find that $L_{\omega_1} \cap \dots \cap L_{\omega_k}$ is core-free in L and hence $K = 1 \times L_{\omega_1}^n$. $\square$

Fig. 2 The graph of groups Y

Let T be the group given by generators and relators

$$T := \langle A, B_1, \dots, B_k \mid \mathcal{R} \rangle,$$

where $\mathcal{R}$ consists only of the relations in $A, B_1, \dots, B_k$ together with the identification of C_i in A and B_i , for every $i \in \{1, \dots, k\}$. We will obtain some basic properties of T , which can be deduced from any textbook on “groups acting on graphs”, such as [1, 2, 5].

We have adopted the notation and terminology of [2] and will follow closely [2, I.4]. Using this terminology, the group T is exactly the fundamental group of the graph of groups Y shown in Fig. 2. The vertices of Y are $A, B_1, \dots, B_k$ and, for each $i \in \{1, \dots, k\}$, there is a (directed) edge C_i from A to B_i .

It follows from [2, I.4.6] that the images of $A, B_1, \dots, B_k, C_1, \dots, C_k$ in T are isomorphic to $A, B_1, \dots, B_k, C_1, \dots, C_k$, respectively. This allows us to identify $A, B_1, \dots, B_k, C_1, \dots, C_k$ with their isomorphic images in T in what follows. In particular, for each $i \in \{1, \dots, k\}$ we still have the equalities $A \cap B_i = C_i$, $|B_i : C_i| = 2$ and $|A : C_i| = |L : L_{\omega_i}|$ in T . Let $\mathcal{T}$ be the graph with vertex set

$$V\mathcal{T} = T/A \sqcup T/B_1 \sqcup \dots \sqcup T/B_k,$$

(where $\sqcup$ denotes the disjoint union) and edge-set

$$E\mathcal{T} = \{\{Ax, B_i x\} \mid x \in T, i \in \{1, \dots, k\}\}.$$

2.2 Results about the group T and the graph $\mathcal{T}$

Clearly, the action of T by right multiplication on $V\mathcal{T}$ induces a group of automorphisms of $\mathcal{T}$. Under this action, the group T has exactly $k + 1$ orbits on $V\mathcal{T}$, namely $T/A, T/B_1, \dots, T/B_k$, and k orbits on $E\mathcal{T}$ with representatives $\{A, B_1\}, \dots, \{A, B_k\}$. This induces a $(k + 1)$ -partition of the graph $\mathcal{T}$.

Observe that the set of neighbours of A in T/B_i is $\{B_i a \mid a \in A\}$. As $|A : (A \cap B_i)| = |A : C_i| = |L : L_{\omega_i}|$, we see that A has $|L : L_{\omega_i}|$ neighbours in T/B_i . It follows that A has valency $\sum_{i=1}^k |L : L_{\omega_i}| = |\Omega|$. A symmetric argument, with the roles of A and B_i reversed, shows that B_i has valency $|B_i : C_i| = 2$. In particular, $\mathcal{T}$ is a $(2, |\Omega|)$ -regular graph.

Lemma 2.2 *The stabiliser of the vertex A in T is the subgroup A and the kernel of the action on the neighbourhood of A is $1 \times L_{\omega_1}^n$.*

Proof The definition of $\mathcal{T}$ immediately shows that A is the stabiliser in T of the vertex A . Moreover, the neighbourhood of A is $\mathcal{T}(A) = \{B_i a \mid i \in \{1, \dots, k\}, a \in A\}$.

Fig. 3 Illustration for the proof of Lemma 2.3

Let K be the kernel of the action of A on $\mathcal{T}(A)$ and let $x \in K$. Clearly, $B_i ax = B_i a$ if and only if $axa^{-1} \in B_i$, that is, $axa^{-1} \in A \cap B_i = C_i$. It follows by Lemma 2.1 that $K = 1 \times L_{\omega_1}^n$. $\square$

One of the most important and fundamental properties of $\mathcal{T}$ is that it is a tree (see [2, I.4.4]). We now deduce some consequences from this pivotal result.

Lemma 2.3 *For each $i \in \{1, \dots, k\}$, we have $A \cap A^{b_i} = C_i$.*

Proof We argue by contradiction and assume that $A \cap A^{b_i} \neq C_i$ for some $i \in \{1, \dots, k\}$. As $|B_i : C_i| = 2$, we see that B_i normalises C_i and hence $C_i < A \cap A^{b_i}$. In particular, there exist $a, a' \in A \setminus C_i$ with $a' = a^{b_i} = b_i^{-1}ab_i$. It follows that A, B_i, Ab_i and B_iab_i are distinct vertices of $\mathcal{T}$. Now, the definition of $\mathcal{T}$ shows that $(A, B_i, Ab_i, B_iab_i, Ab_i^{-1}ab_i = A)$ is a cycle of length 4 in $\mathcal{T}$ (see Fig. 3). This contradicts the fact that $\mathcal{T}$ is a tree and concludes the proof. $\square$

Lemma 2.4 *The subgroup A is core-free in T . In particular, the group T acts faithfully on T/A .*

Proof Let N be the core of A in T . From Lemma 2.3, we obtain $N \leq C_1 \cap \dots \cap C_k$, and it follows from Lemma 2.1 that $N \leq 1 \times L_{\omega_1}^n$. By construction, the group $\langle b_1, b_2 \rangle$ induces a transitive permutation group on the $n+1$ coordinates of $L_{\omega_1} \times L_{\omega_1}^n$. As the first coordinate of the elements of N is 1 and as N is invariant under $\langle b_1, b_2 \rangle$, we see that every coordinate of N must be equal to 1, that is, $N = 1$. The lemma now follows. $\square$

As every vertex of $\mathcal{T}$ not in T/A has valency 2, we see that $\mathcal{T}$ is the subdivision graph of a tree $\mathcal{T}_0$ with vertex set T/A and valency $|\Omega|$. Clearly, T acts transitively and, in view of Lemma 2.4, faithfully on the vertices of $\mathcal{T}_0$. The tree $\mathcal{T}_0$ and the group T are our main ingredients for the proof of Theorem 1.2. (The auxiliary graph $\mathcal{T}$ was introduced mainly to make it more convenient to apply the results from [2].)

Lemma 2.5 *The stabiliser in T of the vertex A of $\mathcal{T}_0$ is the subgroup A and the action induced by A on its neighbourhood is permutation isomorphic to the action of L on Ω .*

Proof Let $\pi : A \rightarrow L$ be the natural projection onto the first coordinate. In other words, if $a = (a_0, a_1, \dots, a_n) \in A$, with $a_0 \in L$ and with $a_1, \dots, a_n \in L_{\omega_1}$, then $\pi(a) = a_0$. Clearly, the kernel of π is $1 \times L_{\omega_1}^n$, which by Lemma 2.2 is also the kernel of the action of A on the neighbourhood of the vertex A . Denote by $\mathcal{T}_0(A)$ the neighbourhood of A in $\mathcal{T}_0$. The definitions of $\mathcal{T}$ and $\mathcal{T}_0$ yield $\mathcal{T}_0(A) = \{Ab_i a \mid i \in$

$\{1, \dots, k\}, a \in A\}$. Let $\varphi : \mathcal{T}_0(A) \rightarrow \Omega$ be the mapping $\varphi : Ab_i a \mapsto \omega_i^{\pi(a)}$. We show that φ is well-defined and injective.

Indeed, $Ab_i a = Ab_i a'$ for some $a, a' \in A$ if and only if $Ab_i a(a')^{-1}b_i^{-1} = A$, that is, $a(a')^{-1} \in A \cap A^{b_i}$. By Lemma 2.3, $A \cap A^{b_i} = C_i$. Clearly, $a(a')^{-1} \in C_i$ if and only if $\pi(a(a')^{-1}) \in L_{\omega_i}$, that is, $\omega_i^{\pi(a)} = \omega_i^{\pi(a')}$. This shows that φ is well-defined and that it is injective.

Clearly, φ is surjective and hence it is a bijection. For every $a, x \in A$ and for every $i \in \{1, \dots, k\}$, we have $\varphi((Ab_i a)x) = (\varphi(Ab_i a))^{\pi(x)}$. As φ is a bijection, this shows that the action of A on $\mathcal{T}_0(A)$ is permutation isomorphic to the action of L on Ω . $\square$

Recall that a group X is said to be *residually finite* if there exists a family $\{X_m\}_{m \in \mathbb{N}}$ of normal subgroups of finite index in X with $\bigcap_{m \in \mathbb{N}} X_m = 1$.

Lemma 2.6 *The group T is residually finite.*

Proof As the groups $A, B_1, \dots, B_k$ are finite, it follows from [2, I.4.7] that there exist a finite group F and a group homomorphism $\pi : T \rightarrow F$ with $\text{Ker } \pi \cap A = 1$ and $\text{Ker } \pi \cap B_i = 1$ for each $i \in \{1, \dots, k\}$. Write $K = \text{Ker } \pi$. Since F is finite, we have $|T : K| < \infty$.

Since $K \trianglelefteq T$, $K \cap A = 1$ and $K \cap B_i = 1$, it follows that the only element of K fixing a vertex of $\mathcal{T}$ is 1 and hence, by [2, I.5.4], K is a free group. In particular, K is residually finite (see [4, 6.1.9] for example). It follows that there exists a family $\{K_m\}_{m \in \mathbb{N}}$ of normal subgroups of finite index in K with $\bigcap_{m \in \mathbb{N}} K_m = 1$.

Let T_m be the core of K_m in T . As $|T : K_m| = |T : K| |K : K_m| < \infty$, we see that $|T : T_m| < \infty$. Moreover, since $T_m \leq K_m$, we have $\bigcap_{m \in \mathbb{N}} T_m = 1$ and the lemma follows. $\square$

2.3 Proof of Theorem 1.2

We now recall the definition of a normal quotient of a graph. Let Γ be a G -vertex-transitive graph and let N be a normal subgroup of G . Let v^N denote the N -orbit containing $v \in V\Gamma$. Then the *normal quotient* Γ/N is the graph whose vertices are the N -orbits on $V\Gamma$, with an edge between distinct vertices v^N and w^N if and only if there is an edge $\{v', w'\}$ of Γ for some $v' \in v^N$ and some $w' \in w^N$. Observe that the group G/N acts transitively on the graph Γ/N .

Lemma 2.7 *There exists a locally L pair (Γ_n, G_n) such that the stabiliser of a vertex of Γ_n in G_n has order $|L||L_{\omega_1}|^n$.*

Proof By Lemma 2.6, T is residually finite and hence there exists a family $\{T_m\}_{m \in \mathbb{N}}$ of normal subgroups of finite index in T with $\bigcap_{m \in \mathbb{N}} T_m = 1$. Consider the set

$$X = \{a_1 b_i^{-1} a_2 b_j a_3 \mid a_1, a_2, a_3 \in A, i, j \in \{1, \dots, k\}\}.$$

Observe that since A is finite, so is X . In particular, as $\bigcap_{m \in \mathbb{N}} T_m = 1$ and $1 \in X$, there exists $m \in \mathbb{N}$ with $X \cap T_m = 1$. Let $G_n = T/T_m$ and $\Gamma_n = \mathcal{T}_0/T_m$. As $|T : T_m| < \infty$,

the group G_n and the graph Γ_n are finite. Note that Γ_n is connected and G_n -vertex-transitive. We first show that Γ_n has valency $|\Omega|$.

We argue by contradiction and suppose that Γ_n has valency less than $|\Omega|$. It follows from the definition of normal quotient that the vertex A of $\mathcal{T}_0$ must have two distinct neighbours in the same T_m -orbit. Recall that the neighbourhood of A in $\mathcal{T}_0$ is $\{Ab_i a \mid i \in \{1, \dots, k\}, a \in A\}$. In particular, $Ab_i a \neq Ab_j a'$ and $Ab_i a n = Ab_j a'$, for some $i, j \in \{1, \dots, k\}$, $a, a' \in A$ and $n \in T_m$. It follows that $n \in a^{-1}b_i^{-1}Ab_j a' \subseteq X$ and hence $n \in X \cap T_m = 1$, which is a contradiction.

Let K be the kernel of the action of G_n on $V\Gamma_n$. Since the valency of Γ_n equals the valency of $\mathcal{T}_0$, we see that Γ_n is a regular cover of $\mathcal{T}_0$. Since Γ_n is connected, it follows that K acts semiregularly on $V\Gamma_n$ and hence $K = T_m$. By Lemma 2.5, $(\mathcal{T}_0, T)$ is locally L and hence so is (Γ_n, G_n) . Finally, the stabiliser of the vertex AT_m of Γ_n is $AT_m/T_m \cong A/(A \cap T_m) \cong A$, which has order $|A| = |L||L_{\omega_1}|^n$. $\square$

Proof of Theorem 1.2 By Lemma 2.7, for every natural integer $n \geq 2$, there exists a locally L pair (Γ_n, G_n) with $|(G_n)_v| = |L||L_{\omega_1}|^n$, for $v \in V\Gamma_n$. As $|L_{\omega_1}| > 1$, this shows that L is not graph-restrictive. $\square$

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