On normal 2-geodesic transitive Cayley graphs

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Abstract We investigate connected normal 2-geodesic transitive Cayley graphs $\text{Cay}(T, S)$. We first prove that if $\text{Cay}(T, S)$ is neither cyclic nor $K_{4[2]}$, then $\langle a \rangle \setminus \{1\} \subsetneq S$ for all $a \in S$. Next, as an application, we give a reduction theorem proving that each graph in this family which is neither a complete multipartite graph nor a bipartite 2-arc transitive graph, has a normal quotient that is either a complete graph or a Cayley graph in the family for a characteristically simple group. Finally we classify complete multipartite graphs in the family.

Keywords Cayley graph · Normal 2-geodesic transitivity

1 Introduction

In this paper, graphs are finite, simple, connected and undirected. In a non-complete graph Γ , a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-arc if $u \neq w$, and a 2-geodesic if in addition u, w are not adjacent. An arc is an ordered pair

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Fig. 1 Two examples

of adjacent vertices. A graph Γ is said to be $(G, 2)$ -arc transitive or $(G, 2)$ -geodesic transitive if its automorphism subgroup G firstly is transitive on arcs, and in addition is transitive on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc. Thus each incomplete $(G, 2)$ -arc transitive graph is $(G, 2)$ -geodesic transitive. The first remarkable result about $(G, 2)$ -arc transitive graphs comes from Tutte [25, 26], and this family of graphs has been studied extensively, see [1, 18, 20, 21, 23]. However, if Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. For example, the graphs in Fig. 1 are the complete multipartite graphs $K_{3[2]}$ and $K_{4[2]}$ which are $(G, 2)$ -geodesic transitive but not $(G, 2)$ -arc transitive. The papers [6, 7] give classifications of all finite graphs Γ which are $(G, 2)$ -geodesic transitive but not $(G, 2)$ -arc transitive, and which have valency 4 or prime valency, respectively.

A general study of $(G, 2)$ -geodesic transitive graphs begun in [13, Chap. 3] highlights the role of a subfamily of ‘basic’ examples, many of which are Cayley graphs. The aim of this paper is to study (not necessarily basic) Cayley graphs which are $(G, 2)$ -geodesic transitive relative to a group G which demonstrates both the transitivity on 2-geodesics, and the property of being a Cayley graph. Such graphs are said to be normal $(G, 2)$ -geodesic transitive (see Definition 1.1). The philosophy behind this approach is similar to that in [24] and [28]. Our analysis identifies both the fundamental role of innately transitive groups G and the exceptional role of complete multipartite graphs in this study. We explore the latter class of graphs in depth.

For a finite group T , and a subset S of T such that $1 \notin S$ and $S = S^{-1}$, the Cayley graph $\Gamma := \text{Cay}(T, S)$ of T with respect to S is the graph with vertex set T and edge set $\{\{g, sg\} \mid g \in T, s \in S\}$. In particular, Γ is connected if and only if $T = \langle S \rangle$. The group $R(T) = \{\sigma_t \mid t \in T\}$ of right translations $\sigma_t : x \mapsto xt$ is a subgroup of the automorphism group $\text{Aut}(\Gamma)$ and acts regularly on the vertex set. We may identify T with $R(T)$. Godsil [9, Lemma 2.1] observed that $N_{\text{Aut}(\Gamma)}(T) = T \rtimes \text{Aut}(T, S)$ where $\text{Aut}(T, S) = \{\sigma \in \text{Aut}(T) \mid S^\sigma = S\}$. If $\text{Aut}(\Gamma) = N_{\text{Aut}(\Gamma)}(T)$, then the graph Γ was called a *normal Cayley graph* by Xu [28] and such graphs have been studied under various additional conditions on their order or valencies, for example in [8, 15, 16]. In any case, $T \rtimes \text{Aut}(T, S)$ is the largest subgroup of $\text{Aut}(\Gamma)$ which witnesses that Γ is a Cayley graph for T . Cayley graphs for which this group is transitive on edges were called *normal edge transitive* and studied in [19, 24].

Definition 1.1 Let $\Gamma = \text{Cay}(T, S)$ for a finite group T and a subset $S \subseteq T \setminus \{1\}$, $S = S^{-1}$. Then Γ is said to be *normal 2-geodesic transitive* if it is $(G, 2)$ -geodesic

transitive for a group G satisfying $T \leq G \leq T \rtimes \text{Aut}(T, S)$. We also say that Γ is *normal* $(G, 2)$ -geodesic transitive if we wish to specify the group G .

There are many normal $(G, 2)$ -geodesic transitive Cayley graphs. One simple example is $\Gamma := \text{Cay}(T, S)$ where $T = \langle a \rangle \cong \mathbb{Z}_r$ for some $r \geq 4$, $S = \{a, a^{-1}\}$, and $G = T \rtimes H (\cong \text{Aut}(\Gamma))$ where $H = \langle \alpha \rangle$ with $\alpha : a \mapsto a^{-1}$. Our analysis of this family of graphs follows two lines of attack, one local and the other a global structural description. The former approach identifies the cycle C_r and the complete multipartite graph $K_{4[2]}$ as exceptions. The complete multipartite graph $K_{m[b]}$ with $m \geq 3, b \geq 2$, has vertex set consisting of m parts of size b , with edges between all pairs of vertices from distinct parts; in particular, $K_{3[2]}$ is the octahedron.

Theorem 1.2 *Let $\Gamma = \text{Cay}(T, S)$ be a connected normal $(G, 2)$ -geodesic transitive Cayley graph. Then one of the following holds.*

- (1) $\Gamma \cong C_r$ and $T \cong \mathbb{Z}_r$ for some $r \geq 4$.
- (2) $\Gamma \cong K_{4[2]}$, $T \cong Q_8$, the quaternion group, and $S = T \setminus Z(T)$.
- (3) *There is a prime q and an integer m such that for all $a \in S$, $|a| = q$ and $\langle a \rangle \setminus \{1\} \subsetneq S$, and $|b| = m$ for each $b \in S^2 \setminus (S \cup \{1\})$.*

Our global approach is similar to that introduced in [24] but our proofs are necessarily more complex. We study normal quotients. Let Γ be a G -vertex transitive graph. If N is an intransitive normal subgroup of G , then the set of N -orbits $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ forms a G -invariant partition of $V(\Gamma)$. The *quotient graph* Γ_N of Γ is the graph with vertex set $\mathcal{B}$ such that B_i, B_j are adjacent in Γ_N if and only if there exist $x \in B_i, y \in B_j$ such that x, y are adjacent in Γ (Γ_N is also called a *normal quotient graph* of Γ). Further, Γ is said to be a *cover* of Γ_N if for each edge $\{B_i, B_j\}$ of Γ_N and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$. (The notation $\Gamma(v)$ denotes the set of vertices of Γ at distance 1 with v .) The next theorem is a reduction result, which highlights the exceptional nature of the graphs $K_{m[b]}$.

Theorem 1.3 *Let $\Gamma = \text{Cay}(T, S)$ be a connected normal $(G, 2)$ -geodesic transitive Cayley graph. If T is not a minimal normal subgroup of G , then there exists a normal subgroup N of G such that $N < T$ and one of the following holds.*

- (1) $\Gamma \cong K_{q^f[p^e]}$ for primes p, q and integers e, f , and $\Gamma_N \cong K_{q^f}$.
- (2) Γ is bipartite, $(G, 2)$ -arc transitive and $\Gamma_N \cong K_2$.
- (3) Γ is a cover of $\Gamma_N \cong \text{Cay}(T/N, SN/N)$, where T/N is a minimal normal subgroup of G/N . Further, either $\Gamma_N \cong K_{|S|+1}$ is G/N -arc transitive, or Γ_N is normal $(G/N, 2)$ -geodesic transitive.

Remark 1.4 (1) Although all the graphs $\Gamma = K_{q^f[p^e]}$ in Theorem 1.3(1) are $(G, 2)$ -geodesic transitive for $G = \text{Aut}(\Gamma)$, the requirement of being normal $(G, 2)$ -geodesic transitive places strong restrictions on both the group T and the parameters p^e, q^f . For example, by Theorem 1.2, $K_{4[2]}$ is a normal 2-geodesic transitive Cayley graph for Q_8 , while the octahedron $K_{3[2]}$ although 2-geodesic transitive is not normal 2-geodesic transitive. We will investigate further those complete multipartite graphs which are normal 2-geodesic transitive, see Theorem 1.5.

(2) As mentioned above, $(G, 2)$ -arc transitive graphs have been studied intensively, so this theorem may genuinely be viewed as a reduction pathway for investigating normal $(G, 2)$ -geodesic transitive Cayley graphs. The study of such graphs arising in Theorem 1.3(3) reduces to the following three problems: investigating the case that T is a minimal normal subgroup of G , studying the 2-geodesic transitive covers of these graphs, and investigating the 2-geodesic transitive covers of complete graphs. In the first problem, the group G is innately transitive on $V(\Gamma)$ with a regular minimal normal subgroup T . A permutation group is *innately transitive* if it has a transitive minimal normal subgroup. Innately transitive groups have been classified in [2] in the same spirit as the O’Nan–Scott Theorem for primitive groups.

Let p be a prime and a, n be positive integers. Then a p -group G is called a *special p -group* if its center Z and derived subgroup coincide and both Z and G/Z are elementary abelian; p is called a *primitive prime divisor* of $a^n - 1$ if p divides $a^n - 1$ but does not divide $a^i - 1$ for any $i < n$. A transitive group G on a set Ω is a *Frobenius group* if $G_x \neq 1$, but $G_{x,y} = 1$ for any $x \neq y \in \Omega$.

Theorem 1.5 *Suppose that $\Gamma \cong K_{m[b]}$ ($m \geq 3, b \geq 2$) is a normal $(G, 2)$ -geodesic transitive Cayley graph $\text{Cay}(T, S)$. Then one of the following holds.*

- (1) $m = 4, b = 2$ and $T = Q_8$.
- (2) mb is a power of a prime p , and T is either elementary abelian or a special p -group of exponent p .
- (3) $m = q, b = p^e$ for distinct primes p, q such that $q - 1$ divides e , q is a primitive prime divisor of $p^{q-1} - 1$ and $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ is a Frobenius group with $\mathbb{Z}_q$ acting as a group of $F_{p^{q-1}}$ -scalar transformations on $\mathbb{Z}_p^e$. Moreover, $G = T.G_0$ with $\mathbb{Z}_p^e \leq G_0 \leq \text{AGL}(e/(q-1), p^{q-1})$ and G_0 induces the full group of field automorphisms of $F_{p^{q-1}}$.

Conversely, for all parameters m, b as in (1), (2) or (3), there exists at least one triple T, S, G such that the graph $K_{m[b]}$ is a normal $(G, 2)$ -geodesic transitive Cayley graph $\text{Cay}(T, S)$.

We comment on the claim of the last sentence of Theorem 1.5. Constructions to prove this claim are given in the proof of Theorem 1.2 for case (1), in Example 4.6 for case (2), and Example 4.10 for case (3). However, for each of the graphs constructed in Example 4.6, the group T is elementary abelian. We give a further construction in Example 4.8 which shows, by Lemma 4.9, that there are examples for extra special p -groups T of order p^3 in Theorem 1.5(2). It would be interesting to know for precisely which p -powers m, b there is a special p -group T giving rise to an example in Theorem 1.5(2).

2 Proof of Theorem 1.2

For a graph Γ and a vertex v , we denote by $\Gamma_2(v)$ the set of vertices of Γ at distance 2 from v . (Recall that $\Gamma(v)$ is the set of vertices of Γ at distance 1 from v .) Also, $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and edges of Γ , respectively.

Throughout this section, we assume that $\Gamma = \text{Cay}(T, S)$ is a connected normal $(G, 2)$ -geodesic transitive Cayley graph. Then $S = S^{-1}$ and $T = \langle S \rangle$. Let G_1 be the stabiliser of 1_T in G . Since $G \leq T \rtimes \text{Aut}(T, S)$, it follows that $G_1 \leq \text{Aut}(T, S)$, and since Γ is $(G, 2)$ -geodesic transitive, G_1 is transitive on both S and $\Gamma_2(1) = S^2 \setminus (S \cup \{1\})$.

Lemma 2.1 *All elements of S have the same order, all elements of $\Gamma_2(1)$ have the same order and $\Gamma_2(1) = \Gamma_2(1)^{-1}$. Moreover, if $\langle a \rangle \setminus \{1\} \subseteq S$ for some $a \in S$, then $\langle a \rangle \setminus \{1\} \subsetneq S$ and Theorem 1.2(3) holds.*

Proof The first assertion follows from the facts that $G_1 \leq \text{Aut}(T, S)$, G_1 is transitive on both S and $\Gamma_2(1) = S^2 \setminus (S \cup \{1\})$ and $S = S^{-1}$. Suppose that $\langle a \rangle \setminus \{1\} \subseteq S$ for some $a \in S$. If $S = \langle a \rangle \setminus \{1\}$, then $T = \langle S \rangle = \langle a \rangle = S \cup \{1\}$, so Γ is complete, contradicting that Γ has 2-geodesics. Hence $\langle a \rangle \setminus \{1\} \subsetneq S$. It follows that $o(a)$ must be a prime number and $\langle x \rangle \setminus \{1\} \subsetneq S$ for all $x \in S$, and Theorem 1.2(3) holds. $\square$

A special p -group is called an *extra special p -group* if its center is cyclic of order p .

Proof of Theorem 1.2 Let $a \in S$. Suppose first that $\langle a \rangle \setminus \{1\} \subseteq S$. Then it follows from Lemma 2.1 that (3) holds. Now assume that $\langle a \rangle \setminus \{1\} \not\subseteq S$. Let i be the smallest positive integer such that $a^i \notin S$, and let $|S| = m \geq 2$. Note that $a^i \neq 1$ since $\langle a \rangle \not\subseteq S \cup \{1\}$, so $a^i = a \times a^{i-1} \in \Gamma_2(1) \cap \Gamma(a)$.

Since Γ is $(G, 2)$ -geodesic transitive, $G_{1,a}$ acts transitively on $\Gamma_2(1) \cap \Gamma(a)$. Since $G_{1,a} \leq G_1 \leq \text{Aut}(T, S)$ and $G_{1,a}$ fixes a , it follows that $G_{1,a}$ fixes a^i . Hence $\Gamma_2(1) \cap \Gamma(a) = \{a^i\}$, and so $\Gamma(a) = (\Gamma(a) \cap S) \cup \{1, a^i\}$. Thus, $|\Gamma_2(1) \cap \Gamma(a)| = 1$, $|\Gamma(1) \cap \Gamma(a)| = m - 2$. Since Γ is G -arc transitive, it follows that for every arc (c, c') , we have $|\Gamma(c) \cap \Gamma(c')| = m - 2$ and $|\Gamma_2(c) \cap \Gamma(c')| = 1$. In particular, $|\Gamma(a^i) \cap \Gamma(a)| = m - 2$ and this set must be contained in S , since it does not contain 1 or a^i . Thus $|\Gamma(a^i) \cap \Gamma(1)| \geq m - 1$, and counting arcs (c, c') with $c \in \Gamma(1)$ and $c' \in \Gamma_2(1)$ yields $|\Gamma_2(1)| \leq \frac{m-1}{m-1} \leq 2$ with equality if and only if $m = 2$.

If $m = 2$, then Γ has valency 2. Since Γ is connected, it follows that $\Gamma \cong C_r$ for some integer $r \geq 3$. Hence $|T| = r$ and $|S| = 2$. Let $S = \{a, s\}$. Since $S = S^{-1}$ and $\langle a \rangle \setminus \{1\} \not\subseteq S$, we must have $a = s^{-1}$. It follows from $T = \langle S \rangle$ that $T \cong \mathbb{Z}_r$, and since $\langle a \rangle \setminus \{1\} \not\subseteq S$, we have $r \geq 4$, so (1) holds.

Now, suppose that $m - 2 \geq 1$. Then $|\Gamma_2(1)| < 2$ and hence $\Gamma_2(1) = \{a^i\}$. This implies that each element of S is adjacent to a^i and hence $\Gamma(a^i) = S$, and $T = S \cup \{1, a^i\}$. It follows that $\Gamma \cong K_{n[2]}$ where $n = \frac{m+2}{2}$.

Suppose that $o(a) = i'$ and $o(a^i) = j$. If $i' = j$, then since all elements of S have the same order and $T \setminus \{1\} = S \cup \{a^i\}$, all elements of $T \setminus \{1\}$ have the same order i' , and hence $i' = q$ is a prime number. Thus T is a q -group. Since $\Gamma \cong K_{n[2]}$, 2 divides $|T|$, and hence $q = 2$. However, if $q = 2$, then $\langle a \rangle \setminus \{1\} = \{a\} \subseteq S$, contradicting the assumption $\langle a \rangle \setminus \{1\} \not\subseteq S$. Thus, $i' \neq j$. Now, since $T = S \cup \{1, a^i\}$, we know that a^i is the unique element with order j . Since $o(a^{-i}) = j$, it follows that $a^i = a^{-i}$, so $j = 2$. Thus, a^i is the unique involution of T , and hence $a^i \in Z(T)$. Moreover, since $\langle a \rangle$ contains a^i of order 2, it follows that $i' = o(a)$ is even. Thus $i' = 2k \geq 4$

and $o(a^2) = k \neq i'$, so $a^2 \in \Gamma_2(1)$. Hence $a^2 = a^{i'}$, and so $i' = 4$. It follows from $T = S \cup \{1, a^2\}$ that T is a 2-group.

Since $m > 2$, we have $|T| > 4$. Thus $|T| \geq 8$. The facts that T has exponent 4 and a^i is the unique involution imply that T is not abelian, $Z(T) = \langle a^i \rangle$, each element $t \in T \setminus \langle a^i \rangle$ has order 4 and $t^2 = a^i$. Hence $T/Z(T)$ has exponent 2 and so is elementary abelian. Therefore T is an extra special group. Since T has a unique involution, it follows that $T \cong Q_8$, and $\Gamma \cong K_{4|2|}$, so (2) holds. $\square$

3 Proof of Theorem 1.3

In all lemmas of this section, we assume that $\Gamma = \text{Cay}(T, S)$ is a normal $(G, 2)$ -geodesic transitive Cayley graph. A subgraph X of Γ is an *induced subgraph* provided two vertices of X are adjacent in X if and only if they are adjacent in Γ . When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U . The first lemma determines some important properties of normal subgroups N of G such that $N < T$.

Lemma 3.1 *Let N be a normal subgroup of G such that $N < T$. Then the following statements hold.*

- (1) $S \cap N = \emptyset$, and $d_\Gamma(x, y) \geq 2$ for any $x, y \in N$.
- (2) $d_\Gamma(x, y) = 2$ for some $x, y \in N$ if and only if either $\Gamma_N \cong K_2$ or $\Gamma \cong K_{m[b]}$ and $\Gamma_N \cong K_m$ for some $m \geq 3, b \geq 2$.
- (3) If $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$, then Γ is a cover of $\Gamma_N \cong \text{Cay}(T/N, SN/N)$. Further, either $\Gamma_N \cong K_{|S|+1}$ is $\overline{G}$ -arc transitive or Γ_N is non-complete, connected, and $(\overline{G}, 2)$ -geodesic transitive, where $\overline{G}$ is the group induced by G on $V(\Gamma_N)$.

Proof (1) Since N is a proper subgroup of T , N is not transitive on $V(\Gamma)$. Hence N is a block of imprimitivity for the action of G on $V(\Gamma)$. Since Γ is connected and edge transitive, it follows that N does not contain edges of Γ . Thus, $S \cap N = \emptyset$ and $d_\Gamma(x, y) \geq 2$ for distinct $x, y \in N$.

(2) Assume first that there exist $x, y \in N$ such that $d_\Gamma(x, y) = 2$. Since Γ is $(G, 2)$ -geodesic transitive, Γ is $(G, 2)$ -distance transitive, and so [5, Lemma 5.2] (i) or (iii) holds. If (i) holds, then N has two orbits in $V(\Gamma)$, and so $\Gamma_N \cong K_2$. If (iii) holds, then $\Gamma \cong K_{m[b]}$ and $\Gamma_N \cong K_m$ for some $m \geq 3, b \geq 2$.

Conversely, assume first that $\Gamma_N \cong K_2$. Then N has two non-trivial orbits $B_1 = N$ and B_2 , which are adjacent in Γ_N . By (1), B_1 contains no edges of Γ . Let (x, y) be an arc of Γ such that $x \in B_1$ and $y \in B_2$. Since Γ has valency at least 2, there exists $z \in B_1 \setminus \{x\}$ such that z, y are adjacent in Γ , and hence $d_\Gamma(x, z) = 2$. Next, suppose $\Gamma \cong K_{m[b]}$ and $\Gamma_N \cong K_m$ for some $m \geq 3, b \geq 2$. Let $x \in N$. Since N does not contain edges of Γ , it follows that $N \setminus \{x\} \subseteq \Gamma_2(x)$. Since $|N| = b \geq 2$, there exists $z \in N$ such that $d_\Gamma(x, z) = 2$.

(3) Suppose that $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$. Since N is a proper normal subgroup of T , it follows that N is not transitive on $V(\Gamma)$ and it is regular on each orbit. Further, by [24, Theorem 3], $\Gamma_N \cong \text{Cay}(T/N, SN/N)$.

Let $B_1 = N$ and B_2 be adjacent vertices of Γ_N . Since N is a normal subgroup and T is transitive, $d_\Gamma(u, v) \geq 3$ for distinct u, v within any fixed orbit of N , so there must be some vertex w such that $d_\Gamma(1, w) = 2$, and $w \notin B_1, B_2$. It follows that N has more than two orbits in $V(\Gamma)$. Thus by [22, Lemma 1.5], Γ is a cover of Γ_N .

Let $\bar{G}$ be the group induced by G on $V(\Gamma_N)$. If Γ_N is complete, then by the facts that Γ is $(G, 2)$ -geodesic transitive and Γ is a cover of Γ_N , $\Gamma_N \cong K_{|S|+1}$ is $\bar{G}$ -arc transitive. Suppose that Γ_N is non-complete. Then for any two 2-geodesics (Nu_i, Nv_i, Nw_i) of Γ_N where $i = 1, 2$, there exist 2-geodesics (x_i, y_i, z_i) of Γ such that $x_i \in Nu_i$, $y_i \in Nv_i$ and $z_i \in Nw_i$, as Γ is a cover of Γ_N . Using the $(G, 2)$ -geodesic transitivity of Γ , $(x_1, y_1, z_1)^g = (x_2, y_2, z_2)$ for some $g \in G$, and so $(Nu_1, Nv_1, Nw_1)^g = (Nu_2, Nv_2, Nw_2)$. Thus Γ_N is $(\bar{G}, 2)$ -geodesic transitive. $\square$

Lemma 3.2 *Let N be a normal subgroup of G such that $N < T$, and G_1 be the stabiliser of 1_T in G . Let $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$ where $\bar{T} = T/N$ and $\bar{S} = SN/N$. Let $\bar{G}$ and $\bar{G}_1$ be the group induced by G and G_1 on $V(\bar{\Gamma})$, respectively. Then the following statements hold.*

(1) $\bar{G}_1 \cong G_1/K \leq \text{Aut}(\bar{T}, \bar{S})$ where $K = \{k \in G_1 \mid t^k t^{-1} \in N, \forall t \in T\}$, and $\bar{G} \cong \bar{T} \rtimes \bar{G}_1 \leq \bar{T} \rtimes \text{Aut}(\bar{T}, \bar{S})$.

(2) If $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$, then $K = 1$, $\bar{G}_1 \cong G_1$ and $\bar{G} \cong G/N$.

Proof (1) Let K be the kernel of the induced G_1 -action on $V(\bar{\Gamma})$. Then $\bar{G}_1 \cong G_1/K$. Now $k \in K$ if and only if $(Nt)^k = Nt$, or equivalently $t^k t^{-1} \in N$, for every $Nt \in \bar{T}$. Thus $K = \{k \in G_1 \mid t^k t^{-1} \in N, \forall t \in T\}$.

Let $\phi(g)$ be the permutation of $\bar{T}$ induced by $g \in G_1$. Then $(Nt)^{\phi(g)} = Nt^g$ for all $Nt \in \bar{T}$. Since $g \in G_1 \leq \text{Aut}(T, S)$ and N is g -invariant, it follows that $\phi(g) \in \text{Aut}(\bar{T}, \bar{S})$. Also $\phi(g) = 1$ is equivalent to $t^g t^{-1} \in N$ for all t , that is, $g \in K$. Thus $\phi: G_1 \mapsto \phi(G_1)$ is a surjective homomorphism with kernel K , and so $\bar{G}_1 \cong G_1/K \cong \phi(G_1) \leq \text{Aut}(\bar{T}, \bar{S})$.

Finally, since $G = T \rtimes G_1$, and since the stabiliser of N in G is $N \rtimes G_1$, which induces $\bar{G}_1$ on $\bar{T}$, it follows that $\bar{G} = \bar{T} \rtimes \bar{G}_1 \leq \bar{T} \rtimes \text{Aut}(\bar{T}, \bar{S})$.

(2) Suppose that $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$. Then $N \cap (S \cup \Gamma_2(1)) = \emptyset$. Let the map ϕ be as in (1), and let $k \in K$, $s \in S$. Then $(Ns)^{\phi(k)} = Ns^k = Ns$ and $s^k s^{-1} \in N \cap (\{1\} \cup S \cup \Gamma_2(1))$. Since $N \cap (S \cup \Gamma_2(1)) = \emptyset$, it follows that $s^k s^{-1} = 1$. Hence $s^k = s$ for all $s \in S$. Since $T = \langle S \rangle$ and $K \leq G_1 \leq \text{Aut}(T)$, it follows that $k = 1$, and hence $K = 1$. Thus $\bar{G}_1 \cong G_1$. By part (1), $\bar{G} \cong \bar{T} \rtimes G_1$, so the kernel of the G -action on Γ_N is N , that is, $\bar{G} \cong G/N$. $\square$

Lemma 3.3 *Let N be a normal subgroup of G such that $N < T$ and suppose that Γ_N is non-complete. Then $\Gamma_N = \text{Cay}(T/N, SN/N)$ is a normal $(G/N, 2)$ -geodesic transitive Cayley graph.*

Proof Let $\bar{T} = T/N$ and $\bar{S} = SN/N$. Since Γ_N is non-complete, it follows from Lemma 3.1(2) that $d_\Gamma(x, y) \geq 3$ for all distinct $x, y \in N$, and so by Lemma 3.1(3), $\Gamma_N \cong \text{Cay}(\bar{T}, \bar{S})$. Let $\bar{G}$ be the group induced by G on $V(\Gamma_N)$. Then by Lemma 3.1(3), Γ_N is $(\bar{G}, 2)$ -geodesic transitive, and by Lemma 3.2, $\bar{T} \leq \bar{G} \cong G/N \leq \bar{T} \rtimes \text{Aut}(\bar{T}, \bar{S})$.

Let $Ns \in \overline{S}$. Since $S = S^{-1}$ and N is a normal subgroup of T , it follows that $s^{-1} \in S$ and $N^s = N$, and hence $(Ns)^{-1} = s^{-1}N = Ns^{-1} \in \overline{S}$. Thus $\overline{S}^{-1} \subseteq \overline{S}$, that is, $\overline{S} = \overline{S}^{-1}$. On the other hand, since Γ is connected, it follows that Γ_N is also connected, and so $\overline{T} = \langle \overline{S} \rangle$. Thus $\Gamma_N = \text{Cay}(T/N, SN/N)$ is a normal $(G/N, 2)$ -geodesic transitive Cayley graph. $\square$

Lemma 3.4 *Let $\Gamma \cong K_{m[b]}$ for some $m \geq 3, b \geq 2$. Then either $T \cong Q_8$ and $(m, b) = (4, 2)$, or there exist (possibly equal) primes p, q such that elements of $S, \Gamma_2(1)$ have orders q, p , respectively, and $|T| = p^e q^f$ for integers e, f .*

Proof By Lemma 2.1, all elements of S have the same order, say q , and all elements of $\Gamma_2(1)$ have the same order, say p . As $V(\Gamma) = T = \{1\} \cup S \cup \Gamma_2(1)$, it follows that the order of each element of $V(\Gamma) \setminus \{1\}$ is either p or q . By Theorem 1.2, either $T \cong Q_8$ with $(m, b) = (4, 2)$ or for each $a \in S$, $\langle a \rangle \setminus \{1\} \subsetneq S$. Assume the latter case. Then q is a prime since all elements of S have the same order.

Suppose that p is not prime and let p_0 be a prime dividing p . If $x \in \Gamma_2(1)$, then $o(x^{p/p_0}) = p_0 \neq p, 1$, and $o(x^{p_0}) = p/p_0 \neq p, 1$. Hence $x^{p_0}, x^{p/p_0} \in S$, and so $p/p_0 = p_0 = q$ and $p = q^2$. Thus T is a q -group of exponent q^2 . In particular, T is not elementary abelian, so T is not a minimal normal subgroup of G . Thus there exists $1 < N < T$ with $N \trianglelefteq G$. By Lemma 3.1(1), $N \cap S = \emptyset$, so $N \setminus \{1\} \subseteq \Gamma_2(1)$. Since $(N \setminus \{1\})^{G_1} = N \setminus \{1\}$ and G_1 is transitive on $\Gamma_2(1)$, we have $N \setminus \{1\} = \Gamma_2(1)$. This contradicts the fact that $x^{p_0} = x^q \in S$ for $x \in \Gamma_2(1)$. Thus p is prime.

Finally, if r is a prime divisor of $|T|$, then T has elements of order r , and so $r = p$ or q , that is, $|T| = p^e q^f$ for integers e, f . $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let $\Gamma = \text{Cay}(T, S)$ be a normal $(G, 2)$ -geodesic transitive Cayley graph. Suppose that T is not a minimal normal subgroup of G , and let $1 < N < T$ with N a minimal normal subgroup of G . By Lemma 3.1(1), $d_\Gamma(x, y) \geq 2$ for any $x, y \in N$. Suppose that $d_\Gamma(x, y) = 2$ for some $x, y \in N$. Then by Lemma 3.1(2), either $\Gamma_N \cong K_2$ or $\Gamma \cong K_{m[b]}$ and $\Gamma_N \cong K_m$ for some $m \geq 3, b \geq 2$. Assume first that $\Gamma_N \cong K_2$. Since N does not contain edges of Γ , it follows that Γ is a bipartite graph, and so its girth is even and at least 4. Hence each 2-arc of Γ is a 2-geodesic, so Γ is $(G, 2)$ -arc transitive, and (2) holds. Next, assume that $\Gamma \cong K_{m[b]}$ and $\Gamma_N \cong K_m$ for some $m \geq 3, b \geq 2$. Then by Lemma 3.4, $|T| = p^e q^f$ for primes p, q and integers e, f . If $T = Q_8$, then (1) holds with $p = q = 2$. So assume that $T \neq Q_8$. Since the valency of Γ is $(m-1)b \geq 4$, it follows from Theorem 1.2 and Lemma 3.4 that all elements of S have the same prime order, say q , and all elements of $\Gamma_2(1)$ have the same prime order, say p . If $p = q$, then (1) holds, so assume that $p \neq q$. Then every element of T of order p lies in $\Gamma_2(1) = N \setminus \{1\}$, and so N is a Sylow p -subgroup of T . Hence $b = p^e$ and $m = q^f$, for some e, f , and (1) holds.

Now suppose that $d_\Gamma(x, y) \geq 3$ for all $x, y \in N$. Let $\overline{G}$ be the group induced by G on $V(\Gamma_N)$. Then by Lemma 3.1(3), Γ is a cover of $\Gamma_N \cong \text{Cay}(T/N, SN/N)$, and either $\Gamma_N \cong K_{|S|+1}$ is $\overline{G}$ -arc transitive or Γ_N is non-complete connected $(\overline{G}, 2)$ -geodesic transitive. By Lemma 3.2(2), $\overline{G} \cong G/N$. In particular, if Γ_N is non-

complete, then by Lemma 3.3, Γ_N is a normal $(G/N, 2)$ -geodesic transitive Cayley graph, so (3) holds. $\square$

4 Complete multipartite graphs

In this section, we investigate complete multipartite graphs arising in Theorem 1.3(1). In the first two lemmas, we assume that $\Gamma = \text{Cay}(T, S)$ is a normal $(G, 2)$ -geodesic transitive Cayley graph, and also $\Gamma \cong K_{m[b]}$ for some $m \geq 3, b \geq 2$.

Lemma 4.1 *The group T has a unique subgroup N such that N is a minimal normal subgroup of G . Moreover, N is elementary abelian and either $N = T$ or $N = \{1\} \cup \Gamma_2(1)$ has order b .*

Proof Let N be a minimal normal subgroup of G contained in T . By Lemma 3.4, $T \cong Q_8$ or $|T| = p^e q^f$ for primes p, q and integers e, f . Hence T is solvable, and so N is elementary abelian.

Suppose that M is a second minimal normal subgroup of G contained in T . Then M, N are proper subgroups of T and by Lemma 3.1, $M \cap S = N \cap S = \emptyset$. Thus both $M \setminus \{1\}$ and $N \setminus \{1\}$ contain an element of $\Gamma_2(1)$. As M, N are normal subgroups of G , $M \setminus \{1\}$ and $N \setminus \{1\}$ are G_1 -invariant which contradicts the fact that G_1 is transitive on $\Gamma_2(1)$. Therefore, N is unique. Since the G_1 -orbits in $T \setminus \{1\}$ are S and $\Gamma_2(1)$, it follows that N contains $\Gamma_2(1)$, and either $N = T$ or $N = \{1\} \cup \Gamma_2(1)$ of order b . $\square$

We recall some definitions. Let F be a field, G be a group and V be an FG -module. Suppose that $V = V_1 \oplus \cdots \oplus V_r$ where V_i are subspaces of V which are transitively permuted by the action of G . We call $V = V_1 \oplus \cdots \oplus V_r$ an *imprimitive decomposition* of V . If V has no imprimitive decomposition with $r > 1$, then V is a *primitive* FG -module.

Lemma 4.2 *Suppose that $\Gamma_2(1)$ contains an element of prime order p and T is not a p -group. Then $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ is a Frobenius group and q divides $p^e - 1$ for some prime $q \neq p$ and integer e , and $m = q, b = p^e$. In particular, $\mathbb{Z}_p^e = \{1\} \cup \Gamma_2(1)$.*

Proof By Lemma 3.4, $|T| = p^e q^f$ for a prime $q \neq p$, and all elements of $S, \Gamma_2(1)$ have order q, p , respectively. Since T is not a p -group, it follows that T is not a minimal normal subgroup of G . By Lemma 4.1, G has a unique minimal normal subgroup P contained in T and $P = \{1\} \cup \Gamma_2(1)$ is a p -group. Since every element of T of order p lies in $\Gamma_2(1) = P \setminus \{1\}$, it follows that P is the unique Sylow p -subgroup of T . As P is elementary abelian, $P = \mathbb{Z}_p^e$ for some e , and $|P| = b = p^e$. By Sylow's theorem, there exists a Sylow q -subgroup Q of T such that $T = P \rtimes Q$. If T is abelian, then $T = P \times Q$, and Q contains a minimal normal subgroup of G , contradicting Lemma 4.1. Thus T is non-abelian.

Since G_1 leaves P invariant and G_1 is transitive on $S = T \setminus P$, it follows that G_1 induces a transitive action on the non-trivial elements of $T/P \cong Q$, and hence Q is

an elementary abelian q -group, say $Q \cong \mathbb{Z}_q^f$ for some f . So $T \cong \mathbb{Z}_p^e \rtimes \mathbb{Z}_q^f$. Since $|T| = mb$ we have $m = q^f$, so $\Gamma \cong K_{q^f[p^e]}$.

Suppose that $1 \neq x \in Q \cap C_G(P)$. Then for each $g \in G_1$, $x^g \in C_G(P)$, and since G_1 is transitive on S and $Q \setminus \{1\} \subset S$, it follows that $Q \subseteq C_G(P)$. This implies that $T = P \times Q$ is abelian which is a contradiction. Thus $Q \cap C_G(P) = 1$. Now $P \subseteq T \cap C_G(P)$ and so $T \cap C_G(P) = P \cdot (Q \cap C_G(P)) = P$. Thus $Q \cong T/P = T/(T \cap C_G(P)) \cong TC_G(P)/C_G(P) \trianglelefteq G/C_G(P)$. Set $H = G/C_G(P)$.

Identify P with the vector space F^e over the field $F := F_p$, and identify the identity with the zero vector. Since P is a normal subgroup of G , it follows that $H \leq \text{Aut}(P) = GL(e, p)$, and P is an FH -module. Since Γ is $(G, 2)$ -geodesic transitive and $\Gamma_2(0) = P \setminus \{0\}$, it follows that G_0 is transitive on $P \setminus \{0\}$, and in particular, P is an irreducible FH -module. Hence $Z(H)$ is cyclic.

Assume that P is not a primitive FH -module. Then $P = P_1 \oplus \cdots \oplus P_r$ for some $r \geq 2$, where the P_i are subspaces of P that are permuted by H . For each i , let $x_i \in P_i \setminus \{0\}$ and let $y = x_1 \oplus \cdots \oplus x_r$. Then H cannot map x_1 to y , contradicting the transitivity of H on $P \setminus \{0\}$. Thus P is a primitive FH -module. Since $Q \trianglelefteq H$, it follows from [17, Corollary 6.13] that $Q \leq Z(H)$, and so Q is cyclic. Since Q has exponent q , this implies that $Q \cong \mathbb{Z}_q$. Thus $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ and as $Z(H)$ is semiregular on $P \setminus \{0\}$, q divides $p^e - 1$, and Q is semiregular on $P \setminus \{0\}$. Therefore T is a Frobenius group. $\square$

We note the following fact without proof.

Lemma 4.3 *Let T be a finite group. Then the action of $\text{Aut}(T)$ on $T/Z(T)$ is permutationally isomorphic to the conjugation action of $\text{Aut}(T)$ on $\text{Inn}(T)$. In particular, we can identify $T/Z(T)$ with $\text{Inn}(T)$, a normal subgroup of $\text{Aut}(T)$.*

Let $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ be a Frobenius group where p, q are distinct odd primes. Then $Z(T) = 1$, by Lemma 4.3, and we may identify T with $\text{Inn}(T)$ a normal subgroup of $\text{Aut}(T)$.

Lemma 4.4 *Let $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ be a Frobenius group where p, q are distinct primes. Let $P = \mathbb{Z}_p^e$ and $Q = \mathbb{Z}_q$. Identify T with $\text{Inn}(T)$. Then the following two statements hold.*

- (1) $C_{\text{Aut}(T)}(P) = P$.
- (2) $\text{Aut}(T)$ and Q induce the subgroups $\overline{A} := \text{Aut}(T)/P$ and $\overline{Q} := T/P \cong Q$ of $\text{Aut}(P)$, respectively, such that $\overline{Q} \trianglelefteq \overline{A}$ and $\overline{A} = N_{\text{Aut}(P)}(\overline{Q})$. Moreover, $\text{Aut}(T)$ is isomorphic to a subgroup of the holomorph of P .

Proof (1) Since P is abelian, $P \leq C := C_A(P)$ where $A := \text{Aut}(T)$. We claim that $P = C$. Let $Q = \langle x \rangle$. Since $Q \cong T/P$, it follows that $T/P = \langle Px \rangle$. Let $\alpha \in C$. Then as $T = \langle P, x \rangle$ and α acts as the identity on P , so α is determined by its action on x . Since α induces an automorphism $\overline{\alpha}$ of T/P , it follows that $(Px)^{\overline{\alpha}} = Px^\alpha = Px^i$ for some i , so $x^\alpha = yx^i$ for some $y \in P$.

Let $X = \langle P, Q, \alpha \rangle$ and $Y = X \cap C$. Let $\overline{T} = T/P$ and $\overline{Y} = Y/P$. Then $\overline{T} \cap \overline{Y} = 1$. Since both $\overline{T}$ and $\overline{Y}$ are normal subgroups of X/P , it follows that $\overline{T} \cdot \overline{Y} = \overline{T} \times \overline{Y}$.

Since $\bar{\alpha} \in \bar{Y}$, $\bar{\alpha}$ commutes with every element of $\bar{T}$, so $\bar{\alpha}Px = Px\bar{\alpha}$, $(Px)^{\bar{\alpha}} = Px$. Recall that $(Px)^{\bar{\alpha}} = Px^{\alpha} = Px^i$, so $Px^i = Px$. Hence $x^{i-1} \in P \cap Q = 1$, so $i = 1$, that is, $x^{\alpha} = yx$. Since there are $|P|$ choices for y , there are also $|P|$ choices for α , and hence $|C| \leq |P|$, so $C = P$.

(2) The group $A = \text{Aut}(T)$ induces a subgroup $\bar{A} := A/C_A(P) = A/P$ of $\text{Aut}(P)$ that contains the subgroup $\bar{Q} \cong Q$ induced by the conjugation action of Q as a normal subgroup (since $\bar{Q} = QP/P = T/P \trianglelefteq \bar{A}$). Hence $\bar{A} \leq N_{\text{Aut}(P)}(\bar{Q})$. We will show that in fact $\bar{A} = N_{\text{Aut}(P)}(\bar{Q})$.

The holomorph of P is the semidirect product $H := R(P) \rtimes \text{Aut}(P)$ with $\text{Aut}(P)$ acting naturally on $R(P)$. The subgroup $R(P) \rtimes \bar{A}$ of H has a normal subgroup $\bar{T} := R(P) \rtimes \bar{Q}$, and $\bar{T} \cong \text{Inn}(T) \cong T$ since the action of $\bar{Q}$ on $R(P)$ is by definition the same as the action of Q on P . Hence $A \cong R(P) \rtimes \bar{A}$, a subgroup of H . Each element $y \in N_{\text{Aut}(P)}(\bar{Q})$ normalises the subgroup $\bar{T}$ of H and thus induces an automorphism of $\bar{T}$ by conjugation. Thus y is an automorphism of T which is induced by an automorphism of $\bar{T} \cong T$, and hence $y \in \bar{A}$. Thus $\bar{A} = N_{\text{Aut}(P)}(\bar{Q})$. $\square$

Let V be a completely reducible FG -module and let M be an irreducible FG -submodule of V . Then the M -homogeneous part W of V is the sum of all those submodules of V which are isomorphic to M . We also say that W is a *homogeneous part* of V relative to M .

Lemma 4.5 *Suppose that $\Gamma \cong K_{m[b]}$ ($m \geq 3, b \geq 2$) is a normal $(G, 2)$ -geodesic transitive Cayley graph $\text{Cay}(T, S)$, where $T = \mathbb{Z}_p^e \rtimes \mathbb{Z}_q$ is a Frobenius group for distinct primes p, q . Then $q - 1$ divides e , q is a primitive prime divisor of $p^{q-1} - 1$, $\text{Aut}(T)$ preserves on $\mathbb{Z}_p^e$ the structure of an F -vector space $F^{e/(q-1)}$ where $F = F_{p^{q-1}}$, $\mathbb{Z}_q$ acts as a subgroup of F -scalar transformations. Moreover, $G = T.G_u$ where G_u is the stabiliser of $u = 1_T$, and $G_u \leq A\Gamma L(e/(q-1), p^{q-1})$ is such that G_u contains both the group of translations and the group of field automorphisms of F .*

Proof Since $Z(T) = 1$, by Lemma 4.3, we can identify T with $\text{Inn}(T)$. Let $P = \mathbb{Z}_p^e$ and $Q = \mathbb{Z}_q$. Since P is a characteristic subgroup of T and $Z(T) = 1$, $\bar{P} \trianglelefteq A := \text{Aut}(T)$ where $\bar{P}$ is the subgroup of the inner automorphisms induced by P . We identify P with an e -dimensional vector space over the finite field F_p and identify 1 with 0. By Lemma 4.2, $P = \{0\} \cup \Gamma_2(0)$, so $S = T \setminus P$, and it follows that $A = \text{Aut}(T, S)$. Since Γ is normal $(G, 2)$ -geodesic transitive, it follows that $G_0 \leq A$ is transitive on both $P \setminus \{0\}$ and S . Hence A is transitive on $P \setminus \{0\}$. Since $\bar{P}$ acts trivially on $P \setminus \{0\}$ by conjugation, it follows that $A/\bar{P}$ acts transitively on $P \setminus \{0\}$. By Lemma 4.4, $A/\bar{P} \cong N_{\text{Aut}(P)}(Q)$, and hence $N_{\text{Aut}(P)}(Q)$ is transitive on $P \setminus \{0\}$.

Since $Q \leq \text{Aut}(P)$, P is an $F_p Q$ -module. Since the characteristic of F_p is p and $p \neq q$, by Maschke's Theorem [17, p. 4], P is completely reducible. Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_r$ where each P_i is an irreducible $F_p Q$ -submodule. Thus (see [12, p. 187, Theorem 7.3]) there exists a divisor f of e such that each $|P_i| = p^f$, we may identify each P_i with a field $F := F_{p^f}$ of order p^f and there exists $a_i \in F^*$ of order q such that Q acts on each $x \in P_i$ by $\alpha : x \mapsto a_i x$. Note that, since Q is irreducible

on each P_i , p has order f modulo q , that is, q is a primitive prime divisor of $p^f - 1$. On the other hand, by Fermat's Little Theorem, $q | (p^{q-1} - 1)$. Hence $f \leq q - 1$.

Suppose that $P = V_1 \oplus \cdots \oplus V_k$ where each V_j is a P_i -homogeneous part of P for some P_i , and suppose that $k \geq 2$. Then by [10, Theorem 4.1], $N_{\text{Aut}(P)}(Q)$ permutes the elements of $\{V_1, \dots, V_k\}$. Thus for non-zero $v_1, v_2 \in V_1$ and $v_3 \in V_2$, no element of $N_{\text{Aut}(P)}(Q)$ maps v_1 to $v_2 + v_3$, contradicting the transitivity of $N_{\text{Aut}(P)}(Q)$ on $P \setminus \{0\}$. Thus P has only one P_i -homogeneous part, and so $a_1 = \cdots = a_r$, that is, Q acts as a subgroup of F -scalar transformation on P , and $A/\overline{P}$ preserves on P the structure of an F -vector space $F^{e/f}$. In particular, $N_{\text{Aut}(P)}(Q) = GL(e/f, p^f) \cdot \mathbb{Z}_f = \Gamma L(e/f, p^f)$ where $\mathbb{Z}_f$ is the group of Frobenius automorphisms of F and $Q \leq Z(GL(e/f, p^f))$.

Note that $S \cup \{0\}$ is the union of all the Sylow q -subgroups of T , $G_0 \leq A$ is transitive on S and A is an extension of $\overline{P}$ by $GL(e/f, p^f) \cdot \mathbb{Z}_f$, by Lemma 4.4(2). Since $\overline{P}$ is transitive (by conjugation) on the Sylow q -subgroups of T it follows, since G_0 must be transitive on S and $G_0 \leq A$, that $N_A(Q)$ is transitive on Q^* . Now $Q \leq Z(GL(e/f, p^f))$, and hence $N_A(Q)/GL(e/f, p^f) \cong \mathbb{Z}_f$ acts transitively on Q^* . Thus G_0 induces the full group $\mathbb{Z}_f$ of field automorphisms of F and $q - 1$ divides f . Since $f \leq q - 1$, we have $f = q - 1$. By Lemma 4.4(2), A is isomorphic to $\overline{P} \cdot N_{\text{Aut}(P)}(Q)$, and hence $A = A\Gamma L(e/(q - 1), p^{q-1})$. Thus $G_0 \leq A\Gamma L(e/(q - 1), p^{q-1})$ and G_0 contains the group of field automorphisms of F .

Now, we prove that G_0 contains the subgroup $R(P)$ of translations of $\text{Hol}(P)$ the holomorph of P . Since $G_0 \cap R(P) \leq G_0$ and G_0 is irreducible on P , it follows that $G_0 \cap R(P) = 1$ or $R(P)$.

Suppose that $G_0 \cap R(P) = 1$. Since $G_0 \leq A \leq \text{Hol}(P)$, it follows that G_0 is isomorphic to a subgroup of $\text{Hol}(P)/R(P) \cong \text{Aut}(P)$. Since Γ is $(G, 2)$ -geodesic transitive, G_0 acts transitively by conjugation on $\Gamma_2(0) = P \setminus \{0\}$. Thus $R(P) \rtimes G_0$ is an affine 2-transitive group of degree p^e . This means that there are groups H, L such that $H \trianglelefteq G_0 \leq L$ and (H, G_0, L, p^e) are as listed in [3, p. 195, Table 7.3]. Let $Z = Z(L)$. Then $p \nmid |Z|$ and $\overline{H} := HZ/Z \trianglelefteq \overline{G_0} := G_0Z/Z \leq L/Z$.

Let $Z_1 = Z \cap G_0$. Since G_0 is transitive on $\Gamma(0) = T \setminus P$, it follows in particular that G_0 acts transitively by conjugation on the set Ω of p^e Sylow q -subgroups of T . Since $Z_1 \trianglelefteq G_0$, the Z_1 -orbits in this action have length dividing p^e , and since $p \nmid |Z_1|$, it follows that Z_1 lies in the kernel of the action. Hence $\overline{G_0}$ is transitive on Ω , so p^e divides $|\overline{G_0}|$.

Suppose first that G_0 is insoluble. Then $\overline{H}$ is a non-abelian simple group. Since $\overline{H} \trianglelefteq \overline{G_0}$, each $\overline{H}$ -orbit Δ has length $|\Delta| = p^m$ for some $0 \leq m \leq e$. If $m = 0$, then $\overline{H}$ acts trivially on Ω , and so p^e divides $|\overline{G_0}/\overline{H}|$ which is impossible by inspecting the candidates of [3, p. 195, Table 7.3]. Thus $m \geq 1$. Hence $\overline{H}$ has a maximal subgroup of index $p^{m'}$ for some m' satisfying $1 \leq m' \leq m$, and by [11, Theorem 1], it follows that $\overline{H}$, $p^{m'}$ and p^e are as in one of the lines of Table 1. Since $(p, (r^n - 1)/(r - 1)) = 1$, it follows that Line 2 of Table 1 does not occur. Suppose that $\overline{H} \cong A_5$ or $PSL(2, 11)$. Then $\overline{H} \trianglelefteq \overline{G_0} \leq \text{Aut}(\overline{H})$. Since p^e does not divide $|\text{Aut}(\overline{H})|$, we have a contradiction, as p^e divides $|\overline{G_0}|$. Thus $\overline{H} \cong PSp(4, 3)$ and $p^e = 3^4$. Then as $H \cong Sp(4, 3) \cong G_0$, we have $\overline{H} \cong \overline{G_0}$. So $\overline{H}$ has a transitive action of degree 3^4 which is not true, see the Atlas [4, p. 26].

Thus G_0 is soluble. Then H is one of the following: $H \cong SL(d, r) \trianglelefteq G_0 \leq \Gamma L(d, r)$ with $d = 1$, or $d = 2$ and $p \in \{2, 3\}$; $H \cong Q_8 \trianglelefteq G_0 \leq N_{GL(2, p)}(H)$ with

Table 1 Insoluble possibilities for the proof of Lemma 4.5

Line	$\overline{H}$	$p^{m'}$	p^e
1	A_5	5	5^2
2	$PSL(n, r), r = p^i, i.n = e$	$(r^n - 1)/(r - 1)$	p^e
3	$PSL(2, 11)$	11	11^2
4	$PSU(4, 2) \cong PSp(4, 3)$	3^3	3^4

$p^e \in \{5^2, 7^2, 11^2, 23^2\}$; $H \cong D_8 \circ Q_8 \trianglelefteq G_0$ with $p^e = 3^4$. In any of the above cases, p^e does not divide $|G_0|$. Therefore $G_0 \cap R(P) \neq 1$, and so $R(P) \leq G_0$. $\square$

Proof of Theorem 1.5 Suppose that $\Gamma \cong K_{m[b]}$ ($m \geq 3, b \geq 2$) is a normal $(G, 2)$ -geodesic transitive Cayley graph $\text{Cay}(T, S)$. Then by Theorem 1.2, either $T \cong Q_8$ as in case (1), or Theorem 1.2(3) holds. Assume the latter case. Then it follows from Lemma 3.4 that all elements of S have the same prime order, say q , all elements of $\Gamma_2(1)$ have the same prime order, say p , and $|T| = p^e q^f$ for some integers e, f . Assume first that $p = q$. Then T has exponent p . If T is abelian, then $T \cong \mathbb{Z}_p^{e+f}$ is elementary abelian, and (2) holds. Assume that T is non-abelian. By Lemma 4.1, T has a unique subgroup N such that N is a non-trivial normal subgroup of G and $N = \{1\} \cup \Gamma_2(1)$. The G_1 -orbits on T are $\{1\}$, $\Gamma_2(1)$ and S . Suppose that T has a proper characteristic subgroup M such that $M \neq N$. Then $M \triangleleft G$ and by Lemma 4.1, $N \leq M$. Since G_1 leaves M invariant and acts transitively on $S = T \setminus N$, it follows that $M = N$. In particular, $N = Z(T) = \Phi(T) = T'$, so T is a special p -group and (2) holds. Finally, if $p \neq q$, then by Lemmas 4.2 and 4.5, case (3) holds.

Conversely, by Theorem 1.2, $(\Gamma, T, S, G) = (K_{4[2]}, Q_8, Q_8 \setminus Z(Q_8), Q_8.\text{Aut}(Q_8))$ is an example for case (1). By Lemma 4.7, for any prime p , and any powers m, b of p , Example 4.6 gives a construction of a triple T, S, G such that $K_{m[b]}$ is a normal $(G, 2)$ -geodesic transitive Cayley graph $\text{Cay}(G, S)$. This proves the ‘converse statement’ for all parameters m, b occurring in case (2). However, for each of these examples, the group T is elementary abelian. We give a further construction in Example 4.8 which shows, by Lemma 4.9, that at least for some prime powers mb there are also examples with T an extra special p -group.

Finally, by Lemma 4.11, Example 4.10 gives a construction of examples for all m, b as in case (3). $\square$

Example 4.6 Let $T = \mathbb{Z}_p^{e+f}$ where p is a prime and e, f are positive integers. Identify T with an $(e+f)$ -dimensional vector space over the field F_p . Let $\{u_1, \dots, u_e, w_1, \dots, w_f\}$ be the standard basis of T and let $W = \langle w_1, \dots, w_f \rangle$. Let $\Gamma := \text{Cay}(T, S)$ where $S := T \setminus W$. Let $G := T.GL(e+f, p)_W$.

Lemma 4.7 Let $\Gamma = \text{Cay}(T, S)$ and G be the graph and group defined in Example 4.6. Then $\Gamma \cong K_{p^e[p^f]}$ is normal $(G, 2)$ -geodesic transitive.

Proof Since $S = T \setminus W$, it follows that $S = -S$ and $S = (W + s_1) \cup \dots \cup (W + s_r)$ where $r = p^e - 1$, for some $s_i \in S$. Let $w \in W$. Then $w = (w + s) - s$ for any $s \in S$. Since $S = -S$, it follows that $-s \in S$, so $w \in S + S$. Hence $W \subseteq S + S$, so Γ has

diameter 2 and $W \setminus \{0\} = \Gamma_2(0)$. It is straightforward to check that $\Gamma \cong K_{p^e[p^f]}$ with respect to the partition of T given by the W -cosets.

Let $A := \text{Aut}(T) = GL(e + f, p)$. Then A_W leaves $S = T \setminus W$ invariant and is transitive on S . Also $u_1 \in S$, and $(A_W)_{u_1} = G_{0,u_1}$ induces $GL(f, p)$ on W and hence is transitive on $W \setminus \{0\} = \Gamma_2(0) \cap \Gamma(u_1)$. Hence Γ is normal $(G, 2)$ -geodesic transitive. $\square$

Example 4.8 Let p be an odd prime and $T = \langle a, b | a^p = b^p = 1, [a, b] = c, ac = ca, bc = cb \rangle$ be an extra special group of order p^{1+2} and exponent p . Let $Z := Z(T) = \langle c \rangle$, $S := T \setminus Z$ and $\Gamma := \text{Cay}(T, S)$. Let $G := R(T) \rtimes \text{Aut}(T)$.

Lemma 4.9 Let $\Gamma = \text{Cay}(T, S)$ and G be the graph and group defined in Example 4.8. Then $\Gamma \cong K_{p^2[p]}$ is normal $(G, 2)$ -geodesic transitive.

Proof Note that $Z := Z(T) = \langle c \rangle \cong \mathbb{Z}_p$. Since $S = T \setminus Z$, it follows that $\Gamma_2(1) \subseteq Z \setminus \{1\}$, $S = S^{-1}$ and $S = Zs_1 \cup \dots \cup Zs_r$ for some $s_i \in S$ with $r = p^2 - 1$. Let $z \in Z$. Then $z = (zs)s^{-1}$ for any $s \in S$. Since $S = S^{-1}$, it follows that $s^{-1} \in S$, so $Z \subseteq S^2$ and also that $\Gamma_2(1) = Z \setminus \{1\}$, and $\Gamma \cong K_{p^2[p]}$ with respect to the partition of T given by the Z -cosets.

Let $A := \text{Aut}(T)$. Since Z is a characteristic subgroup of T , A leaves Z , and hence also S invariant. Thus $A \leq \text{Aut}(\Gamma)$ so we have $G \leq \text{Aut}(\Gamma)$. By [14, Proposition 4.6.2], $C_A(Z) \cong \mathbb{Z}_p^2 \cdot Sp(2, p)$, and by [27, Theorem 1], $A = C_A(Z) \cdot \text{Aut}(Z) \cong C_A(Z) \cdot \mathbb{Z}_{p-1} \cong \mathbb{Z}_p^2 \cdot GL(2, p)$. Now $GL(2, p)$ induces a transitive action on the $p^2 - 1$ non-trivial Z -cosets, and the stabiliser of Za in A contains the group of inner automorphisms of T induced by $\langle b \rangle$. Since, for each i , $ac^i \in aZ = Za$ is such that $(ac^i)^b = a^b c^i = ac^{i+1}$, it follows that A acts transitively on S . Moreover, the stabiliser $G_{1,a}$ of the arc $(1, a)$ of Γ is equal to $C_A(a)$, which is an extension of the group of inner automorphisms of T induced by $\langle a \rangle$, by the subgroup B of $GL(2, p)$ stabilising the non-zero vector corresponding to Za in the space $T/Z \cong F_p^2$. Since $GL(2, p) = Sp(2, p)B$, we have $A = C_A(Z)B$. It follows that the conjugation action of B on T has $Z \setminus \{1\} = \Gamma_2(1) \cap \Gamma(a)$ as a B -orbit. Thus $G_{1,a}$ is transitive on $\Gamma_2(1) \cap \Gamma(a)$, and hence Γ is normal $(G, 2)$ -geodesic transitive. $\square$

Example 4.10 Let p, q be primes such that q is a primitive prime divisor of $p^{q-1} - 1$ and let $e = (q - 1)r$ for some r . Let P be the additive group of an r -dimensional vector space $V = F^r$ over a field F of order p^{q-1} , and let Q be the subgroup of scalar transformations of V with $|Q| = q$. Let $T = P \cdot Q \leq AGL(V)$, let $S = T \setminus P$ and define $\Gamma = \text{Cay}(T, S)$ and $G = R(T) \rtimes \text{Aut}(T)$, the holomorph of T .

Lemma 4.11 Let $\Gamma = \text{Cay}(T, S)$ and G be the graph and group defined in Example 4.10. Then the group T is a Frobenius group of order $p^e q$, $\text{Aut}(T) \cong A\Gamma L(r, p^{q-1})$, $\Gamma \cong K_{q[p^e]}$ and Γ is a normal $(G, 2)$ -geodesic transitive graph.

Proof Let $Q = \langle x \rangle$. Since $S = T \setminus P$, it follows that $\Gamma_2(1) \subseteq P \setminus \{1\}$, $S = S^{-1}$ and $S = Px \cup Px^2 \cup \dots \cup Px^{q-1}$. Then for any $a \in P$, we have $a = (ax)x^{-1}$ where $ax \in Px \subseteq S$ and $x^{-1} \in S$, so $P \subseteq S^2$ and $\Gamma_2(1) = P \setminus \{1\}$. Hence $\Gamma \cong K_{q[p^e]}$ with respect to the partition of T given by the P -cosets.

Since P is the additive group of the vector space $V = (F_{p^{q-1}})^r$, and Q is a subgroup of scalar transformations of V , it follows that $T = P.Q$ is a Frobenius group of order $p^e q$ and $Q \leq Z(GL(V)) \cong Z(GL(r, p^{q-1}))$. Hence $N_{\text{Aut}(P)}(Q) = GL(r, p^{q-1}).\mathbb{Z}_{q-1}$ where $\mathbb{Z}_{q-1} = \langle \alpha \rangle$ is the automorphism group of the field $F_{p^{q-1}}$. By Lemma 4.4, $N_{\text{Aut}(P)}(Q) = \text{Aut}(T)/P$, so $\text{Aut}(T) \cong P.(GL(r, p^{q-1}).\mathbb{Z}_{q-1}) \cong A\Gamma L(r, p^{q-1})$. Since q is a primitive prime divisor of $p^{q-1} - 1$, the images x^{α^i} and x^{α^j} are distinct for $0 \leq i < j \leq q - 2$. This is because $x^{\alpha^i} = x^{\alpha^j}$ is equivalent to $x = x^{\alpha^{j-i}} = x^{p^{j-i}}$, which implies that $q = |x|$ divides $p^{j-i} - 1$, and this is impossible since $1 \leq j - i \leq q - 2$. Hence $\{x, x^2, \dots, x^{q-1}\} = \{x, x^p, x^{p^2}, \dots, x^{p^{q-2}}\}$. Therefore, $\langle \alpha \rangle$ is transitive on the set $\{x, x^2, \dots, x^{q-1}\}$, and so $\text{Aut}(T)$ is transitive on $Q \setminus \{1\}$. Let $s \in S$. Since S is the set of all elements of order q of T , it follows that $S \cup \{1\}$ is the union of all Sylow q -subgroups of T . Hence there exists a Sylow q -subgroup Q_1 of T such that $s \in Q_1 \setminus \{1\}$. Since $Q = Q_1^t$ for some $t \in T$, it follows that $s^t \in Q \setminus \{1\}$. Since $\text{Aut}(T)$ is transitive on $Q \setminus \{1\}$, it follows that $(s^t)^g = x$ for some $g \in \text{Aut}(T)$. Thus $\text{Aut}(T)$ is transitive on S .

Now $G_1 = \text{Aut}(T)$ acts by conjugation on T , and $\text{Aut}(T) = A\Gamma L(r, p^{q-1}) = \overline{P}.H.K$ with $\overline{P}$ the group of inner automorphisms of T induced by P , $H \cong GL(r, p^{q-1})$ and K the group of field automorphisms. The element $x \in Q \setminus \{1\} \subset S$ is centralised by H and so $G_{1,x}$ contains H . Since H acts transitively on P^* , it follows that $G_{1,x}$ is transitive on $\Gamma_2(1) \cap \Gamma(x) = P^*$. Therefore, Γ is normal $(G, 2)$ -geodesic transitive. $\square$

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