Finite polytopes have finite regular covers

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Abstract We prove that any finite, abstract n -polytope is covered by a finite, abstract regular n -polytope.

Keywords Abstract polytopes · Regular polytopes · Covers · Monodromy group

1 Introduction

Any point or line segment is a regular convex polytope, admittedly of modest dimension. And every polygon with p vertices is combinatorially equivalent to a regular p -gon. Moving further into the combinatorial domain, it is clear that a typical map $\mathcal{M}$ on a compact surface is nonregular. Even so, it is folklore (and not so hard to prove, see [5, Corollary 6.2]) that $\mathcal{M}$ is a quotient of a regular map $\mathcal{R}$ of finite type $\{p, q\}$, most likely on some new surface, and which, from another point of view, covers the base map $\mathcal{M}$.

On the other hand, it is certainly true that any n -polytope is a quotient of the universal regular n -polytope $\mathcal{U}_n = \{\infty, \dots, \infty\}$, which is indeed infinite if the rank $n \geq 2$.

In this note we look at all these situations from an abstract point of view and prove a reasonable but hitherto unaddressed result: every finite abstract n -polytope $\mathcal{Q}$

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has a finite regular cover $\mathcal{R}$ (Theorem 3.4). This particular finite cover problem has been raised by one of us (B.M.) at various conferences (see also [7] for some related questions). In essence Theorem 3.4 says that there exists a covering map from the infinite polytope $\mathcal{U}_n$ onto $\mathcal{Q}$ which factors through a finite regular n -polytope. Thus the covering relationship between $\mathcal{U}_n$ and $\mathcal{Q}$ can be described as a sequence of two covering relationships: the first between two regular polytopes, namely $\mathcal{U}_n$ and $\mathcal{R}$, and the second between two finite polytopes, namely $\mathcal{R}$ and $\mathcal{Q}$.

2 Abstract polytopes and their automorphism groups

An abstract n -polytope $\mathcal{P}$ has certain key combinatorial properties of the face lattice of a convex n -polytope; in general, however, $\mathcal{P}$ need not be a lattice or have a familiar geometric realization. Let us summarize some general definitions and results, referring to [4] for details. An *abstract n -polytope* $\mathcal{P}$ is a partially ordered set with properties **A**, **B** and **C** below.

A: $\mathcal{P}$ has a strictly monotone rank function with range $\{-1, 0, \dots, n\}$. Moreover, $\mathcal{P}$ has a unique least face F_{-1} and unique greatest face F_n .

An element $F \in \mathcal{P}$ with $\text{rank}(F) = j$ is called a *j -face*; often F_j will indicate a j -face. Each maximal chain or *flag* in $\mathcal{P}$ therefore contains $n + 2$ faces, so that n is the number of *proper* faces in each flag. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags in $\mathcal{P}$. Naturally, faces of ranks 0, 1 and $n - 1$ are called vertices, edges and facets, respectively.

B: Whenever $F < G$ with $\text{rank}(F) = j - 1$ and $\text{rank}(G) = j + 1$, there are exactly two j -faces H with $F < H < G$.

For $0 \leq j \leq n - 1$ and any flag Φ , there thus exists a unique *adjacent* flag Φ^j , differing from Φ in just the face of rank j . With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the vertex set of the *flag graph* for $\mathcal{P}$. If $F \leq G$ are incident faces in $\mathcal{P}$, we call

$$G/F := \{H \in \mathcal{P} \mid F \leq H \leq G\}$$

a *section* of $\mathcal{P}$.

C: $\mathcal{P}$ is *strongly flag-connected*, that is, the flag graph for each section is connected.

It follows that G/F is a $(k - j - 1)$ -polytope in its own right if $F \leq G$ with $\text{rank}(F) = j \leq k = \text{rank}(G)$. For example, if F is a vertex, then the section F_n/F is called the *vertex-figure* over F . Likewise, it is useful to think of the k -face G as having the structure of the k -polytope G/F_{-1} .

A map $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$ between $\mathcal{P}$ and another polytope $\mathcal{Q}$ is called a *homomorphism* if φ preserves incidence in one direction; that is, $F \leq G$ in $\mathcal{P}$ implies $F\varphi \leq G\varphi$ in $\mathcal{Q}$. An *automorphism* of $\mathcal{P}$ is a bijective homomorphism of $\mathcal{P}$ onto itself whose inverse is also a homomorphism.

The *automorphism group* $\Gamma(\mathcal{P})$ consists of all order-preserving bijections on $\mathcal{P}$. We say $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flag set $\mathcal{F}(\mathcal{P})$. In this case we may

choose any one flag $\Phi \in \mathcal{F}(\mathcal{P})$ as *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j , for $0 \leq j \leq n-1$. Each ρ_j has period 2. From [4, 2B] we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*, meaning that it has the following properties **SC1** and **SC2**:

SC1: $\Gamma(\mathcal{P})$ is generated by $\{\rho_0, \dots, \rho_{n-1}\}$. These involutory generators satisfy the commutativity relations typical of a Coxeter group with string diagram, namely

$$(\rho_j \rho_k)^{p_{jk}} = 1 \quad \text{for } 0 \leq j \leq k \leq n-1, \quad (1)$$

where $p_{jj} = 1$ and $p_{jk} = 2$ whenever $|j - k| > 1$. In other words, $\Gamma(\mathcal{P})$ is a *string group generated by involutions* or an *sggi*.

SC2: $\Gamma(\mathcal{P})$ satisfies the *intersection condition*

$$\begin{aligned} \langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle &= \langle \rho_k : k \in I \cap J \rangle \\ \text{for any } I, J &\subseteq \{0, \dots, n-1\}. \end{aligned} \quad (2)$$

The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [4, 2E]. Thus a regular polytope can be identified naturally with a string C-group. We frequently make use of this fact, and depending on context, work with string C-groups rather than polytopes.

The periods $p_j := p_{j-1, j}$ in (1) satisfy $2 \leq p_j \leq \infty$ and are assembled into the *Schläfli symbol* $\{p_1, \dots, p_{n-1}\}$ for the regular polytope $\mathcal{P}$. We also say that $\mathcal{P}$ has *type* indicated by the same symbol and that the group $\Gamma(\mathcal{P})$ has rank n .

In the same way, any sggi $G = \langle g_0, \dots, g_{n-1} \rangle$ also has a Schläfli symbol, type and rank, although $p_j = 1$ can occur (if $g_{j-1} = g_j$; this is impossible for regular polytopes by (2)).

The *dual* $\mathcal{P}^*$ of the polytope $\mathcal{P}$ is obtained by simply reversing the partial order on the underlying set of faces. If $\mathcal{P}$ is regular of type $\{p_1, \dots, p_{n-1}\}$, then $\mathcal{P}^*$ is also regular of type $\{p_{n-1}, \dots, p_1\}$.

Definition 2.1 [4, 2D] Let $\mathcal{R}$ and $\mathcal{P}$ be n -polytopes. A *covering* is a rank and adjacency preserving homomorphism $\eta : \mathcal{R} \rightarrow \mathcal{P}$. (This means that η induces a mapping $\mathcal{F}(\mathcal{R}) \rightarrow \mathcal{F}(\mathcal{P})$ that sends any j -adjacent pair of flags in $\mathcal{R}$ to another such pair in $\mathcal{P}$; it is easy to show that η must then be surjective.) We also say that $\mathcal{R}$ is a *cover* of $\mathcal{P}$ and write $\mathcal{R} \rightarrow \mathcal{P}$.

In verifying that a given sggi (a group satisfying (1)) is a string C-group, the following *quotient criterion* is frequently useful (see [4, Thm. 2E17]).

Lemma 2.2 Let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be an sggi, and let $\Delta = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ be a string C-group (with respect to the distinguished generators σ_i). If the mapping $\rho_j \rightarrow \sigma_j$ for $j = 0, \dots, n-1$ induces a homomorphism $\pi : \Gamma \rightarrow \Delta$, which is one-to-one on $\langle \rho_0, \dots, \rho_{n-2} \rangle$ or on $\langle \rho_1, \dots, \rho_{n-1} \rangle$, then Γ is also a string C-group, and π induces a covering $\mathcal{R} \rightarrow \mathcal{P}$ between the regular polytopes $\mathcal{R}$ and $\mathcal{P}$ corresponding to Γ and Δ , respectively.

If a polytope $\mathcal{R}$ covers a polytope $\mathcal{P}$, then from another point of view, $\mathcal{P}$ will be a quotient of $\mathcal{R}$; see [4, 2D], [3] or [5]. One way to understand how $\mathcal{P}$ arises by identifications in $\mathcal{R}$ is to exploit the monodromy group:

Definition 2.3 Let $\mathcal{P}$ be a polytope of rank $n \geq 1$. For $0 \leq j \leq n - 1$, let g_j be the bijection on $\mathcal{F}(\mathcal{P})$ that maps each flag Φ to the j -adjacent flag Φ^j . Then the *monodromy group* for $\mathcal{P}$ is

$$\text{Mon}(\mathcal{P}) = \langle g_0, \dots, g_{n-1} \rangle$$

(a subgroup of the symmetric group on $\mathcal{F}(\mathcal{P})$).

It is easy to see that $\text{Mon}(\mathcal{P})$ is an sgg. Let us quote from [5] some useful and fairly easily proved results. We let $\text{Stab}_{\text{Mon}(\mathcal{P})}\Phi$ denote the stabilizer of a flag Φ under the action of $\text{Mon}(\mathcal{P})$.

Theorem 2.4 Let $\mathcal{R}$ be a regular n -polytope with base flag Φ , automorphism group $\Gamma(\mathcal{R}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$, and monodromy group $\text{Mon}(\mathcal{R}) = \langle g_0, \dots, g_{n-1} \rangle$. Then there is an isomorphism $\Gamma(\mathcal{R}) \simeq \text{Mon}(\mathcal{R})$ mapping each ρ_j to g_j .

Theorem 2.5 Suppose that $\mathcal{R}$ and $\mathcal{Q}$ are n -polytopes and that

$$\bar{\eta} : \text{Mon}(\mathcal{R}) \rightarrow \text{Mon}(\mathcal{Q})$$

is an epimorphism of sgg's (i.e. mapping specified generators to specified generators, in order). Suppose also that there are flags Λ' of $\mathcal{R}$ and Λ of $\mathcal{Q}$ such that

$$(\text{Stab}_{\text{Mon}(\mathcal{R})}\Lambda')\bar{\eta} \subseteq \text{Stab}_{\text{Mon}(\mathcal{Q})}\Lambda. \quad (3)$$

Then there is a unique covering $\eta : \mathcal{R} \rightarrow \mathcal{Q}$ that maps Λ' to Λ .

Remark 2.6 If $\mathcal{R}$ is regular, then condition (3) is fulfilled automatically, since all flags Λ' are equivalent, with trivial stabilizer, in $\Gamma(\mathcal{R}) \simeq \text{Mon}(\mathcal{R})$ (Theorem 2.4).

3 Regular covers of general polytopes

A key step in our construction is provided by extension results for regular polytopes. By an *extension* of an m -polytope $\mathcal{K}$ we mean an $(m + 1)$ -polytope all of whose facets are isomorphic to $\mathcal{K}$. There are many ways of extending a polytope.

First there is the trivial way. Any m -polytope $\mathcal{K}$ is the facet type of an $(m + 1)$ -polytope with only two facets, here denoted $\{\mathcal{K}, 2\}$ (for example, see [4, 8]). If $\mathcal{K}$ is regular, then $\{\mathcal{K}, 2\}$ is also regular, and $\Gamma(\{\mathcal{K}, 2\}) = \Gamma(\mathcal{K}) \times C_2$, where the C_2 -factor is generated by the $(m + 1)$ -st distinguished generator.

The construction of extensions becomes more complicated if further conditions are imposed on the new polytope. The following lemma and its proof are a variation of Danzer's 2^K construction (see [4, Ch. 8C]). Even though later we mainly require the trivial extension of $\mathcal{K}$ just described (rather than that of Lemma 3.1), we will reproduce here some of the details of the 2^K construction in a form that is slightly different from presentations that have appeared elsewhere.

Lemma 3.1 Suppose $\mathcal{K}$ is a regular m -polytope with automorphism group $\Gamma(\mathcal{K}) = \langle \rho_0, \dots, \rho_{m-1} \rangle$, Schläfli type $\{p_1, \dots, p_{m-1}\}$ and facet set A . Then there is a regular

$(m+1)$ -polytope $\tilde{\mathcal{K}}$ of type $\{p_1, \dots, p_{m-1}, 4\}$, with facets isomorphic to $\mathcal{K}$, and such that $|\Gamma(\tilde{\mathcal{K}})| = |\Gamma(\mathcal{K})| \cdot 2^{|A|}$, when $\mathcal{K}$ is finite. Indeed, $\tilde{\mathcal{K}}$ is finite if and only if $\mathcal{K}$ is finite.

Proof As hinted above, we could simply refer to [4, Theorem 8C2] and take $\tilde{\mathcal{K}} = (2^{\mathcal{K}^*})^*$. But for the reader's convenience, we shall rework here the essentials of that construction. First off, let N be the group of all sequences

$$x : A \rightarrow C_2 := \{\pm 1\},$$

with component-wise multiplication. (If A is infinite, take sequences of finite “support”, meaning that -1 can occur only finitely often.) Thus N is generated by the indicator functions δ_F , $F \in A$, where for facets $F, H \in A$, we have

$$(H)\delta_F = \begin{cases} -1 & \text{if } H = F, \\ +1 & \text{if } H \neq F. \end{cases}$$

We require $\rho_m := \delta_{F_{m-1}}$, the indicator function for the base facet of $\mathcal{K}$.

Now for $x \in N$, $\gamma \in \Gamma(\mathcal{K})$, $F \in A$, we let

$$(F)x^\gamma := (F)\gamma^{-1}x.$$

This defines an action $x \mapsto x^\gamma$ of $\Gamma(\mathcal{K})$ on N , and we let

$$S := N \rtimes \Gamma(\mathcal{K})$$

be the corresponding semidirect (indeed, wreath) product. For ready computation, we abuse notation a bit, taking $N \triangleleft S$ and $\Gamma(\mathcal{K}) < S$, so that $x \cdot \gamma = \gamma \cdot x^\gamma$ for $x \in N$, $\gamma \in \Gamma(\mathcal{K})$. Note in particular that $\delta_F^\gamma = \delta_{(F)\gamma}$ for any facet F . Since the base facet F_{m-1} is stabilized by ρ_j , $0 \leq j \leq m-2$, we have

$$\rho_j \rho_m = \rho_j \delta_{F_{m-1}} = \delta_{F_{m-1}}^{\rho_j} \rho_j = \delta_{(F_{m-1})\rho_j} \rho_j = \delta_{(F_{m-1})} \rho_j = \rho_m \rho_j.$$

Similarly we find that $(\rho_{m-1} \rho_m)^2$ is an element of N and so has period 2. In fact,

$$S = \langle \rho_0, \dots, \rho_{m-1}, \rho_m \rangle$$

is an sggi of rank $m+1$ and type $\{p_1, \dots, p_{m-1}, 4\}$. Using the unique factorization in S given by $S = N\Gamma(\mathcal{K})$, together with the intersection property (2) for $\mathcal{K}$, we soon verify the intersection property for S , too [4, Lemma 8B5]. Thus S is a string C-group of rank $m+1$, so $S = \Gamma(\tilde{\mathcal{K}})$ for just the sort of regular $(m+1)$ -polytope $\tilde{\mathcal{K}}$ which we seek. $\square$

Remark 3.2 In [6], Daniel Pellicer uses ‘CPR-graphs’ to generalize the results in Lemma 3.1. One can construct a regular polytope $\tilde{\mathcal{K}}$ of type $\{p_1, \dots, p_{m-1}, 2s\}$, with facets isomorphic to $\mathcal{K}$, for any integer $s \geq 2$.

Lemma 3.3 *Let $G = \langle g_0, \dots, g_{n-1} \rangle$ be an sggi, and let $0 \leq i \leq n-2$. Suppose there exists a string C-group $H = \langle h_0, \dots, h_i \rangle$ that covers $\langle g_0, \dots, g_i \rangle$. Then there also exists a string C-group $L = \langle l_0, \dots, l_{i+1} \rangle$ that covers $\langle g_0, \dots, g_{i+1} \rangle$. Furthermore, we may choose L in such a way that, if $j \leq i$ and $\langle g_0, \dots, g_j \rangle \simeq \langle h_0, \dots, h_j \rangle$, then $\langle g_0, \dots, g_j \rangle \simeq \langle l_0, \dots, l_j \rangle$ as well. Moreover, if $\langle g_0, \dots, g_{i+1} \rangle$ and H are finite, then we can take L to be finite, too.*

Proof Let $\mathcal{K}$ be a regular $(i + 1)$ -polytope with $\Gamma(\mathcal{K}) \simeq H$. Then there is a regular $(i + 2)$ -polytope $\tilde{\mathcal{K}}$ with facets isomorphic to $\mathcal{K}$. In fact, we can choose any such regular $(i + 2)$ -polytope for $\tilde{\mathcal{K}}$; for example, the trivial extension $\{K, 2\}$ described earlier,¹ or the polytope of Lemma 3.1. Either way, we may suppose that $\Gamma(\tilde{\mathcal{K}}) = \langle h_0, \dots, h_i, h_{i+1} \rangle$.

Now we exploit the fact that the mix of two sggi's is the smallest sggi that covers both sggi's. More explicitly, the *mix* $L = \langle g_0, \dots, g_{i+1} \rangle \diamond \langle h_0, \dots, h_{i+1} \rangle$ (of $\langle g_0, \dots, g_{i+1} \rangle$ and $\langle h_0, \dots, h_{i+1} \rangle$) is the subgroup of the direct product

$$\langle g_0, \dots, g_{i+1} \rangle \times \langle h_0, \dots, h_{i+1} \rangle = \langle g_0, \dots, g_{i+1} \rangle \times \Gamma(\tilde{\mathcal{K}}) \quad (4)$$

generated by all $l_t := (g_t, h_t)$, $0 \leq t \leq i + 1$. Clearly, L is also an sggi of rank $i + 2$. (We refer to [2, 4, 5, 10] for other useful properties of this operation.)

Since H covers $\langle g_0, \dots, g_i \rangle$, we have $H \simeq \langle l_0, \dots, l_i \rangle$, too. Thus we can apply the quotient criterion of Lemma 2.2 to the second natural projection $L \rightarrow \Gamma(\tilde{\mathcal{K}})$ (onto the second component of the mix), and conclude that L is a string C-group. The first natural projection shows that L covers $\langle g_0, \dots, g_{i+1} \rangle$.

Note that if $j \leq i$ and $f : \langle g_0, \dots, g_j \rangle \rightarrow \langle h_0, \dots, h_j \rangle$ is an isomorphism of sggi's, then the map $g_t \mapsto (g_t, (g_t)f) = l_t$, $0 \leq t \leq j$, clearly induces an isomorphism of $\langle g_0, \dots, g_j \rangle$ with $\langle l_0, \dots, l_j \rangle$.

Appealing once more to Lemma 3.1, we see that L is finite if both factors in the direct product (4) are finite. $\square$

The last lemma is just what we need to prove our main result.

Theorem 3.4

- (a) Every finite sggi G is covered by a finite string C-group G' .
- (b) Every finite n -polytope $\mathcal{Q}$ is covered by a finite regular n -polytope $\mathcal{R}$. If $\mathcal{Q}$ has all its k -faces isomorphic to some regular k -polytope $\mathcal{K}$, then we may choose $\mathcal{R}$ to have its k -faces isomorphic to $\mathcal{K}$.

Proof Let $G = \langle g_0, \dots, g_{n-1} \rangle$ be any finite sggi. Clearly, $H_0 := \langle g_0 \rangle$ is a string C-group, as is the dihedral group $H_1 := \langle g_0, g_1 \rangle$. Thus we can begin an inductive construction. Suppose H_j is a finite string C-group covering $\langle g_0, \dots, g_j \rangle$. Take $i = j$ and $H = H_j$ in Lemma 3.3. We obtain a finite string C-group L covering $\langle g_0, \dots, g_{j+1} \rangle$. Now let $H_{j+1} = L$ and iterate. Eventually we get a finite string C-group $G' = H_{n-1}$ that covers G . Note that if $\langle g_0, \dots, g_{k-1} \rangle$ happens to be a string C-group, we can start the iteration with this subgroup as our choice for H . Then it is clear from Lemma 3.3 that at each stage of the iteration the corresponding subgroup of the mix is isomorphic to $\langle g_0, \dots, g_{k-1} \rangle$ and that therefore we end with the corresponding subgroup of G' isomorphic to $\langle g_0, \dots, g_{k-1} \rangle$ as well.

For part (b), we merely apply part (a) to the finite sggi $G = \text{Mon}(\mathcal{Q}) = \langle g_0, \dots, g_{n-1} \rangle$. Let $\mathcal{R}$ be the finite regular n -polytope whose automorphism group

¹We thank an anonymous referee for suggesting the use of the trivial extension. This greatly simplifies many explicit calculations.

is G' constructed in (a). From Theorem 2.5 we conclude that $\mathcal{R}$ covers $\mathcal{Q}$. If all k -faces are isomorphic to $\mathcal{K}$, it is also true that $\langle g_0, \dots, g_{k-1} \rangle \simeq \Gamma(\mathcal{K})$ [5]. We conclude that $\mathcal{R}$ has isomorphic k -faces. $\square$

Remark 3.5 It is clear that a dual result concerning co- k -faces must hold in Theorem 3.4(b).

Corollary 3.6 *Every convex n -polytope $\mathcal{Q}$ has a finite abstract regular cover $\mathcal{R}$. If $\mathcal{Q}$ is simplicial (or simple), then $\mathcal{R}$ is likewise simplicial (or simple).*

The monodromy group G of a polytope $\mathcal{Q}$ always gives rise to a regular *pre-polytopal* cover $\mathcal{V}$, constructed from G as a coset geometry in much the same way as a regular polytope can be rebuilt from a given string C-group Γ . However, as the following example shows, this object $\mathcal{V}$ can fail condition C concerning strong flag-connectedness.

Example 3.7 Suppose $\mathcal{Q}$ is a pyramid over the toroidal base $\{4, 4\}_{(3,0)}$. Recall that $\{4, 4\}_{(3,0)}$ is the map on the 2-torus obtained from a 3×3 chessboard by identifying opposite sides (see [1]). Now the ‘lateral’ facets of the self-dual 4-polytope $\mathcal{Q}$ are the nine ordinary pyramids over the square faces in the toroid. Using GAP [9], we find that the monodromy group $G = \langle g_0, g_1, g_2, g_3 \rangle$ is an ssgi of type $\{12, 12, 12\}$ and order $2^{12} \cdot 3^{11} \cdot 5$. However, the intersection condition (2) fails since $\langle g_1, g_2 \rangle$ has index 2 in $\langle g_0, g_1, g_2 \rangle \cap \langle g_1, g_2, g_3 \rangle$. Following our earlier remarks, we could manufacture a regular *pre-polytopal* cover $\mathcal{V}$ of $\mathcal{Q}$ with automorphism group G . We find, however, that the section between a typical vertex and facet of $\mathcal{V}$ consists of two disjoint copies of a dodecagon $\{12\}$.

If we want a finite, regular *polytopal* cover $\mathcal{R}$, then we must appeal to Theorem 3.4 and its proof. Since the subgroup $\langle g_0, g_1, g_2 \rangle$ is a string C-group (of order $2^{12} \cdot 3^3$), we actually need to employ an extension of the corresponding regular 3-polytope $\mathcal{K}$ just once. This polytope $\mathcal{K}$ has 4608 facets $\{12\}$.

The regular extension $\tilde{\mathcal{K}}$ of $\mathcal{K}$ described in Lemma 3.1 has type $\{12, 12, 4\}$ with group order $|\Gamma(\tilde{\mathcal{K}})| = 2^{12} \cdot 3^3 \cdot 2^{4608}$. The resulting regular cover $\mathcal{R}$ still has type $\{12, 12, 12\}$, with facets isomorphic to $\mathcal{K}$. Note that this particular regular cover of $\mathcal{Q}$ is quite large, with a group order bounded by

$$|G \times \Gamma(\tilde{\mathcal{K}})| = 2^{4632} \cdot 3^{14} \cdot 5.$$

On the other hand, if $\tilde{\mathcal{K}}$ is the trivial regular extension $\{\mathcal{K}, 2\}$ described earlier, then $\tilde{\mathcal{K}}$ is of type $\{12, 12, 2\}$. The corresponding regular cover $\mathcal{R}$ again has type $\{12, 12, 12\}$, but now its automorphism group is much smaller than in the previous case and has order $2^{23} \cdot 3^{14} \cdot 5$.

It can be shown that there are (at least) two mutually non-isomorphic minimal regular covers of the 4-polytope $\mathcal{Q}$. Each has type $\{12, 12, 12\}$ and a group of order $2^{13} \cdot 3^{11} \cdot 5$, which is twice the order of the monodromy group of $\mathcal{Q}$. It may be conjectured that these two are the only minimal regular covers of $\mathcal{Q}$.

Remark 3.8 One might ask whether there is some sort of extension of Theorem 3.4(b) to the class of infinite polytopes $\mathcal{Q}$. Perhaps each has a regular cover with finite ‘covering index’.

To see that there is no hope for a general statement of this sort, we begin with $\mathcal{T} = \{4, 4\}$. This familiar tiling of the Euclidean plane $\mathbb{R}^2$ by unit squares is an infinite regular 3-polytope. Next, for every odd integer $n \geq 3$, we subdivide the square with southwest vertex $(n, 0)$, using an $(n+2)$ -gon together with n triangles. The southeast vertex now has degree $n+3$. (The figure illustrates the case $n = 5$.) The resulting 3-polytope $\mathcal{Q}$ clearly has a trivial automorphism group $\Gamma(\mathcal{Q})$.

Even though $\mathcal{Q}$ is locally finite, it is also clear that any regular cover $\mathcal{R}$ must have Schläfli type $\{\infty, \infty\}$. But, in any case, each fibre over a face of $\mathcal{Q}$ induced by the regular cover $\mathcal{R} \rightarrow \mathcal{Q}$ must have infinite cardinality.

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