2-Arc-transitive cyclic covers of $K_{n,n} - nK_2$

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Abstract Du et al. (in J. Comb. Theory B 74:276–290, 1998 and J. Comb. Theory B 93:73–93, 2005), classified regular covers of complete graph whose fiber-preserving automorphism group acts 2-arc-transitively, and whose covering transformation group is either cyclic or isomorphic to $\mathbb{Z}_p^2$ or $\mathbb{Z}_p^3$ with p a prime. In this paper, a complete classification is achieved of all the regular covers of bipartite complete graphs minus a matching $K_{n,n} - nK_2$ with cyclic covering transformation groups, whose fiber-preserving automorphism groups act 2-arc-transitively.

Keywords Arc-transitive graph · Covering graph · Lifting · 2-Transitive group · Linear group

1 Introduction

Throughout this paper graphs are finite, simple and undirected. For the group- and graph-theoretic terminology we refer the reader to [14, 16]. For a graph X , let $V(X)$, $E(X)$, $A(X)$ and $\text{Aut } X$ denote the vertex set, edge set, arc set and the full automorphism group of X , respectively. For an arc $(u, v) \in A(X)$, we denote the corresponding undirected edge by uv . An s -arc of X is a sequence $(v_0, v_1, \dots, v_s)$ of $s + 1$ vertices such that $(v_i, v_{i+1}) \in A(X)$ and $v_i \neq v_{i+2}$, and X is said to be 2-arc-transitive if $\text{Aut } X$ acts transitively on the set of 2-arcs of X .

Let X be a graph, and let $\mathcal{P}$ be a partition of $V(X)$ into independent sets of equal size m . The quotient graph $Y := X/\mathcal{P}$ is the graph with vertex set $\mathcal{P}$ and two vertices P_1 and P_2 of Y are adjacent if there is at least one edge between a vertex of P_1 and a vertex of P_2 in X . We say that X is an m -fold cover of Y if the edge set between P_1 and P_2 in X is a matching whenever $P_1 P_2 \in E(Y)$. In this case Y is called the

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base graph of X and the sets P_i are called the *fibers* of X . An automorphism of X which maps a fiber to a fiber is said to be *fiber-preserving*. The subgroup K of all those automorphisms of X which fix each of the fibers setwise is called the *covering transformation group*. It is easy to see that if X is connected then the action of K on the fibers of X is necessarily semiregular; that is, $K_v = 1$ for each $v \in V(X)$. In particular, if this action is regular we say that X is a *regular cover* of Y .

By [21, Theorem 4.1], the class of finite 2-arc-transitive graphs can be divided into the following two subclasses: (i) the 2-arc-transitive graphs with the property that every normal subgroup N of a 2-arc-transitive subgroup G of $\text{Aut } X$ has at most two orbits on vertices; (ii) the 2-arc-transitive regular covers of the graphs given in the case (i).

A finite connected 2-arc-transitive graph X is bipartite if and only if $\text{Aut } X$ has a normal subgroup N having two orbits on vertices. If every nontrivial normal subgroup of $\text{Aut } X$ is transitive on vertices, then $\text{Aut } X$ is said to be *quasiprimitive*. In particular, all primitive groups are quasiprimitive. During the past ten years, a lot of results regarding the primitive, quasiprimitive, bipartite 2-arc-transitive graphs have appeared, see [8, 9, 17–19, 21, 22]. However, the known results concerning the 2-arc-transitive covers are very few. To the best knowledge of the authors, even for complete graphs, it is very difficult to determine all 2-arc-transitive covers of them. In [4], a classification of covers of a complete graph is given, whose fiber-preserving automorphism groups act 2-arc-transitively and whose covering transformation groups are either $\mathbb{Z}_d$ or $\mathbb{Z}_p^2$, and it is generalized in [5] when covering transformation groups are $\mathbb{Z}_p^3$. In this paper, the classification is achieved of all the regular covers of bipartite complete graphs minus a matching $K_{n,n} - nK_2$ with cyclic covering transformation group, whose fiber-preserving automorphism groups act 2-arc-transitively.

The present work has the following motivation. First, in [6], 2-arc-transitive Cayley graphs on dihedral group are classified. In the proof, a special case of cyclic regular covers X of $K_{n,n} - nK_2$ are determined, that is, $\text{Aut}(X)$ contains a dihedral regular subgroup. Hence classifying general 2-arc-transitive cyclic regular covers of $K_{n,n} - nK_2$ without such restriction is one of interesting problems. Second motivation is related to our other recent research problem, that is, to classify covers of a complete graph, whose fiber-preserving automorphism groups act 2-arc-transitively and whose covering transformation groups are metacyclic, while many cases can be concluded to the cyclic regular covers of $K_{n,n} - nK_2$. Therefore, the present work will be an important and necessary step to classify such metacyclic covers of complete graphs. For the other papers on the covers of complete graphs, see [11–13].

A purely combinatorial description of a covering is introduced through a voltage graph, see the next section.

Let Q_3 be the cube as shown in Fig. 1(left), and fix a spanning tree T in Q_3 as shown in Fig. 1(right).

First, we introduce two covers of Q_3 : $X_1(3) := Q_3 \times_{\phi} \mathbb{Z}_3$ and $X(6) := Q_3 \times_{\phi} \mathbb{Z}_6$, with voltage assignments ϕ such that $\phi_{i,j} = 0$ for any tree arc (i, j) , and the voltages on cotree arcs are, respectively, given by

- (1) $X_1(3)$: $\phi_{b,y} = \phi_{c,w} = \phi_{d,x} = \phi_{d,w} = 1$ and $\phi_{c,x} = 2$;
- (2) $X(6)$: $\phi_{b,y} = \phi_{c,w} = \phi_{d,x} = 1$, $\phi_{d,w} = 4$, and $\phi_{c,x} = 5$.

Fig. 1 (Left): the graph Q_3 ; (right): a spanning tree T of Q_3

Next, we introduce a 3-fold cover of $Y := K_{5,5} - 5K_2$, where

$$V(Y) = \{1, 2, 3, 4, 5\} \cup \{1', 2', 3', 4', 5'\},$$

$$E(Y) = \{ij' \mid i \neq j, i, j' \in V(Y)\}.$$

Define $X_2(3) := Y \times_f \mathbb{Z}_3$, with the voltage assignment $f : A(Y) \rightarrow \mathbb{Z}_3$ such that

$$\begin{aligned} f_{1,2'} &= f_{1,3'} = f_{1,4'} = f_{1,5'} = f_{2,1'} = f_{2,3'} \\ &= f_{3,1'} = f_{3,2'} = f_{4,1'} = f_{4,5'} = f_{5,1'} = f_{5,4'} = 0, \\ f_{2,5'} &= f_{3,4'} = f_{4,3'} = f_{5,2'} = 1, \quad f_{2,4'} = f_{3,5'} = f_{4,2'} = f_{5,3'} = 2. \end{aligned}$$

To introduce another family of covers, let $q = r^l$ for an odd prime r and $\mathbb{F}_q^* = \langle \theta \rangle$ the multiplicative group of the field $\mathbb{F}_q$ of order q . Let $K_{q+1,q+1} - (q+1)K_2$ be a complete bipartite graph minus a matching whose vertex set is two copies of the projective line $\text{PG}(1, q)$, where the missing matching consists of all pairs $[i, i']$, $i \in \text{PG}(1, q)$. For any $d \mid q-1$ and $d \geq 2$, define a voltage graph $K_{q+1}^{2d} = (K_{q+1,q+1} - (q+1)K_2) \times_f \mathbb{Z}_d$, where

$$f_{\infty',i} = f_{\infty,j'} = \bar{0} \quad \text{for } i, j \neq \infty; \quad f_{i,j'} = \bar{h} \quad \text{if } j - i = \theta^h, \text{ for } i, j \neq \infty.$$

Now we are ready to state the main result of this paper.

Theorem 1.1 *Let X be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ ($n \geq 4$) with a nontrivial cyclic covering transformation group of order d , whose fiber-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:*

- (1) $n = 4$ and $X \cong K_4^4$ for $d = 2$; $X_1(3)$ for $d = 3$; or $X(6)$ for $d = 6$;
- (2) $n = 5$ and $X \cong X_2(3)$ for $d = 3$;
- (3) $n = q + 1 \geq 6$ and $X \cong K_{q+1}^{2d}$, where $d \mid q - 1$ and $d \geq 2$.

Remark 1.2 For the graph K_{q+1}^{2d} , one may extract from Lemmas 2.9, 3.4 and 3.6 that

- (1) The subgroup $\text{P}\Gamma\text{L}(2, q) \times \langle \sigma \rangle$ in $\text{Aut}(K_{1+q})$ lifts, where σ exchanges each matching pair;
- (2) Let $K \cong \mathbb{Z}_d$ be the covering transformation group and let $\text{PSL}(2, q) \leq \text{Aut}(K_{q+1})$. Then
 - if $d \mid \frac{q-1}{2}$, then $\text{PSL}(2, q)$ is lifted to $\text{PSL}(2, q) \times K$;
 - if $d \mid q - 1$ but $d \nmid \frac{q-1}{2}$ (now d is even), then $\text{PSL}(2, q)$ is lifted to $\text{SL}(2, q)K$, where $\text{SL}(2, q) \cap K \cong \mathbb{Z}_2$.

2 Preliminaries

In this section we introduce some preliminary results needed in Sect. 3.

To describe a covering graph, we need the following definition. A purely combinatorial description of a covering was introduced through a voltage graph by Gross and Tucker [14, 15]. Let Y be a graph and K a finite group. A *voltage assignment* (or, K -*voltage assignment*) of the graph Y is a function $f : A(Y) \rightarrow K$ with the property that $f(u, v) = f(v, u)^{-1}$ for each $(u, v) \in A(Y)$. For convenience, we denote $f(u, v)$ by $f_{u,v}$. The values of f are called *voltages*, and K is called the *voltage group*. The *derived graph* $Y \times_f K$ from a voltage assignment f has as its vertex set $V(Y) \times K$ and as its edge set $E(Y) \times K$, so that an edge (e, g) of $Y \times_f K$ joins a vertex (u, g) to $(v, gf_{u,v})$ for $(u, v) \in A(Y)$ and $g \in K$, where $e = uv$. Clearly, the graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p : Y \times_f K \rightarrow Y$, which is called the *natural projection*. For each $u \in V(Y)$, $\{(u, g) \mid g \in K\}$ is a fiber of u . Moreover, by defining $(u, g')^g := (u, g^{-1}g')$ for any $g \in K$ and $(u, g') \in V(Y \times_f K)$, K can be identified with a subgroup of $\text{Aut}(Y \times_f K)$ fixing each fiber setwise and acting regularly on each fiber. Therefore, p can be viewed as a K -covering. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f . Given a spanning tree T of the graph Y , a voltage assignment f is said to be T -*reduced* if the voltages on the tree arcs are the identity. Gross and Tucker ([15]) showed that every regular cover X of a graph Y can be derived from a T -reduced voltage assignment f with respect to an arbitrary fixed spanning tree T of Y . Moreover, the voltage assignment f naturally extends to walks in Y . For any walk W of Y , let f_W denote the voltage of W . Finally, we say that an automorphism α of Y *lifts* to an automorphism $\bar{\alpha}$ of X if $\alpha p = p\bar{\alpha}$, where p is the covering projection from X to Y .

The first proposition is related to a lifting criterion of an automorphism of a base graph with respect to a voltage assignment.

Proposition 2.1 [20, Corollary 4.3] *Let Y be a connected graph and let X be a cover of Y derived from a voltage assignment f . Then, an automorphism α of Y can be lifted to an automorphism of X if and only if, for each closed walk W in Y , we have $f_W^\alpha = 1$ implies $f_W = 1$.*

The following result may be deduced from the classification of doubly transitive groups (see [1] and [2, Corollary 8.3]).

Proposition 2.2 *Let G be a 3-transitive permutation group of degree at least 5. Then one of the following occurs.*

- (i) $\text{soc}(G)$ is 4-transitive;
- (ii) $\text{soc}(G) = M_{22}$ or A_5 , which are 3-transitive but not 4-transitive;
- (iii) $\text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$, where the projective special linear group $\text{PSL}(2, q)$ is the socle of G which does not act 3-transitively, and G acts on the projective geometry $\text{PG}(1, q)$ in a natural way, having degree $q + 1$, with $q \geq 5$ an odd prime power;

- (iv) $G = \text{AGL}(m, 2)$, with $m \geq 3$;
- (v) $G = \mathbb{Z}_2^4 \rtimes A_7 < \text{AGL}(4, 2)$.

Let G be a finite group and H a proper subgroup of G with $\text{Core}(H) = 1$. Let $D = D^{-1}$ be inverse-closed union of some double cosets of H in G . Then the *coset graph* $X = X(G; H, D)$ is defined by taking $V(X) = \{Hg \mid g \in G\}$ as the vertex set and $D(X) = \{(Hg_1, Hg_2) \mid g_2g_1^{-1} \in D\}$ as the edge set. It follows that G acts by right multiplication faithfully and vertex-transitively on X . In particular, if $H = 1$ we get a *Cayley graph*. Conversely, each vertex-transitive graph is isomorphic to a coset graph (see [23]).

Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G , namely, $D = \bigcup_i Rd_iL$. By $[G : L]$ and $[G : R]$, we denote the set of cosets G relative to L and R , respectively. Define a bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{[Lg, Rdg] \mid g \in G, d \in D\}$. This graph is called the *bicoset graph* of G with respect to L, R and D (see [7]).

Proposition 2.3 [7, Lemmas 2.3, 2.4]

- (i) The bicoset graph $X = \mathbf{B}(G, L, R; D)$ is connected if and only if G is generated by elements of $D^{-1}D$.
- (ii) Let Y be a bipartite graph with bipartition $V(Y) = U(Y) \cup W(Y)$, let G be a subgroup of $\text{Aut}(Y)$ acting transitively on both U and W , let $u \in U(Y)$ and $w \in W(Y)$, and set $D = \{g \in G \mid w^g \in Y_1(u)\}$. Then D is a union of double cosets of G_w and G_u in G , and $Y \cong \mathbf{B}(G, G_u, G_w; D)$. In particular, if $uw \in E(Y)$ and G_u acts transitively on its neighbor, then $D = G_wG_u$.

The next two propositions deal with two basic group-theoretic results.

Proposition 2.4 [16, Satz 4.5] Let H be a subgroup of a group G . Then $C_G(H)$ is a normal subgroup of $N_G(H)$ and the quotient $N_G(H)/C_G(H)$ is isomorphic with a subgroup of $\text{Aut } H$.

Proposition 2.5 (Gaschütz [16, Satz 17.5]) Let G be a finite group. Let A and B be two subgroups of G such that A is abelian normal in G , $A \leq B \leq G$ and $(|A|, |G : B|) = 1$. If A has a complement in B , then A has a complement in G .

For a group G , we let G' denote the commutator subgroup of G . Recall that a group G is an *extension* of N by H if G has a normal subgroup N such that the quotient group G/N is isomorphic to H . In particular, G is a *proper central extension* of N by H if $N \leq Z(G) \cap G'$ is a central subgroup. Such central subgroups are all quotients of a largest group, called the *Schur multiplier* $\text{Mult}(G)$ of G .

Proposition 2.6 [3, page xv] The Schur multiplier of simple group $\text{PSL}(2, q)$ is $\mathbb{Z}_2$ for $q \neq 9$; and $\mathbb{Z}_6$ for $q = 9$.

The next proposition is quoted from [6].

Proposition 2.7 *Let $X \rightarrow Y$ be a regular cyclic covering of connected graphs such that some 2-arc-transitive group $G \leq \text{Aut } X$ projects along $X \rightarrow Y$. Then there exists a regular prime cyclic covering $X_1 \rightarrow Y$ such that some 2-arc-transitive group $G_1 \leq \text{Aut } X_1$ projects along $X_1 \rightarrow Y$.*

Proposition 2.8 *There exists no regular cover X of $K_{n,n} - nK_2$ whose fiber-preserving automorphism groups act 2-arc-transitively and whose covering transformation groups are cyclic, provided either $\text{soc}(G) = M_{22}$ or $\text{soc}(G)$ acts 4-transitively on both parts, where G is the subgroup of the fiber-preserving group fixing both parts setwise.*

Proof In [6], a classification of 2-arc-transitive Cayley graphs on dihedral groups was given. The cyclic regular covers $K_{n,n} - nK_2$ was dealt with in [6, Theorem 6.1]. However, when either $\text{soc}(G) = M_{22}$ or $\text{soc}(G)$ acts 4-transitively on both parts, the proof was independent on that $\text{Aut}(X)$ contains a regular dihedral subgroup. $\square$

Lemma 2.9 *For the graph $X = K_{q+1}^{2d}$, $\text{P}\Gamma\text{L}(2, q) \times \langle \sigma \rangle$ lifts.*

Proof It is well known that $\text{P}\Gamma\text{L}(2, q) = \langle \text{PGL}(2, q), \tau \rangle$, where τ is the field automorphism of $\text{PSL}(2, q)$ (induced by the generator of the full automorphism group of the field $\mathbb{F}_q$, that is, $\theta \rightarrow \theta^r$, where $q = r^l$ and $\mathbb{F}_q^* = \langle \theta \rangle$). It has been shown that in [6] $\text{PGL}(2, q)$ and σ lift. To prove the lemma, it suffices to show that τ lifts.

Let us first recall the voltage assignment of K_{q+1}^{2d} , that is $f(\infty', i) = f(\infty, j') = 1$, for $i, j \neq \infty$; and $f(i, j') = a^{h_{ij}}$, where $K = \langle a \rangle$ and $j - i = \theta^{h_{ij}}$.

Let W be any closed walk in X with $f_W = 1$. Assume that the arc (i, j') (resp. (j', i)) appears $l_{i,j'}$ (resp. $l_{j',i}$) times in W . Then

$$f_W = \prod_{(i,j') \in A(X)} f(i, j')^{l_{i,j'}} = \prod_{(i,j') \in A(X)} a^{h_{ij} l_{i,j'}} = a^{\sum_{(i,j') \in A(X)} l_{i,j'} h_{ij}} = 1.$$

Therefore, we have

$$f_{W^\tau} = \prod_{(i,j') \in A(X)} f(i^\tau, j'^\tau)^{l_{i,j'}} = \prod_{(i,j') \in A(X)} a^{r h_{ij} l_{i,j'}} = (a^{\sum_{(i,j') \in A(X)} l_{i,j'} h_{ij}})^r = 1.$$

By Proposition 2.1, τ lifts, and so $\text{P}\Gamma\text{L}(2, q) \times \langle \sigma \rangle$ lifts. $\square$

From now on, let $q = r^l \geq 5$ for an odd prime r and write $l = 2^n m$, where m is odd. Let $\mathbb{F}_q^* = \langle \theta \rangle$, and let $\tau^r : \theta \rightarrow \theta^r$, the generator of $\text{Aut}(\mathbb{F}_q)$. Then τ^r induces an automorphism, say τ of $\text{PSL}(2, q)$.

The following two lemmas are related to the automorphism group of $\text{Aut}(Y)$, where $Y = K_{n,n} - nK_2$.

Lemma 2.10 *Let G be a minimal 3-transitive permutation group with $\text{soc}(G) = \text{PSL}(2, q)$, where $q = r^l \geq 5$ is an odd prime power. Then $G = \text{PSL}(2, q) \langle \tau^{im} x \rangle$, where x is any given element in $\text{PGL}(2, q) \setminus \text{PSL}(2, q)$ and $0 \leq i \leq 2^n - 1$.*

Proof Let G be a minimal 3-transitive subgroup with $\text{soc}(G) = \text{PSL}(2, q)$. Set $T = \text{soc}(G)$. Then $T \leq G \leq \text{P}\Gamma\text{L}(2, q) = T\langle x, \tau \rangle$, where x is any given element in $\text{P}\Gamma\text{L}(2, q) \setminus T$. By the minimal 3-transitivity of G , we may let $G \leq T\langle \tau^j x \rangle$, for some integer j , noting that $T \rtimes \langle \tau \rangle$ is not 3-transitive.

Note that $\text{P}\Gamma\text{L}(2, q)/T = \langle \bar{x} \rangle \times \langle \bar{\tau} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_l$. Since every minimal subgroup of $\langle \bar{x} \rangle \times \langle \bar{\tau} \rangle$ not contained in $\langle \bar{\tau} \rangle$ must be of the form $\langle \bar{\tau}^{im} \bar{x} \rangle$, where $0 \leq i \leq 2^n - 1$, it follows that $G = \text{PSL}(2, q)\langle \tau^{im} x \rangle$. $\square$

Lemma 2.11 *Let $Y = K_{n,n} - nK_2$ and $\text{Aut}(Y) = S_n \times \langle \sigma \rangle$, where σ is an involution exchanging two parts. Let A be a minimal 2-arc-transitive subgroup of $\text{Aut}(Y)$ and let $G \leq A$ be the corresponding index 2 subgroup of A fixing two parts setwise. Suppose that $\text{soc}(G) = \text{PSL}(2, q)$, where $q = r^l \geq 5$ is an odd prime power. Then one of the following holds:*

- (i) $G = \text{PGL}(2, q)$ and $A = G \times \langle \sigma \rangle$;
- (ii) $G = \text{PGL}(2, q)\langle \tau^{2im} \rangle$ and $A = G\langle \tau^{im} \sigma \rangle$, $1 \leq i \leq 2^n - 1$;
- (iii) $G = \text{PSL}(2, q)\langle \tau^{im} x \rangle$ and $A = G \times \langle \sigma \rangle$, $1 \leq i \leq 2^n - 1$;
- (iv) $G = \text{PSL}(2, q)\langle \tau^{im} x \rangle$ and $A = G\langle \tau^{im} \sigma \rangle$, $1 \leq i \leq 2^n - 1$,

where x is any given element in $\text{PGL}(2, q) \setminus \text{PSL}(2, q)$, and l is even for (ii), (iii) and (iv).

Proof Since A is a minimal 2-arc-transitive subgroup of $\text{Aut}(Y)$, G must be a 3-transitive subgroup on both parts with $T := \text{soc}(G) = \text{PSL}(2, q)$. By Proposition 2.2(iii), we have $T \leq G \leq \text{P}\Gamma\text{L}(2, q)$. Since A is a 2-arc-transitive subgroup of $\text{Aut}(Y) = S_n \times \langle \sigma \rangle$ and $|A : G| = 2$, we get $A \leq \text{P}\Gamma\text{L}(2, q)\langle b\sigma \rangle$, where $b \in S_n$ and $b^2 \in \text{P}\Gamma\text{L}(2, q)$. Since $b\sigma$ normalizes $\text{P}\Gamma\text{L}(2, q)$, b normalizes $\text{P}\Gamma\text{L}(2, q)$. However, it is well known that $N_{S_n}(\text{P}\Gamma\text{L}(2, q)) = \text{P}\Gamma\text{L}(2, q)$, which follows that $b \in \text{P}\Gamma\text{L}(2, q)$ and so $A \leq \text{P}\Gamma\text{L}(2, q) \times \langle \sigma \rangle$. Therefore, $A = G\langle c\sigma \rangle$, where $c \in \text{P}\Gamma\text{L}(2, q)$.

By the minimality of A and $A = G\langle c\sigma \rangle$, where $c \in \text{P}\Gamma\text{L}(2, q)$, we get $|c\sigma| = 2^k$ for some integer k . Since $\text{P}\Gamma\text{L}(2, q)/T = \langle \bar{x} \rangle \times \langle \bar{\tau} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_l$, we may set $c = \tau^{jm} x^s$ for some integer j and $s \in \{0, 1\}$. Let $G_1 \leq G$ be the minimal 3-transitive subgroup in G . Then by Lemma 2.10, $G_1 = T\langle \tau^{im} x \rangle$, where $0 \leq i \leq 2^n - 1$. Set $A_1 := G_1\langle c\sigma \rangle = T\langle \tau^{im} x \rangle\langle \tau^{jm} x^s \sigma \rangle$. Obviously, A_1 is also a 2-arc-transitive group. By the minimality of A again, we get $A = A_1$. Naturally, we consider the following two cases:

Case 1: $\langle \tau^{im} \rangle \leq \langle \tau^{2jm} \rangle$: First suppose that $\tau^{2jm} = 1$. Then, either $\tau^{jm} = 1$ and then $A = \text{PGL}(2, q) \times \langle \sigma \rangle$, that is, case (i) in the lemma; or $\tau^{jm} = \tau^{\frac{l}{2}}$ and then $A = \text{PGL}(2, q) \rtimes \langle \tau^{\frac{l}{2}} \sigma \rangle$, that is, case (ii) in the lemma. Secondly suppose that $\tau^{2jm} \neq 1$. Since $(\bar{\tau}^{jm} \bar{x}^s \bar{\sigma})^2 = \bar{\tau}^{2jm}$ and $\langle \tau^{im} \rangle \leq \langle \tau^{2jm} \rangle$, we get

$$\begin{aligned} A &= \langle G_1, \tau^{2jm} \rangle \langle \tau^{jm} x^s \sigma \rangle = \langle \text{PGL}(2, q), \tau^{2jm} \rangle \langle \tau^{jm} x^s \sigma \rangle \\ &= \text{PGL}(2, q) \langle \tau^{2jm} \rangle \langle \tau^{jm} \sigma \rangle, \end{aligned}$$

that is, case (ii).

Case 2: $\langle \tau^{im} \rangle > \langle \tau^{2jm} \rangle$: In this case, $\langle \tau^{2im} \rangle \geq \langle \tau^{2jm} \rangle$, which follows that $\bar{\tau}^{2jm} \in \langle \bar{\tau}^{im} \bar{x} \rangle$. Thus, $|A_1 : G_1| = 2$, note that $A_1 = A$, which forces that $G_1 = G$. So

G must be a minimal 3-transitive subgroup. First suppose that $s = 0$. Then, one may get $A = (T\langle\tau^{im}x\rangle) \times \langle\sigma\rangle$ while $\langle\tau^{2im}\rangle > \langle\tau^{2jm}\rangle$, which is case (iii); and $A = T\langle\tau^{im}x\rangle\langle\tau^{im}\sigma\rangle$ while $\langle\tau^{2im}\rangle = \langle\tau^{2jm}\rangle$, which is case (iv). Secondly suppose that $s = 1$. Then we get $A = T\langle\tau^{im}x\rangle\langle\tau^{im}\sigma\rangle$ while $\langle\tau^{2im}\rangle > \langle\tau^{2jm}\rangle$, which is case (iv); and $A = (T\langle\tau^{im}x\rangle) \times \langle\sigma\rangle$ while $\langle\tau^{2im}\rangle = \langle\tau^{2jm}\rangle$, which is case (iii). $\square$

3 Proof of Theorem 1.1

Now we prove Theorem 1.1. Let $U = \{1, 2, \dots, n\}$ and $W = \{1', 2', \dots, n'\}$. Set $Y = K_{n,n} - nK_2$ ($n \geq 4$) with the vertex set $V(Y) = U \cup W$ and edge set $E(Y) = \{ij' \mid i \neq j, i, j = 1, 2, \dots, n\}$. Let X be a cover of Y with covering projection $\phi: X \rightarrow Y$ and the covering transformation group $K = \langle a \rangle \cong \mathbb{Z}_d$.

Let A be a 2-arc-transitive group of automorphisms of the base graph Y and let $G = A_U = A_W$. Let $\tilde{A}$ and $\tilde{G}$ be the respective lift of A and G . Clearly, $\text{Aut}(Y) = S_n \times \langle\sigma\rangle$, where σ is the involution exchanging every pair i and i' .

Since G acts 2-arc-transitively on Y , G has a faithful 3-transitive representations on $V(Y)$ and thus G must be one of 3-transitive groups listed in Proposition 2.2. Theorem 1.1 will be proved by two subsections: for the cases: $n = 4, 5$ or $n \geq 6$ but $\text{soc}(G) \neq \text{PSL}(2, q)$ where $q \geq 5$, it will be proved in Sect. 3.1 that $X \cong Q_3, X_1(3), X(6)$ and $X_2(3)$; and for the case $n \geq 6$ and $\text{soc}(G) = \text{PSL}(2, q)$ where $q \geq 5$ it will be proved in Sect. 3.2 that $X \cong K_{q+1}^{2d}$ defined in Sect. 1.

3.1 $n = 4, 5$ or $n \geq 6$ but $\text{soc}(G) \neq \text{PSL}(2, q)$ for $q \geq 5$

Suppose that either $n = 4, 5$ or $n \geq 6$ but $\text{soc}(G) \neq \text{PSL}(2, q)$ for $q \geq 5$. By Proposition 2.8, there exist no covers when either $\text{soc}(G)$ is 4-transitive or $\text{soc}(G) \cong M_{22}$. So, by Proposition 2.2, we only need to consider four cases: $n = 4$; $n = 5$; $G = \text{AGL}(m, 2)$, with $m \geq 3$; and $G = \mathbb{Z}_2^4 \rtimes A_7 < \text{AGL}(4, 2)$. For the second case, we get the cover $X_2(3)$, while the detail is omitted, as its proof is similar to the case when $n \geq 6$ and $\text{soc}(G) = \text{PSL}(2, q)$.

Lemma 3.1 *Suppose that $n = 4$, that is, $Y = Q_3$. Then $X \cong K_4^4, X_1(3)$ or $X(6)$.*

Proof In [10], the coverings of cube whose covering transformation groups are cyclic groups K and whose fiber-preserving automorphism groups act s -regularly are determined. In particular, if $s = 2$, that is, fiber-preserving automorphism groups act 2-arc-transitively, then there are, respectively, only one 2-fold cover, 3-fold cover and 6-fold cover. Since K_4^4 is such a cover (see Lemma 2.9), it follows that $X \cong K_4^4$ for $K \cong \mathbb{Z}_2$. If $K \cong \mathbb{Z}_3$ and $\mathbb{Z}_6$, then it had been proved in [10] that $X \cong X_1(3)$ and $X(6)$ defined in Sect. 1, respectively. $\square$

Lemma 3.2 *Suppose that either $G \cong \text{AGL}(m, 2)$, where $m \geq 3$ or $G = \mathbb{Z}_2^4 \rtimes A_7 < \text{AGL}(4, 2)$. Then there are no connected graph X arising as regular cyclic covers of Y , and the group of fiber-preserving automorphisms of X acts 2-arc-transitively.*

Proof For these two cases, we have the completely same arguments and so here we just discuss the first case in detail.

In view of Proposition 2.7, it suffices to show that no such regular $\mathbb{Z}_p$ -covers, p a prime, exist. Since $C_{\tilde{G}}(K)/K$ is normal in $\tilde{G}/K$, we have $C_{\tilde{G}}(K)/K = 1$, $\mathbb{Z}_2^m$ or $\tilde{G}/K$. If the previous two cases happen, then by Proposition 2.4, we get $\tilde{G}/C_{\tilde{G}}(K) \cong (\tilde{G}/K)/(C_{\tilde{G}}(K)/K) \leq \text{Aut}(K) \cong \mathbb{Z}_{p-1}$, that is, either G or $\text{GL}(m, 2)$ is contained in a cyclic group, a contradiction. Thus $C_{\tilde{G}}(K) = \tilde{G}$, that is, $K \leq Z(\tilde{G})$. Let $\tilde{A}$ be the group of fiber-preserving automorphisms of X acting 2-arc-transitively.

Let $\tilde{U}$ and $\tilde{W}$ be the two parts of X . Take a fiber $F = \{u_1, u_2\}$ in $\tilde{U}$. Set $K = \langle k \rangle \cong \mathbb{Z}_p$, $\tilde{M} := \tilde{G}_{u_1} \cong \text{GL}(m, 2)$ and $\tilde{T}/K = \text{soc}(\tilde{G}/K) \cong \mathbb{Z}_2^m$. Then $\tilde{G} = \tilde{T} \rtimes \tilde{M}$. Let $F' = \{w_1, w_2\}$ denote the unique corresponding fiber in $\tilde{W}$ without edges leading to F . Then $\tilde{G}_F = \tilde{G}_{F'}$. Since $\tilde{M}$ is the unique subgroup which is isomorphic to $\text{GL}(m, 2)$ in $K \times \tilde{M}$, we may let $\tilde{G}_{w_1} = \tilde{M}$. Then we divide the proof into two cases.

Step 1: Case $p \neq 2$

Now $\tilde{G}_F = \tilde{M} \times K$. Then K has a complement in $\tilde{G}$, so that $\tilde{G} = K \times (\tilde{L} \rtimes \tilde{M})$, where $\tilde{L} \cong \mathbb{Z}_2^m$. Since $\tilde{G}$ is transitive on $\tilde{W}$, there exists $x \in \tilde{G}$ such that $(u_1, w_1^x) \in E(X)$. By Proposition 2.3(ii), X is isomorphic to a bicoset graph $\mathbf{B}(\tilde{G}, \tilde{M}, \tilde{M}^x; D)$, where $D = \tilde{M}\tilde{M}^x$. Since $\langle D^{-1}D \rangle = \langle \tilde{M}, \tilde{M}^x \rangle \leq \tilde{L}\tilde{M} \neq \tilde{G}$, it follows from Proposition 2.3(i) that X is disconnected.

Step 2: Case $p = 2$

Since $\tilde{T}/K$ is abelian, we get $\tilde{T}' \leq K$ and so we know that either $\tilde{T}' = 1$ or $\tilde{T}' = K$.

Suppose that $\tilde{T}' = K = \langle k \rangle \cong \mathbb{Z}_2$. Then there exist $\tilde{x}_1$ and $\tilde{x}_2 \in \tilde{T}$, such that $[\tilde{x}_1, \tilde{x}_2] = k$. Take $\tilde{x}_3 \in \tilde{T}$ such that $\tilde{x}_3 K \notin \langle \tilde{x}_1 K, \tilde{x}_2 K \rangle$. As $MK/K \cong \text{GL}(m, 2)$ and the conjugacy action of MK/K on $\tilde{T}/K \setminus K$ is 2-transitive, there exist $\tilde{y}_1 K, \tilde{y}_2 K \in MK/K$, such that

$$(\tilde{x}_1 K, \tilde{x}_3 K)^{\tilde{y}_1} = (\tilde{x}_1 K, \tilde{x}_2 K), \quad (\tilde{x}_1 K, \tilde{x}_2 \tilde{x}_3 K)^{\tilde{y}_2} = (\tilde{x}_1 K, \tilde{x}_2 K).$$

Since $K \leq Z(\tilde{G})$, we get

$$[\tilde{x}_1, \tilde{x}_3]^{\tilde{y}_1} = [\tilde{x}_1, \tilde{x}_2] = k, \quad [\tilde{x}_1, \tilde{x}_2 \tilde{x}_3]^{\tilde{y}_2} = [\tilde{x}_1, \tilde{x}_2] = k,$$

and so

$$[\tilde{x}_1, \tilde{x}_3] = k^{\tilde{y}_1^{-1}} = k, \quad [\tilde{x}_1, \tilde{x}_2 \tilde{x}_3] = k^{\tilde{y}_2^{-1}} = k.$$

However, $[\tilde{x}_1, \tilde{x}_2 \tilde{x}_3] = [\tilde{x}_1, \tilde{x}_2][\tilde{x}_1, \tilde{x}_3] = k^2 = 1$, a contradiction. Hence, we conclude that $\tilde{T}' = 1$, which means that $\tilde{T}$ is an abelian group.

Suppose that there exists an element $\tilde{x}$ of order 4 in $\tilde{T}$. Then for any $\tilde{y} \in \tilde{T} \setminus K$, there exists an element $\tilde{z}K \in MK/K$ such that $(\tilde{y}K)^{\tilde{z}} = \tilde{x}K$. Then $\tilde{y}^{\tilde{z}} = \tilde{x}k^i$, where $K = \langle k \rangle$ and $i \in \{0, 1\}$, which implies $|\tilde{y}| = 4$. Therefore, $\tilde{T}$ has only one involution k . However, from the structure of abelian groups, we know that $\tilde{T}$ has more than one involutions. Therefore, every nontrivial element in $\tilde{T}$ is of order 2, which means that $K \leq \tilde{T} = \mathbb{Z}_2^{m+1}$ and $\tilde{G} = \tilde{T} \rtimes \tilde{M} \leq \text{AGL}(m+1, 2)$.

Now X is a Bi-Cayley Graph, that is, $\tilde{T}$ acts regularly on both parts. Denote the vertices by

$$\tilde{U} = \{v \mid v \in V(m+1, 2)\} \quad \text{and} \quad \tilde{W} = \{\bar{v} \mid v \in V(m+1, 2)\},$$

where let $u_1 = 0$ and $w_1 = \bar{0}$ so that $\tilde{G}_0 = \tilde{G}_{\bar{0}} = \tilde{M}$ and let u_2 and $w_2 = \bar{u}_2$ be the second vectors in the fiber F and F' , respectively.

Let Δ be the neighbor $\bar{0}$, while $\bar{\Delta} = \{\bar{v} \mid v \in \Delta\}$ is the neighbor of 0 . Then the edge set E of X is

$$E = \{\{u, \overline{u+v}\} \mid u \in U, v \in \Delta\}.$$

Now $\tilde{M}$ has eight orbits on $V(X)$, that is,

$$\{0\}, \{u_2\}, \Delta, u_2 + \Delta; \{\bar{0}\}, \{\bar{u}_2\}, \bar{\Delta}, \overline{u_2 + \Delta}.$$

For any positive integer i , by $X_i(v)$ we denote the set of vertices which have the distance i from v . Then

$$X_1(0) = \bar{\Delta} \quad \text{and} \quad X_2(0) = \{v_i + v_j \mid v_i, v_j \in \Delta, i \neq j\}.$$

Since $\tilde{M}$ acts 2-transitively on the arcs initiate from 0 , it follows that $X_2(0)$ is an orbit of $\tilde{M}$, that is, either $X_2(0) = \Delta$ or $X_2(0) = u_2 + \Delta$. For the first case, $X_3(0) = \bar{u}_2$ and for the second case, $X_3(0) = \bar{0}$. Clearly, $X_4(0) = \emptyset$ and then X is disconnected. $\square$

3.2 $n \geq 6$ and $\text{soc}(G) = \text{PSL}(2, q)$ for $q \geq 5$

In this subsection, suppose that $n \geq 6$ and $\text{soc}(G) = \text{PSL}(2, q)$ where $q \geq 5$. The main result in this subsection is to show $X \cong K_{q+1}^{2d}$.

The following lemma gives a characterization for group G .

Lemma 3.3 *Let $\text{soc}(G) = \text{PSL}(2, q)$ and $\tilde{G}$ a lift of G so that $\tilde{G}/K = G$. Then $\tilde{G}$ contains a characteristic subgroup T such that either*

- (i) $T \cong \text{PSL}(2, q)$ and $T \cap K = 1$; or
- (ii) $T \cong \text{SL}(2, q)$ and $T \cap K = \mathbb{Z}_2$.

Proof Since $(\text{P}\Gamma\text{L}(2, q))' = \text{PSL}(2, q)$, we get $G' = \text{PSL}(2, q)$ and so $\tilde{G}$ is insoluble. Hence there exists a positive integer m such that $\tilde{G}^{(m)} = \tilde{G}^{(m+1)}$. Therefore, we have $\tilde{G}^{(m)} \cap K \leq \tilde{G}^{(m)} = (\tilde{G}^{(m)})'$, and moreover,

$$\tilde{G}^{(m)} / (\tilde{G}^{(m)} \cap K) \cong \tilde{G}^{(m)} K / K \cong (\tilde{G}/K)^{(m)} \cong G^{(m)} \cong \text{PSL}(2, q).$$

In what follows, we set $T = \tilde{G}^{(m)}$ and take $\tilde{u} \in \tilde{U}$, so that $T/(T \cap K) \cong \text{PSL}(2, q)$ and $(TK)_{\tilde{u}} \cong \mathbb{Z}_r^l \rtimes \mathbb{Z}_{\frac{q-1}{2}}$, where $q = r^l$.

By Proposition 2.4, we have $\tilde{G}/C_{\tilde{G}}(K) \lesssim \text{Aut}(K)$. As $\text{Aut}(K)$ is abelian, it follows that $T \leq \tilde{G}' \leq C_{\tilde{G}}(K)$, which is equivalent to $K \leq C_{\tilde{G}}(T)$. Thus, $K \cap T \leq Z(T)$. So we conclude that T is a proper central extension of $T \cap K$ by $\text{PSL}(2, q)$. Moreover, by Proposition 2.6, the Schur Multiplier of $\text{PSL}(2, q)$ is either $\mathbb{Z}_2$ for $q \neq 9$ or $\mathbb{Z}_6$ for $q = 9$.

Clearly, $T \cap K = 1$ or $\mathbb{Z}_2$ for $q \neq 9$. In fact, it is also true for $q = 9$. For the contrary, suppose that $3 \mid |T \cap K|$. Take $K_1 \in \text{Syl}_3(K)$. Then $(TK)_{\tilde{u}} \cong \mathbb{Z}_3^2 : \mathbb{Z}_4$. Let $\mathbb{Z}_3^2 \leq M \leq (TK)_{\tilde{u}}$. Since K_1 has a complement in MK and $(|TK : MK|, |K_1|) = 1$, we get from Proposition 2.5 that K_1 has a complement in TK , say N . Thus, $TK = K_1 \times N$ so that $(TK)' = N'$. Since $[K, T] = 1$, one may get

$$N' = (TK)' = [TK, TK] = [T, T] = T' = T \geq T \cap K.$$

As $3 \mid |T \cap K|$ and K_1 is the Sylow 3-subgroup of the cyclic group K , it follows that $3 \mid |N' \cap K_1|$, contradicting to $K_1 \cap N = 1$. Therefore we have either $T \cap K = 1$ or $T \cap K = \mathbb{Z}_2$.

Suppose that $T \cap K = 1$. Then $T \cong \text{PSL}(2, q)$, that is, case (i).

Suppose that $T \cap K = \mathbb{Z}_2$. Then $T \cong \text{SL}(2, q)$, that is, case (ii). $\square$

To find all the possible covers, we need to consider all the minimal 2-arc-transitive groups A , that is, four cases (i)–(iv) in Lemma 2.11. In all cases, it will be proved that $X \cong K_{q+1}^{2d}$. Since we shall have similar arguments for these four cases, for the sake of the length of the paper, we just deal with the case (iv) for $T \cap K = 1$ in Sect. 3.2.1 and the case (i) for $T \cap K = \mathbb{Z}_2$ in Sect. 3.2.2.

Before the proof, let us recall again our notation: $q = r^l \geq 5$ is an odd prime power, $l = 2^n m$ where m is odd. Let $\mathbb{F}_q^* = \langle \theta \rangle$, and let $\tau' : \theta \rightarrow \theta^r$, the generator of $\text{Aut}(\mathbb{F}_q)$. Then τ' induces an automorphism, say τ of $\text{PSL}(2, q)$. Moreover, set $f = \tau^s$, where $s = 2^{n-e}m$ so that f is of order 2^e . Finally, let $K = \langle a \rangle \cong \mathbb{Z}_d$, our covering transformation group.

3.2.1 $G = \text{PSL}(2, q)\langle fx \rangle$ and $A = G\langle f\sigma \rangle$

Lemma 3.4 *Suppose that $G = \text{PSL}(2, q)\langle fx \rangle$, $A = G\langle f\sigma \rangle$ and $\tilde{A}$ contains a characteristic subgroup $T \cong \text{PSL}(2, q)$, which is lift of $\text{PSL}(2, q)$ such that $T \cap K = 1$, where $f = \tau^s$ for $s = 2^{n-e}m$. Let X be a connected graph arising as a regular cyclic cover of $K_{n,n} - nK_2$. Then $n = q + 1$ for some odd prime power $q = r^l$, and $X \cong K_{q+1}^{2d}$, for some d dividing $\frac{q-1}{2}$.*

Proof Suppose that $G = \text{PSL}(2, q)\langle fx \rangle$ and $A = G\langle f\sigma \rangle$. Then we shall first determine the structure of the lifted group $\tilde{G}$ and $\tilde{A}$, and then move on to determine the corresponding coset graphs, and finally we shall show that these graphs are isomorphic to the graphs K_{q+1}^{2d} .

We have $T \cong \text{PSL}(2, q)$ and identify T with $\text{PSL}(2, q)$. In $\text{PGL}(2, q)$, set

$$t_i = \overline{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}}, \quad x = \overline{\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}}, \quad y = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}},$$

where $i \in \mathbb{F}_q$. Let $\overline{Q} = \langle t_i \mid i \in \mathbb{F}_q \rangle \cong \mathbb{Z}_r^l$ and $Q \leq T$ the lift of $\overline{Q}$. Acting on $\text{PG}(1, q)$, we have $G_\infty = \overline{Q} \rtimes \langle x^2, fx \rangle$ and the other points $i \in \text{PG}(1, q) \setminus \{\infty\}$ correspond to the cosets $G_\infty y t_i$.

Let τ_1 and τ_2 be lifts of fx and $f\sigma$, respectively. Then

$$\tilde{G} = (T \times K)\langle \tau_1 \rangle, \quad \tilde{A} = (T \times K)\langle \tau_1, \tau_2 \rangle.$$

In what follows, the proof is divided into the following four steps, reminding the reader of $K = \langle a \rangle$.

Step 1: Determination of defining relations of $\tilde{A}$ and $\tilde{G}$

Since τ_1 is the lift of fx , we have $\tau_1 K = fx$ and so $\tau_1^{2^e} = (fx)^{2^e} a^{i_1}$ for some integer i_1 . Now, $\tilde{G}$ is an extension of $T \times K$ by τ_1 , which can be determined by

$$\tau_1^{2^e} = z_1 a^{i_1}, \quad t^{\tau_1} = t^{fx}, \quad a^{\tau_1} = a^{i_2}, \quad (1)$$

where $t \in T$, $(i_2, d) = 1$ and $z_1 = (fx)^{2^e} = \begin{pmatrix} \theta^{\frac{p^{s2^e}-1}{p^s-1}} & 0 \\ 0 & 1 \end{pmatrix} \in T$, recalling f is of order 2^e .

Since $a^{\tau_1} = a^{i_2}$ and $[T, K] = 1$, we have

$$a = a^{z_1 a^{i_1}} = a^{\tau_1^{2^e}} = a^{i_2^{2^e}}.$$

Thus, $i_2^{2^e} \equiv 1 \pmod{d}$. As $\tau_1^{2^e} = z_1 a^{i_1}$ and $z_1 = (fx)^{2^e}$, we get

$$z_1 a^{i_1} = (z_1 a^{i_1})^{\tau_1} = (z_1)^{fx} a^{i_1 i_2} = z_1 a^{i_1 i_2}.$$

Hence, $a^{i_1 i_2} = a^{i_1}$ forcing $(i_2 - 1)i_1 \equiv 0 \pmod{d}$. Thus, by the theory of group cyclic extension one may get Eq. (1) is equivalent to

$$i_2^{2^e} \equiv 1 \pmod{d}, \quad (i_2 - 1)i_1 \equiv 0 \pmod{d}. \quad (2)$$

Also, $\tilde{A}$ is an extension of $\tilde{G}$ by τ_2 , which can be determined by

$$\tau_2^2 = \tau_1^2 z_2 a^{j_1}, \quad t^{\tau_2} = t^{f\sigma}, \quad a^{\tau_2} = a^{j_2}, \quad \tau_1^{\tau_2} = z_3 \tau_1 a^{j_3}, \quad (3)$$

where j_1, j_2, j_3 are integers and $(j_2, d) = 1$; $z_2 = (x^f x)^{-1}$, the first relation in Eq. (3) comes from the relations $(f\sigma)^2 = f^2$ and $(fx)^2 = f^2 x^f x$ in G ; $z_3 = x(x^{-1})^{f^{-1}}$ and the fourth relation in Eq. (3) comes from the relation $(fx)^{f\sigma} = x(x^{-1})^{f^{-1}}(fx)$ in G . Again it can be computed by the theory of group cyclic extension that all the relations in Eq. (3) are equivalent to

$$\begin{aligned} j_2^2 &\equiv i_2^2 \pmod{d}, & j_3(j_2 + 1) + (i_2 - 1)j_1 &\equiv 0 \pmod{d}, \\ j_3(i_2 + 1) + j_1(j_2 - 1) &\equiv 0 \pmod{d}. \end{aligned} \quad (4)$$

Step 2: Determination of point stabilizers $\tilde{A}_{\tilde{u}} = \tilde{G}_{\tilde{u}}$

Take $\tilde{u} \in \phi^{-1}(\infty)$ and set $H := \tilde{A}_{\tilde{u}} = \tilde{G}_{\tilde{u}}$. Since $G_\infty = \overline{Q} \rtimes \langle x^2, fx \rangle$ and H is a lift of G_∞ , we set $H = Q_1 \rtimes \langle x^2 a^{u_1}, \tau_1 a^{u_2} \rangle$, for some integers u_1 and u_2 , and $Q_1 \leq Q \times K$. Actually, we are showing $Q_1 = Q$ below.

Suppose the contrary, $Q_1 \neq Q$. Then there exist two nontrivial elements $c_1 \in Q$ and $k \in K$ such that $c_1 k \in Q_1$. Since the maximal elementary abelian r -subgroup of $Q \times K$ is of order r^{l+1} where $l \geq 2$, we get $|Q \cap Q_1| = r^{l-1} \neq 1$. Take a nontrivial element c_2 in $Q \cap Q_1$. Since $\langle x^2 \rangle$ has two orbits both with length $\frac{q-1}{2}$ on $Q \setminus \{1\}$ by conjugacy action, $\langle x^2 a^{u_1} \rangle$ has the same property on $Q_1 \setminus \{1\}$, whose two orbits must be $B_1 := \{(c_1 k)^{\langle x^2 a^{u_1} \rangle}\} = \{(c_1)^{\langle x^2 a^{u_1} \rangle} k\}$ and $B_2 := \{c_2^{\langle x^2 a^{u_1} \rangle}\}$. Therefore, $Q_1 = B_1 \cup B_2 \cup \{1\}$. Noting $r \geq 3$, the inverse $(c_1 k)^{-1}$ of $c_1 k \in Q_1$ is not contained in $B_1 \cup B_2 \cup \{1\}$, a contradiction.

We continue our proof. Since $H \cap K = 1$, $(x^2 a^{u_1})^{\frac{q-1}{2}} = 1$ and $(\tau_1 a^{u_2})^{2^e} \in \langle x^2 a^{u_1} \rangle$. Then by computation, we get

$$\frac{q-1}{2} u_1 \equiv 0 \pmod{d}, \quad i_1 + \frac{i_2^{2^e} - 1}{i_2 - 1} u_2 \equiv \frac{p^{s2^e} - 1}{2(p^s - 1)} u_1 \pmod{d}. \quad (5)$$

Step 3: Determination of coset graphs of $\tilde{A}$ relative to H

From Steps 1 and 2, the coset $[\tilde{A} : H]$ of $\tilde{A}$ relative to H consists of two parts:

$$\begin{aligned} U(H) &= \{Hk \mid k \in K\} \cup \{Hy\tau_i k \mid i \in \mathbb{F}_q, k \in K\}, \\ W(H) &= \{H\tau_2 k \mid k \in K\} \cup \{Hy\tau_2 \tau_i k \mid i \in \mathbb{F}_q, k \in K\}. \end{aligned}$$

Since there is only one edge from H to the block $\{Hy\tau_2 k \mid k \in K\}$, we may assume that the neighbor of H corresponds to the bicoset $D = Hy\tau_2 k_1 H$ for some $k_1 \in K$. By X' we denote this coset graph $X(\tilde{A}; H, D)$. Then X' should satisfy the following three conditions.

(i) *Undirected property*:

The graph X' is undirected if and only if $D^{-1} = D$. Then

$$\begin{aligned} D^{-1} &= H(y\tau_2 k_1)^{-1} H = Hk_1^{-1} \tau_2^{-1} y H = H(\tau_2 k_1)^{-2} y \tau_2 k_1 H \\ &= H(\tau_2 k_1 \tau_2 k_1)^{-1} y \tau_2 k_1 H = H(\tau_2^2 k_1^{j_2+1})^{-1} y \tau_2 k_1 H = Hk_1^{-(j_2+1)} \tau_2^{-2} y \tau_2 k_1 H \\ &= Hk_1^{-(j_2+1)} a^{-j_1} \tau_2^{-1} \tau_1^{-2} y \tau_2 k_1 H = Hk_1^{-(j_2+1)} a^{-j_1} (x^2)^{\frac{p^s+1}{2}} \tau_1^{-2} y \tau_2 k_1 H \\ &= H(x^2 a^{u_1})^{\frac{p^s+1}{2}} k_1^{-(j_2+1)} a^{-j_1-u_1(\frac{p^s+1}{2})} \tau_1^{-2} y \tau_2 k_1 H \\ &= Hk_1^{-(j_2+1)} a^{-j_1-u_1(\frac{p^s+1}{2})} \tau_1^{-2} y \tau_2 k_1 H \\ &= H\tau_1^{-2} k_1^{-(j_2+1)} i_2^{-2} a^{(-j_1-u_1(\frac{p^s+1}{2}))i_2^{-2}} y \tau_2 k_1 H \\ &= H(\tau_1 a^{u_2})^{-2} k_1^{-(j_2+1)} i_2^{-2} a^{(-j_1-u_1(\frac{p^s+1}{2})+u_2(i_2+1))i_2^{-2}} y \tau_2 k_1 H \\ &= Hk_1^{-(j_2+1)} i_2^{-2} a^{(-j_1-u_1(\frac{p^s+1}{2})+u_2(i_2+1))i_2^{-2}} y \tau_2 k_1 H \\ &= Hk_0 y \tau_2 k_1 H = Hy\tau_2 k_1 k_0^{\tau_2} H = D, \end{aligned}$$

where $k_0 = k_1^{-(j_2+1)} i_2^{-2} a^{(-j_1-u_1(\frac{p^s+1}{2})+u_2(i_2+1))i_2^{-2}}$. Since H joins with just one vertex in the fiber $\{Hy\tau_2 k \mid k \in K\}$, we get $k_0 = k_1^{-(j_2+1)} i_2^{-2} a^{(-j_1-u_1(\frac{p^s+1}{2})+u_2(i_2+1))i_2^{-2}} = 1$, that is,

$$k_1^{-(j_2+1)} a^{-j_1-u_1(\frac{p^s+1}{2})+u_2(i_2+1)} = 1. \quad (6)$$

(ii) $d(X') = q$:

Since the length of the orbit of H containing the vertex $Hy\tau_2 k_1$ is q , both $x^2 a^{u_1}$ and $\tau_1 a^{u_2}$ fix the vertex $Hy\tau_2 k_1$, that is,

$$\begin{aligned} Hy\tau_2 k_1 &= Hy\tau_2 k_1 (x^2 a^{u_1}) = Hy\tau_2 k_1 (x^2 a^{u_1}) (y\tau_2 k_1)^{-1} y \tau_2 k_1 \\ &= Hy\tau_2 x^2 a^{u_1} \tau_2^{-1} y^{-1} y \tau_2 k_1 \\ &= Hy(x^2)^{f^{-1}} a^{u_1 j_2^{-1}} y^{-1} y \tau_2 k_1 = H(x^{-2})^{f^{-1}} a^{u_1 j_2^{-1}} y \tau_2 k_1 \\ &= H(x^{-2})^{(p^s)^{-1}} a^{-u_1(p^s)^{-1}} a^{u_1((p^s)^{-1}+j_2^{-1})} y \tau_2 k_1 = Ha^{u_1((p^s)^{-1}+j_2^{-1})} y \tau_2 k_1, \end{aligned}$$

which forces $a^{u_1((p^s)^{-1}+j_2^{-1})} = 1$, that is,

$$u_1((p^s)^{-1} + j_2^{-1}) \equiv 0 \pmod{d}. \quad (7)$$

By Eq. (3), we have $\tau_1^{\tau_2^{-1}} = (t_3^{\tau_2^{-1}})^{-1} \tau_1 a^{-j_3 j_2^{-1}}$, and then

$$\begin{aligned} Hy\tau_2 k_1 &= Hy\tau_2 k_1 (\tau_1 a^{u_2}) = Hy\tau_2 k_1 (\tau_1 a^{u_2}) (y\tau_2 k_1)^{-1} y\tau_2 k_1 \\ &= Hy\tau_2 k_1 (\tau_1 a^{u_2}) k_1^{-1} \tau_2^{-1} y^{-1} y\tau_2 k_1 = Hy\tau_1^{\tau_2^{-1}} k_1^{(i_2-1)j_2^{-1}} a^{u_2 j_2^{-1}} y\tau_2 k_1 \\ &= Hy(t_3^{-1})^{\tau_2^{-1}} \tau_1 y a^{-j_3 j_2^{-1}} k_1^{(i_2-1)j_2^{-1}} a^{u_2 j_2^{-1}} y\tau_2 k_1 \\ &= Hy(t_3^{-1})^{\tau_2^{-1}} y\tau_1^{-1} \tau_1 a^{(u_2-j_3)j_2^{-1}} k_1^{(i_2-1)j_2^{-1}} y\tau_2 k_1 \\ &= H(x^2 a^{u_1})^{-\frac{(p^s)^{-1}((p^s)^{-1}+1)}{2}} \tau_1 a^{u_1 i_2} \frac{(p^s)^{-1}((p^s)^{-1}+1)}{2} + (u_2-j_3)j_2^{-1} k_1^{(i_2-1)j_2^{-1}} y\tau_2 k_1 \\ &= H a^{u_1 i_2} \frac{(p^s)^{-1}((p^s)^{-1}+1)}{2} + (u_2-j_3)j_2^{-1} - u_2 k_1^{(i_2-1)j_2^{-1}} y\tau_2 k_1 \end{aligned}$$

which forces

$$a^{u_1 i_2} \frac{(p^s)^{-1}((p^s)^{-1}+1)}{2} + (u_2-j_3)j_2^{-1} - u_2 k_1^{(i_2-1)j_2^{-1}} = 1. \quad (8)$$

(iii) *Connectedness property:*

Since X' is connected if and only if $\langle D \rangle = \tilde{A}$, we have

$$\begin{aligned} \langle D \rangle &= \langle H, y\tau_2 k_1 \rangle = \langle Q, x^2 a^{u_1}, \tau_1 a^{u_2}, y\tau_2 k_1 \rangle \\ &= \langle \langle Q, Q^{y\tau_2 k_1} \rangle, x^2 a^{u_1}, \tau_1 a^{u_2}, y\tau_2 k_1 \rangle \\ &= \langle T, a^{u_1}, \tau_1 a^{u_2}, \tau_2 k_1 \rangle = (T \times \langle a^{u_1} \rangle) \langle \tau_1 a^{u_2} \rangle \langle \tau_2 k_1 \rangle = \tilde{A}. \end{aligned}$$

By comparing the order, one may get $K = \langle a^{u_1} \rangle$ and so $(u_1, d) = 1$. Then by Eq. (5), we have

$$d \mid \frac{q-1}{2}. \quad (9)$$

Step 4: Show that $X' \cong K_{q+1}^{2d}$.

For any $i \neq j \in \mathbb{F}_q$, set $(j-i)^{f^{-1}} = \theta^h$. First we compute the neighborhood of $Hy t_i$ and $H\tau_2$, where $i \in \mathbb{F}_q$.

Note that for any $k_2 \in K$,

$$\{Hy t_i, Hy\tau_2 t_j k_2\} \in E(X') \quad \text{if and only if} \quad \{H, Hy\tau_2 t_j k_2 t_i^{-1} y^{-1}\} \in E(X').$$

Moreover, by computation, we know that

$$\begin{aligned} Hy\tau_2 t_j k_2 t_i^{-1} y^{-1} &= Hy\tau_2 t_{j-i} k_2 y^{-1} \\ &= H(y t_{\theta^h} y^{-1}) \tau_2 k_2 = H \left(\begin{pmatrix} 1 & 0 \\ -\theta^h & 1 \end{pmatrix} \right) \tau_2 k_2 \\ &= H \left(t_{\theta^{-h}} \left(\begin{pmatrix} 1 & 0 \\ -\theta^h & 1 \end{pmatrix} \right) \right) \tau_2 k_2 = H \left(\begin{pmatrix} 0 & \theta^{-h} \\ -\theta^h & 1 \end{pmatrix} \right) \tau_2 k_2 \\ &= H \left(\begin{pmatrix} \theta^{-h} & 0 \\ 0 & \theta^h \end{pmatrix} \right) y t_{-\theta^{-h}} \tau_2 k_2 = H(x^2)^{-h} (a^{u_1})^{-h} a^{u_1 h} y t_{-\theta^{-h}} \tau_2 k_2 \\ &= Hy t_{-\theta^{-h}} \tau_2 a^{u_1 h j_2} k_2 \\ &= Hy\tau_2 t_{-\theta^{-h}} a^{u_1 h j_2} k_2. \end{aligned}$$

Since $\{H, Hy\tau_2t_{-\theta-fh}k_1\} \in E(X')$, we get $a^{u_1hj_2}k_2 = k_1$, that is, $k_2 = a^{-u_1hj_2}k_1$, and so $Hy\tau_i$ is adjacent to $Hy\tau_2t_ja^{-u_1hj_2}k_1$.

From $\{H\tau_2, Hy\tau_2t_jk_1\tau_2\} \in E(X')$, we get from Eq. (6) that

$$\begin{aligned} Hy\tau_2t_jk_1\tau_2 &= Hy\tau_2k_1\tau_2t_{jf} = Hy\tau_2^2k_1^{j_2}t_{jf} \\ &= H\tau_1^2(x^f x)^{-1}a^{j_1}yk_1^{j_2}t_{jf} = H(x^2)^{-\frac{p^s+1}{2}}a^{j_1-(i_2+1)u_2}yk_1^{j_2}t_{jf} \\ &= Hyt_{jf}k_1^{j_2}a^{j_1-(i_2+1)u_2+u_1\frac{p^s+1}{2}} = Hyt_{jf}k_1^{-1}, \end{aligned}$$

and so $H\tau_2$ is adjacent to $Hyt_{jf}k_1^{-1}$.

Aroused by the above adjacent relation of the vertices, we define a voltage graph $X'' = Y \times_{f'} K$ by the following rule:

$$f'(\infty, i') = f'(i, \infty') = k_1, \quad f'(i, j') = r^h k_1,$$

for $i \neq j \in \mathbb{F}_q$ and $(j-i)^{f-1} = \theta^h$, where $r = a^{-u_1j_2}$. Note that $(u_1j_2, d) = 1$, so r is also the generator of K .

Define a map $\eta: V(X') \rightarrow V(X'')$ by the rule

$$\begin{aligned} \eta(Hk) &= (\infty, k), & \eta(Hy\tau_i k) &= (i, k), \\ \eta(H\tau_2 k) &= (\infty', k), & \eta(Hy\tau_2 t_i k) &= (i', k), \end{aligned}$$

for any $k \in K$. Clearly, η is an isomorphism from the graph X' to X'' .

Finally, define another map $\zeta: V(X'') \rightarrow V(K_{q+1}^{2d})$ by the rule

$$\begin{aligned} \zeta(\infty, k) &= (\infty, k^{p^s}), & \zeta(i, k) &= (i, k^{p^s}), \\ \zeta(\infty', k) &= (\infty', k^{p^s}(k_1^{-1})^{p^s}), & \zeta(i', k) &= (i', k^{p^s}(k_1^{-1})^{p^s}), \end{aligned}$$

recalling that s is given by $f: \theta \mapsto \theta^{p^s}$. Then ζ gives an isomorphism from X'' to K_{q+1}^{2d} , and therefore $\zeta\eta$ gives an isomorphism from X' to K_{q+1}^{2d} , as required. $\square$

Remark 3.5 From the above arguments, we know that such graph X' exists if and only if Eqs. (2), (4), (5), (6), (7), (8), and (9) hold. By Lemma 2.9, for the graph K_{q+1}^{2d} , $\text{P}\Gamma\text{L}(2, q) \times \langle \sigma \rangle$ lifts and so our group A lifts. Therefore, there exists at least one solution for the above equations.

3.2.2 $T \cap K = \mathbb{Z}_2$, $G = \text{PGL}(2, q)$ and $A = G \times \langle \sigma \rangle$

Lemma 3.6 Suppose that $G = \text{PGL}(2, q)$, $A = G \times \langle \sigma \rangle$ and $\tilde{A}$ contains a characteristic subgroup $T \cong \text{SL}(2, q)$, which is lift of $\text{PSL}(2, q)$ such that $T \cap K = \mathbb{Z}_2$. Then $X \cong K_{q+1}^{2d}$, with $d \mid q-1$ and $d \nmid \frac{q-1}{2}$.

Proof Under the hypothesis, we identify T with $\text{SL}(2, q)$. In $\text{GL}(2, q)$, set

$$\begin{aligned} e &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & t_i &= \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, & x &= \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}, \\ c &= \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, & y &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

where $q = r^l$ and $i \in \mathbb{F}_q$. Let $Q = \langle t_i | i \in \mathbb{F}_q \rangle \leq T \leq \tilde{G}$ and let $\overline{Q} = \langle t_i e | i \in \mathbb{F}_q \rangle \leq \text{PSL}(2, q)$, we have $\overline{Q} \cong Q \cong \mathbb{Z}_r^l$. By τ_1 and τ_2 , we denote the respective lifts of $\bar{x} = xZ(\text{GL}(2, p))$ and σ in $\tilde{A}$. Then

$$\tilde{G} = (TK)\langle \tau_1 \rangle, \quad \tilde{A} = (TK)\langle \tau_1, \tau_2 \rangle,$$

where $[T, K] = 1$ and $T \cap K = \langle e \rangle$, as mentioned above. In what follows, we divide the proof into the following four steps, reminding the reader that $K = \langle a \rangle$.

Step 1: Determination of defining relations of $\tilde{A}$ and $\tilde{G}$

Now, $\tilde{G}$ is an extension of TK by τ_1 , which can be determined by

$$\tau_1^2 = ca^{i_1}, \quad t^{\tau_1} = t^x, \quad a^{\tau_1} = a^{i_2}, \quad (10)$$

where $(i_2, d) = 1$ and noting that $\bar{x}^2 = \bar{c}$. By the theory of group cyclic extension, Eq. (10) is equivalent to

$$i_2^2 \equiv 1 \pmod{d}, \quad (i_2 - 1)i_1 \equiv 0 \pmod{d}. \quad (11)$$

Also, $\tilde{A}$ is an extension of $\tilde{G}$ by τ_2 , which can be determined by

$$t_2^2 = a^{j_1}, \quad t^{\tau_2} = tk_0, \quad a^{\tau_2} = a^{j_2}, \quad \tau_1^{\tau_2} = \tau_1 a^{j_3}, \quad (12)$$

where $t \in T$, j_1, j_2, j_3 are integers and $(j_2, d) = 1$.

We first claim that $k_0 = 1$. Actually, since T is a characteristic subgroup of $\tilde{G}$, it follows that $t^{\tau_2} \in T$. This forces $k_0 = t^{-1}t^{\tau_2} \in T \cap K = \langle e \rangle$, that is, $t^{\tau_2} = te^i$ where $i = 0, 1$. In particular, for any elements t of odd order of T , we have $t^{\tau_2} = t$. Since T can be generated by all the elements of odd order, we have $t^{\tau_2} = t$ for any $t \in T$.

Again, by the theory of group cyclic extension, Eq. (12) is equivalent to

$$\begin{aligned} j_2^2 &\equiv 1 \pmod{d}, & j_3(j_2 + 1) + (i_2 - 1)j_1 &\equiv 0 \pmod{d}, \\ j_1(j_2 - 1) &\equiv 0 \pmod{d}. \end{aligned} \quad (13)$$

In summary, our groups $\tilde{G}$ and $\tilde{A}$ are determined by Eqs. (10) and (11), Eqs. (12) and (13), respectively.

Step 2: Determination of point stabilizers $\tilde{A}_{\tilde{u}} = \tilde{G}_{\tilde{u}}$

Set $H := \tilde{A}_{\tilde{u}} = \tilde{G}_{\tilde{u}}$, which is isomorphic to $Q \rtimes \mathbb{Z}_{q-1}$, a Frobenius group. Then we first show that up to the conjugacy in $\tilde{G}$, we may let $Q \leq H$.

In fact, if $(r, d) = 1$, where $q = r^l$, then Q is a Sylow r -subgroup of $\tilde{A}$ and then we may let $Q \leq H$. Now suppose that $r \mid d$ and K_1 is the Sylow r -subgroup of $K \cong \mathbb{Z}_d$. Then $Q \times K_1$ is a Sylow r -subgroup of $\tilde{A}$. Up to conjugacy, let $H = Q_1 \rtimes \langle h \rangle$, where $\mathbb{Z}_r^l \cong Q_1 \leq Q \times K_1$ and $|h| = q - 1$. Suppose that $Q \neq Q_1$. Then $|Q_1 \cap Q| = r^{l-1}$.

If $l \geq 2$, then there exists a nontrivial element $c_1 \in Q_1 \cap Q$. By the property of a Frobenius group, $Q_1 \setminus \{1\} = (c_1)^{\langle h \rangle} \leq T$, which forces $Q_1 = Q$, a contradiction.

If $l = 1$, we get $Q_1 \cap Q = 1$. As $q = r^l = r \geq 5$, there exist two nontrivial elements $c_2 \in Q$ and $k \in K$ such that $c_2 k \in Q_1$. And it is easy to see that $Q_1 = \{(c_2)^{\langle h \rangle} k\} \cup \{1\}$ or $Q_1 = \{(c_2)^{\langle h \rangle} k\} \cup \{(c_2)^{\langle h \rangle} k^{i_2}\} \cup \{1\}$ depending on whether $h \in TK$ or $h \in \tilde{G} \setminus TK$. Take $k^s \in K \setminus \{1, k, k^{i_2}\}$ for some integer s . Then, $(c_2 k)^s = c_2^s k^s \in Q_1$ is neither contained in $\{(c_2)^{\langle h \rangle} k\} \cup \{1\}$ nor $\{(c_2)^{\langle h \rangle} k\} \cup \{(c_2)^{\langle h \rangle} k^{i_2}\} \cup \{1\}$, a contradiction.

From the above arguments, we get $Q = Q_1 \leq H \leq N_{\tilde{G}}(Q)$. Moreover, a direct checking shows that $N_{\tilde{G}}(Q) = (Q \rtimes \langle \tau_1 \rangle)K$. Therefore, one may assume $H = Q \rtimes \langle \tau_1 a^u \rangle$, for some integer u .

As $|\tau_1 a^u| = q - 1$, we get

$$1 = (\tau_1 a^u)^{q-1} = (ca^{i_1+u+ui_2})^{\frac{q-1}{2}} = ea^{(i_1+u+ui_2)\frac{q-1}{2}}.$$

Hence, $a^{(i_1+u+ui_2)\frac{q-1}{2}} = e$. It follows that

$$d \mid (i_1 + u + ui_2)(q - 1), \quad d \nmid (i_1 + u + ui_2)\frac{q - 1}{2}. \quad (14)$$

Step 3: Determination of coset graphs of $\tilde{A}$ relative to H

From Steps 1 and 2, the coset $[\tilde{A} : H]$ of $\tilde{A}$ relative to H consists of two parts:

$$U(H) = \{Hk \mid k \in K\} \cup \{Hy\tau_i k \mid i \in \mathbb{F}_q, k \in K\},$$

$$W(H) = \{H\tau_2 k \mid k \in K\} \cup \{Hy\tau_2 t_i k \mid i \in \mathbb{F}_q, k \in K\}.$$

Since there is only one edge from H to the block $\{Hy\tau_2 k \mid k \in K\}$, we may assume that the neighbor of H corresponds to the bicoset $D = Hy\tau_2 k_1 H$ for some $k_1 \in K$. Let X' denote the corresponding coset graph $X(\tilde{A}; H, D)$. Then this coset graph should satisfy the following three conditions.

(i) *Undirected property:*

The graph X' is undirected if and only if $D^{-1} = D$. Then

$$\begin{aligned} D^{-1} &= H(y\tau_2 k_1)^{-1}H = H(\tau_2 k_1)^{-2}ey\tau_2 k_1 H \\ &= Ha^{-j_1}k_1^{-(j_2+1)}ey\tau_2 k_1 H = Hk_0y\tau_2 k_1 H \\ &= Hy\tau_2 k_1 k_0^{\tau_2} H = D, \end{aligned}$$

where $k_0 = a^{-j_1}k_1^{-(j_2+1)}e$. Now, the two vertex $Hy\tau_2 k_1$ and $Hy\tau_2 k_1 k_0^{\tau_2}$ in the neighbor of H are contained in the same fiber. Therefore, we have $k_0 = 1$, that is,

$$a^{j_1}k_1^{j_2+1} = e. \quad (15)$$

(ii) $d(X') = q$:

By Eq. (12), we get $\tau_1 \tau_2^{-1} = \tau_1 a^{-j_2 j_3}$. Since the length of the orbit of H containing the vertex $Hy\tau_2 k_1$ is q , it follows that $\tau_1 a^u$ fixes the vertex $Hy\tau_2 k_1$, that is,

$$\begin{aligned} Hy\tau_2 k_1 &= Hy\tau_2 k_1 (\tau_1 a^u) = Hy\tau_2 k_1 \tau_1 a^u (y\tau_2 k_1)^{-1} y\tau_2 k_1 \\ &= Hyk_1^{j_2} \tau_1^{\tau_2^{-1}} a^{uj_2} k_1^{-j_2} y^{-1} y\tau_2 k_1 = Hyk_1^{j_2} \tau_1 a^{-j_2 j_3 + uj_2} k_1^{-j_2} y^{-1} y\tau_2 k_1 \\ &= H\tau_1 y^{\tau_1} y^{-1} a^{-j_2 j_3 + uj_2} k_1^{-j_2 + j_2 i_2} y\tau_2 k_1 \\ &= H\tau_1 c^{-1} a^{-j_2 j_3 + uj_2} k_1^{-j_2 + j_2 i_2} y\tau_2 k_1 \\ &= H(\tau_1 a^u)^{-1} a^{i_1 + ui_2 + uj_2 - j_2 j_3} k_1^{j_2 i_2 - j_2} y\tau_2 k_1 \\ &= Ha^{i_1 + ui_2 + uj_2 - j_2 j_3} k_1^{j_2 i_2 - j_2} y\tau_2 k_1 \end{aligned}$$

which forces

$$a^{i_1+u i_2+u j_2-j_2 j_3} k_1^{j_2 i_2-j_2} = 1. \quad (16)$$

(iii) *Connectedness property*:

Since X' is connected if and only if $\langle D \rangle = \tilde{A}$, that is,

$$\langle D \rangle = \langle H, y \tau_2 k_1 \rangle = \langle Q, Q^y, \tau_1 a^u, y \tau_2 k_1 \rangle = \langle T, \tau_1 a^u, \tau_2 k_1 \rangle = \tilde{A}.$$

By Eq. (15) and Eq. (16), we get

$$\begin{aligned} (\tau_1 a^u)^{\tau_2 k_1} &= k_1^{-1} \tau_1^{\tau_2} a^{u j_2} k_1 = k_1^{-1} \tau_1 a^{u j_2+j_3} k_1 = \tau_1 k_1^{-i_2+1} a^{u j_2+j_3} \\ &= \tau_1 a^{u j_2+j_3} a^{j_1(i_2-1)} k_1^{j_2(i_2-1)} = \tau_1 a^{u j_2+j_3+j_1 i_2-j_1} a^{j_2 j_3-u j_2-i_1-u i_2} \\ &= \tau_1 a^u a^{-(u+i_1+u i_2)} = (\tau_1 a^u)^{-1} c \in \langle T, \tau_1 a^u \rangle. \end{aligned}$$

Thus, $\langle D \rangle = \langle T, \tau_1 a^u \rangle \langle \tau_2 k_1 \rangle$. Now,

$$\begin{aligned} 2d|T| &= 2|T||K| = 4|TK| = |\tilde{A}| = |\langle D \rangle| = 2|T \langle \tau_1 a^u \rangle| \\ &= 2|T||\tau_1 a^u|/|T \cap \langle \tau_1 a^u \rangle| = 2(q-1)|T|/|T \cap \langle \tau_1 a^u \rangle|, \end{aligned}$$

which implies $|T \cap \langle \tau_1 a^u \rangle| = \frac{q-1}{d}$, that is,

$$|T \cap \langle \tau_1 a^u \rangle| = |T \cap \langle (\tau_1 a^u)^2 \rangle| = |T \cap \langle ca^{u+i_2 u+i_1} \rangle| = \frac{q-1}{d}.$$

Set $w = u + i_2 u + i_1$ and $s = |a^w|$. Then $s = \frac{d}{(d, w)}$. In what follows we first show that $(w, d) = 1$.

If s is odd, then we get $T \cap \langle ca^w \rangle = \langle c^s \rangle$. Therefore, $|T \cap \langle ca^w \rangle| = \frac{q-1}{(q-1, s)} = \frac{q-1}{d}$, we have $(q-1, \frac{d}{(d, w)}) = d$, which forces $(d, w) = 1$. Note that d is even, we get s is even, a contradiction. Thus, s should be even, which follows that $T \cap \langle ca^w \rangle = \langle c^{\frac{s}{2}} e \rangle$.

If $|c^{\frac{s}{2}}|$ is odd, we get $|c^{\frac{s}{2}} e| = 2|c^{\frac{s}{2}}| = \frac{2(q-1)}{(q-1, \frac{s}{2})} = \frac{q-1}{d}$, that is, $(q-1, \frac{d}{2(d, w)}) = 2d$, a contraction.

If $2 \parallel |c^{\frac{s}{2}}|$, then $|c^{\frac{s}{2}} e| = \frac{|c^{\frac{s}{2}}|}{2} = \frac{(q-1)}{2(q-1, \frac{s}{2})} = \frac{q-1}{d}$. Hence, $(q-1, \frac{d}{2(d, w)}) = \frac{d}{2}$, forcing $(w, d) = 1$.

If $4 \parallel |c^{\frac{s}{2}}|$, then $|c^{\frac{s}{2}} e| = |c^{\frac{s}{2}}| = \frac{(q-1)}{(q-1, \frac{s}{2})} = \frac{q-1}{d}$, that is, $(q-1, \frac{d}{2(d, w)}) = d$, a contraction.

Since $(w, d) = 1$, Eq. (14) can be reformed as

$$d \mid q-1, \quad d \nmid \frac{q-1}{2}. \quad (17)$$

Moreover, by Eq. (11), we have

$$(u i_2 + u + i_1)(i_2 - 1) = u(i_2^2 - 1) + i_1(i_2 - 1) \equiv 0 \pmod{d},$$

forcing $i_2 \equiv 1 \pmod{d}$, then $a^{\tau_1} = a$, that is, $K \leq Z(\tilde{G})$.

Step 4: Show that $X' \cong K_{q+1}^{2d}$.

For any $i \neq j \in \mathbb{F}_q$, we set $j - i = \theta^h$. Then for any where $k_2 \in K$, we have

$$\{Hyt_i, Hyt_j \tau_2 k_2\} \in E(X') \quad \text{if and only if} \quad \{H, Hyt_j \tau_2 k_2 t_i^{-1} y^{-1}\} \in E(X').$$

Then,

$$\begin{aligned} Hyt_j \tau_2 k_2 t_i^{-1} y^{-1} &= H(yt_{\theta^h} y^{-1}) \tau_2 k_2 = H \begin{pmatrix} 1 & 0 \\ -\theta^h & 1 \end{pmatrix} \tau_2 k_2 \\ &= H t_{\theta^{-h}} \begin{pmatrix} 1 & 0 \\ -\theta^h & 1 \end{pmatrix} \tau_2 k_2 \\ &= H \begin{pmatrix} \theta^{-h} & 0 \\ 0 & \theta^h \end{pmatrix} y t_{-\theta^{-h}} \tau_2 k_2 = H c^{-h} y t_{-\theta^{-h}} \tau_2 k_2 \\ &= H (c a^{(ui_2+u+i_1)})^{-h} a^{(ui_2+u+i_1)h} y t_{-\theta^{-h}} \tau_2 k_2 \\ &= H y t_{-\theta^{-h}} \tau_2 a^{(ui_2+u+i_1)j_2 h} k_2. \end{aligned}$$

Set $r = a^{-(ui_2+u+i_1)j_2}$. Since $(d, (ui_2 + u + i_1)j_2) = 1$, we get $K = \langle r \rangle$ and then

$$Hyt_j \tau_2 k_2 t_i^{-1} y^{-1} = Hyt_{-\theta^{-h}} \tau_2 r^{-h} k_2.$$

Since $\{H, Hyt_{-\theta^{-h}} \tau_2 k_1\} \in E(X')$, we get $r^{-h} k_2 = k_1$, that is, $k_2 = r^h k_1$ and so Hyt_i is adjacent to $Hyt_j \tau_2 r^h k_1$.

From $\{H\tau_2, Hyt_j \tau_2 k_1 \tau_2\} \in E(X')$, we get $Hyt_j \tau_2 k_1 \tau_2 = Hyt_j a^{j_1} k_1^{j_2}$, and so $H\tau_2$ is adjacent to $Hyt_j a^{j_1} k_1^{j_2}$.

Define a map $\eta: V(X') \rightarrow K_{q+1}^{2d}$ by the rule

$$\begin{aligned} \eta(Hk) &= (\infty, k), & \eta(Hyt_i k) &= (i, k), \\ \eta(H\tau_2 k) &= (\infty', a^{j_1} k_1^{j_2} k), & \eta(Hyt_j \tau_2 k) &= (j', k_1^{-1} k), \end{aligned}$$

for any $k \in K$. Then η gives an isomorphism from X' to K_{q+1}^{2d} , as required. $\square$

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