Combinatorics of certain higher q, t -Catalan polynomials: chains, joint symmetry, and the Garsia–Haiman formula

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Abstract The higher q, t -Catalan polynomial $C_n^{(m)}(q, t)$ can be defined combinatorially as a weighted sum of lattice paths contained in certain triangles, or algebraically as a complicated sum of rational functions indexed by partitions of n . This paper proves the equivalence of the two definitions for all $m \geq 1$ and all $n \leq 4$. We also give a bijective proof of the joint symmetry property $C_n^{(m)}(q, t) = C_n^{(m)}(t, q)$ for all $m \geq 1$ and all $n \leq 4$. The proof is based on a general approach for proving joint symmetry that dissects a collection of objects into chains, and then passes from a joint symmetry property of initial points and terminal points to joint symmetry of the full set of objects. Further consequences include unimodality results and specific formulas for the coefficients in $C_n^{(m)}(q, t)$ for all $m \geq 1$ and all $n \leq 4$. We give analogous results for certain rational-slope q, t -Catalan polynomials.

Keywords q, t -Catalan polynomials · Joint symmetry · Lattice paths

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1 Introduction

1.1 The q, t -Catalan polynomials

The q, t -Catalan polynomials $C_n(q, t)$, introduced by Garsia and Haiman [2] in 1996, play a prominent role in combinatorics, symmetric function theory, and algebraic geometry. These polynomials can be defined combinatorially as follows. A sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})$ is called a *Dyck word* if and only if $\gamma_0 = 0$, each $\gamma_i \in \mathbb{N} = \{0, 1, 2, \dots\}$, and $\gamma_i \leq \gamma_{i-1} + 1$ for $1 \leq i < n$. Let W_n be the set of Dyck words of length n . For $\gamma \in W_n$, define $\text{area}(\gamma) = \sum_{i=0}^{n-1} \gamma_i$, and define $\text{dinv}(\gamma)$ to be the number of (i, j) with $0 \leq i < j < n$ and $\gamma_i - \gamma_j \in \{0, 1\}$. Then

$$C_n(q, t) = \sum_{\gamma \in W_n} q^{\text{area}(\gamma)} t^{\text{dinv}(\gamma)}.$$

For example, when $n = 3$, $W_3 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 2)\}$, and $C_3(q, t) = t^3 + qt + qt^2 + q^2t + q^3$.

Part of the interest of the q, t -Catalan polynomials is that there are many different ways of defining $C_n(q, t)$; the equivalence of these definitions is a deep result of algebraic combinatorics due to Garsia, Haglund, and Haiman [6]. We have defined $C_n(q, t)$ as a weighted sum of Dyck words. Another combinatorial formula, first proposed by Haglund [5], expresses $C_n(q, t)$ as a sum of Dyck paths weighted by area and Haglund's bounce statistic. Garsia and Haiman's original definition [2] presented $C_n(q, t)$ as a complicated sum of rational functions in q and t indexed by integer partitions of n (see below for more details). The Garsia–Haiman formula identifies $C_n(q, t)$ as the coefficient of the sign character in $\nabla(e_n)$, where e_n is an elementary symmetric polynomial and ∇ is the nabla operator of Bergeron and Garsia [1]. In turn, Haiman proved that $\nabla(e_n)$ is the Frobenius series of the diagonal harmonics module in $2n$ variables [9], so that $C_n(q, t)$ can be defined algebraically as the Hilbert series of the module of diagonal harmonic alternants. There are other geometric manifestations of $C_n(q, t)$ involving Hilbert schemes [8] and, more recently, compactified Jacobians of plane curve singularities [3, 4].

1.2 The higher q, t -Catalan polynomials

The *higher q, t -Catalan polynomials* are generalizations of the q, t -Catalan polynomials that depend on two integer parameters m and n ; they reduce to ordinary q, t -Catalan polynomials when $m = 1$. We first review the combinatorial definition of the higher q, t -Catalan polynomials given in [10], which generalizes the formula for $C_n(q, t)$ as a weighted sum of Dyck words. Fix an integer $m \geq 1$. An m -Dyck word is a sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})$ such that $\gamma_i \in \mathbb{N}$, $\gamma_0 = 0$, and $\gamma_i \leq \gamma_{i-1} + m$ for $1 \leq i < n$. Denote by $W_n^{(m)}$ the set of m -Dyck words of length n . For $\gamma \in W_n^{(m)}$, define $\text{area}(\gamma) = \sum_{i=0}^{n-1} \gamma_i$, and define $\text{dinv}_m(\gamma) = \sum_{0 \leq i < j < n} \text{sc}_m(\gamma_i - \gamma_j)$, where

$$\text{sc}_m(p) = \begin{cases} m+1-p, & \text{if } 1 \leq p \leq m; \\ m+p, & \text{if } -m \leq p \leq 0; \\ 0, & \text{for all other } p. \end{cases}$$

Define $C_n^{(m)}(q, t) = \sum_{\gamma \in W_n^{(m)}} q^{\text{area}(\gamma)} t^{\text{din}_m(\gamma)}$. See [10] for an equivalent combinatorial definition of $C_n^{(m)}(q, t)$ using m -Dyck paths weighted by area and a suitable m -bounce statistic.

One can also give algebraic definitions of the higher q, t -Catalan polynomials. However, for $m > 1$, the algebraic definitions are not yet known to be equivalent to the combinatorial definitions. Thus we will use the notation $AC_n^{(m)}(q, t)$ to denote the algebraic version of the higher q, t -Catalan polynomials. These can be defined in terms of the nabla operator by setting

$$AC_n^{(m)}(q, t) = \langle \nabla^m(e_n), s_{(1^n)} \rangle.$$

Garsia and Haiman [2] gave an explicit formula for $AC_n^{(m)}(q, t)$ in their original paper on q, t -Catalan polynomials, which we now describe.

Recall that a *partition of n* is a sequence $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ of positive integers with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s$ and $\mu_1 + \dots + \mu_s = n$. Let $\text{Par}(n)$ be the set of partitions of n . For $\mu \in \text{Par}(n)$, the *diagram of μ* is the set

$$D(\mu) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq s, 1 \leq j \leq \mu_i\},$$

which can be visualized as a set of left-justified cells with μ_i squares in the i th row from the top. For example, $\mu = (4, 4, 3, 1)$ is a partition in $\text{Par}(12)$ with

$$D(\mu) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline * & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}.$$

Fix a cell $c \in D(\mu)$. The *arm of c* , denoted $a(c)$, is the number of cells in $D(\mu)$ to the right of c in the same row. The *coarm of c* , denoted $a'(c)$, is the number of cells in $D(\mu)$ to the left of c in the same row. The *leg of c* , denoted $l(c)$, is the number of cells in $D(\mu)$ below c in the same column. The *coleg of c* , denoted $l'(c)$, is the number of cells in $D(\mu)$ above c in the same column. In the example shown above, the cell $c = (2, 1)$ (marked by an asterisk in the figure) has $a(c) = 3$, $a'(c) = 0$, $l(c) = 2$, and $l'(c) = 1$.

The Garsia–Haiman formula for $AC_n^{(m)}(q, t)$ is a sum of rational functions in q and t indexed by partitions of n , which is assembled from the following ingredients. For each $\mu \in \text{Par}(n)$, define:

$$T_\mu = \prod_c [q^{a'(c)} t^{l'(c)}]; \quad B_\mu = \sum_c q^{a'(c)} t^{l'(c)}; \quad \Pi_\mu = \prod_{c \neq (1,1)} (1 - q^{a'(c)} t^{l'(c)});$$

$$w_\mu = \prod_c [(q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1})].$$

All sums and products here range over cells c in $D(\mu)$, except the summation for Π_μ excludes the upper-left corner cell $c = (1, 1)$. Garsia and Haiman's original definition

of the higher q, t -Catalan polynomials is:

$$AC_n^{(m)}(q, t) = \sum_{\mu \in \text{Par}(n)} \frac{T_\mu^{m+1}(1-q)(1-t)B_\mu \Pi_\mu}{w_\mu}. \quad (1)$$

The following conjecture has been open since approximately 2001. (The $m = 1$ case follows from the difficult theorem of Garsia, Haglund, and Haiman mentioned above.)

Conjecture 1 (Haiman/Loehr [10]) *For all $m, n \in \mathbb{N}^+$,*

$$C_n^{(m)}(q, t) = AC_n^{(m)}(q, t).$$

The first main goal of this paper is to prove this conjecture for all $m \geq 1$ and all $n \leq 4$. Our proof is rather intricate, but it requires only elementary combinatorial operations on m -Dyck words and algebraic manipulations of expressions involving q and t . The proof will evolve as a consequence of our combinatorial investigation of joint symmetry, which we describe next.

1.3 Joint symmetry

One notable feature of the higher q, t -Catalan polynomials is the *joint symmetry* $C_n^{(m)}(q, t) = C_n^{(m)}(t, q)$. The joint symmetry property for $AC_n^{(m)}(q, t)$ follows fairly easily from the Garsia–Haiman definition. For, letting μ' denote the conjugate of the partition μ , it is immediate from the definitions that $T_{\mu'}(q, t) = T_\mu(t, q)$, $B_{\mu'}(q, t) = B_\mu(t, q)$, $\Pi_\mu(q, t) = \Pi_{\mu'}(t, q)$, and $w_{\mu'}(q, t) = w_\mu(t, q)$. Joint symmetry of $AC_n^{(m)}$ now follows by replacing the summation variable μ by μ' in (1) and using the preceding identities.

On the other hand, the joint symmetry property for the combinatorially defined polynomials $C_n^{(m)}(q, t)$ is a great mystery. It is an open problem to define a collection of involutions $f : W_n^{(m)} \rightarrow W_n^{(m)}$ for all $m, n \in \mathbb{N}^+$, such that $\text{area}(f(\gamma)) = \text{dinv}_m(\gamma)$ and $\text{dinv}_m(f(\gamma)) = \text{area}(\gamma)$ for all $\gamma \in W_n^{(m)}$. This problem is open even for $m = 1$. For all $m, n \in \mathbb{N}^+$, a bijection is known [6, 10] proving the weaker *univariate symmetry property* $C_n^{(m)}(q, 1) = C_n^{(m)}(1, q)$.

The second main goal of this paper is to develop a combinatorial framework for understanding the joint symmetry of $C_n^{(m)}(q, t)$ and the *rational-slope q, t -Catalan polynomials* $C_{r,s,n}(q, t)$ (to be defined later). We introduce a strategy for producing bijective proofs of joint symmetry that involves dissecting a set of objects into a disjoint union of chains. We show that if the set of initial points and the set of terminal points of all the chains are linked by a certain joint symmetry property, then the generating function for the entire set is jointly symmetric. We use our method to give a combinatorial proof of the joint symmetry of $C_n^{(m)}(q, t)$ and $C_{r,s,n}(q, t)$ for triangles of height at most 4 and all choices of the slope parameters m, r , and s . Gorsky and Mazin [4] recently gave a combinatorial proof of joint symmetry for triangles of height 3 using a different approach. Our method has the added benefit of

providing explicit formulas for all the coefficients of $C_n^{(m)}(q, t)$ for $n \leq 4$, as well as providing the foundation for our proof of Conjecture 1 for these values of n . We also obtain some unimodality results for the coefficient sequences obtained by looking at monomials in the higher q, t -Catalan polynomials of a given total degree. Many of the ingredients in our proof extend to triangles of arbitrary sizes, although we cannot yet prove joint symmetry in full generality.

The rest of the paper is organized as follows. Section 2 describes our general approach of building chains and then passing from the q, t -symmetry of chain endpoints to q, t -symmetry of the full set of combinatorial objects. Section 3 defines a map f_0 that will be used to construct chains of m -Dyck words. Section 4 proves joint symmetry of $C_n^{(m)}(q, t)$ for all $n \leq 4$. Section 5 proves Conjecture 1 for all $n \leq 4$. Section 6 gives the definition of rational-slope q, t -Catalan polynomials and extends the previous constructions to this situation. We also compare our method to the Gorsky–Mazin proof for triangles of height 3. Finally, Sect. 7 gives a conjectured chain map for triangles of height 5 and further discussion of the challenges that arise for larger n .

2 Joint symmetry via chains

This section considers the following general situation. We are given a finite set W and two statistics $a : W \rightarrow \mathbb{N}$ and $d : W \rightarrow \mathbb{N}$. For any subset S of W , define

$$C_S(q, t) = \sum_{w \in S} q^{a(w)} t^{d(w)}.$$

We are also given a set I of *initial objects* in W , a set T of *terminal objects* in W , and a bijection $f : W \setminus T \rightarrow W \setminus I$ such that

$$a(f(w)) = a(w) - 1 \quad \text{and} \quad d(f(w)) = d(w) + 1 \quad \text{for all } w \in W \setminus T.$$

Then W is the disjoint union of f -chains, each of which proceeds from an object in I to an object in T . More specifically, for each $w_0 \in I$, there is an f -chain

$$w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_k,$$

where $w_{i+1} = f(w_i)$ for $0 \leq i < k$, and $k \in \mathbb{N}$ is minimal such that $w_k \in T$. (If $w_0 \in I \cap T$, then $k = 0$ and the f -chain consists of w_0 alone.)

Theorem 2 *With the above notation, if $C_T(q, t) = C_I(t, q)$, then $C_W(q, t) = C_W(t, q)$.*

The next two subsections give an algebraic proof and a bijective proof of this theorem.

2.1 Algebraic proof of Theorem 2

The following lemma and its proof were communicated to us by Mikhail Mazin. Theorem 2 immediately follows from this lemma.

Lemma 3 (a) For any finite set W with a chain map f , we have

$$C_W(q, t) = \frac{C_I(q, t)}{1 - t/q} + \frac{C_T(q, t)}{1 - q/t}.$$

In particular, if $C_I(q, t) = C_T(q, t)$ then $C_W(q, t) = C_I(q, t)$.

(b) Assume $C_T(q, t) = C_I(t, q)$. Then

$$C_W(q, t) = \frac{C_I(q, t)}{1 - t/q} + \frac{C_I(t, q)}{1 - q/t} = \sigma\left(\frac{C_I(q, t)}{1 - t/q}\right),$$

where $\sigma : \mathbb{Q}(q, t) \rightarrow \mathbb{Q}(q, t)$ is the map given by $\sigma(F(q, t)) = F(q, t) + F(t, q)$.

Proof (a) For each f -chain $C_i: w_0^i \rightarrow w_1^i \rightarrow w_2^i \rightarrow \dots \rightarrow w_{k_i}^i$, apply the formula for geometric progressions to obtain

$$C_{C_i}(q, t) = \frac{C_{\{w_0^i\}}(q, t)}{1 - t/q} - \frac{C_{\{w_{k_i}^i\}}(q, t) \cdot t/q}{1 - t/q} = \frac{C_{\{w_0^i\}}(q, t)}{1 - t/q} + \frac{C_{\{w_{k_i}^i\}}(q, t)}{1 - q/t}.$$

After summing up over all chains, one gets

$$C_W(q, t) = \frac{C_I(q, t)}{1 - t/q} + \frac{C_T(q, t)}{1 - q/t}.$$

Part (b) is immediate from (a). $\square$

Remark 4 Define $|C_I(q, t)|_{q\text{-deg} \geq j}^{\deg=j+k}$ to be the number of terms in the sum defining $C_I(q, t)$ whose total degree is $j+k$ and whose q -degree is at least j , and define $|C_I(q, t)|_{t\text{-deg} > j}^{\deg=j+k}$ similarly. From Lemma 3(a),

$$C_W(q, t) = \frac{C_I(q, t)}{1 - t/q} - \frac{C_T(q, t) \cdot t/q}{1 - t/q}.$$

Using power series expansion we conclude that the coefficient of $q^j t^k$ in $C_W(q, t)$ is

$$|C_I(q, t)|_{q\text{-deg} \geq j}^{\deg=j+k} - |C_T(q, t)|_{q\text{-deg} > j}^{\deg=j+k}.$$

(A more explanatory proof: the coefficient is equal to the number of chains with the same total degree starting “weakly before” the monomial minus the number of such chains ending “strictly before” the monomial.) In particular if $C_T(q, t) = C_I(t, q)$, then the coefficient is

$$|C_I(q, t)|_{q\text{-deg} \geq j}^{\deg=j+k} - |C_I(q, t)|_{t\text{-deg} > j}^{\deg=j+k}.$$

2.2 Bijective proof of Theorem 2

We now show that any bijective proof of the hypothesis $C_T(q, t) = C_I(t, q)$ of Theorem 2 can be converted to a bijective proof of the conclusion $C_W(q, t) =$

$C_W(t, q)$. More specifically, assume that we are given a bijection $h : T \rightarrow I$ such that $a(h(w)) = d(w)$ and $d(h(w)) = a(w)$ for all $w \in T$. We will use h and f to build a canonical involution $J : W \rightarrow W$ such that $a(J(w)) = d(w)$ and $d(J(w)) = a(w)$ for all $w \in W$.

First, since $f : W \setminus T \rightarrow W \setminus I$ and $h : T \rightarrow I$ are bijections, $f \cup h$ is a bijection from W to W (here we are viewing functions as sets of ordered pairs). The *digraph* of $f \cup h$ is the directed graph G with vertex set W and directed edges $w \rightarrow f(w)$ for all $w \in W \setminus T$ and $w \rightarrow h(w)$ for all $w \in T$. Because $f \cup h$ is a bijection, G is a disjoint union of directed cycles.

To build the involution J , we perform the following construction on each directed cycle C in G (see the examples below for illustrations). If all vertices w on C satisfy $a(w) = d(w)$ —which must occur when C consists of just one vertex—define $J(w) = w$ for all such w . For all other C , we will create a drawing of the cycle C in the first quadrant of the xy -plane, as follows. First observe that the value of $a(w) + d(w)$ for all vertices w on C is constant, because of the properties of f and h . In our drawing of C , each $w \in C$ will be drawn at a lattice point $(x(w), y(w))$ such that: $y(w) = |a(w) - d(w)|$; the lattice point for w is colored black if $a(w) - d(w) > 0$; and the lattice point for w is colored white if $a(w) - d(w) < 0$. (When $y(w) = 0$, the lattice point for w may be black or white.)

To create the drawing, pick any $w_0 \in I \cap C$ that maximizes $a(w) - d(w)$, and draw a black dot for w_0 at $(0, a(w_0) - d(w_0))$. One may routinely check that $a(w_0) - d(w_0) > 0$. Let the distinct vertices on the cycle C (in order) be $w_0, w_1, \dots, w_k$. Assume by induction, for some fixed $i < k$, that we have already drawn dots for $w_0, w_1, \dots, w_i$ at respective coordinates $(0, y_0), (1, y_1), \dots, (i, y_i)$. To continue the drawing, consider various cases.

- Case 1: $w_i \notin T$, so $w_{i+1} = f(w_i)$.
 - Case 1a: (i, y_i) is a black dot and $y_i > 1$. Draw a black dot for w_{i+1} at $(i + 1, y_i - 2)$.
 - Case 1b: (i, y_i) is a black dot and $y_i = 1$. Draw a white dot for w_{i+1} at $(i + 1, 1)$.
 - Case 1c: (i, y_i) satisfies $y_i = 0$. Draw a white dot for w_{i+1} at $(i + 1, 2)$.
 - Case 1d: (i, y_i) is a white dot. Draw a white dot for w_{i+1} at $(i + 1, y_i + 2)$.
- Case 2: $w_i \in T$, so $w_{i+1} = h(w_i) \in I$. Draw a dot for w_{i+1} of the opposite color as the dot for w_i at $(i + 1, y_i)$.

In the examples pictured below, we also draw line segments between successive dots to help visualize the cycle. One sees that dots are black when the line segments are moving down, and dots are white when the line segments are moving up. Color changes occur at horizontal line segments and also due to reflection off the bottom boundary $y = 0$.

One may now check (by induction on i) that the properties stated earlier regarding $y(w_i)$ and the color of the dot for w_i are indeed true. One should further check that the dot for w_k is a white dot at y -coordinate $y_k = y_0$ (since $h(w_k) = w_0$) and that no intermediate y -coordinate y_i exceeds y_0 . Finally, for each black dot (i, y_i) in the drawing with $y_i > 0$, there is a unique leftmost white dot (j, y_i) at the same level (with $j > i$) that can be found pictorially by moving due east from (i, y_i) until hitting the next dot. Define $J(w_i) = w_j$ and $J(w_j) = w_i$ for each such “matched pair” of a

Fig. 1 Example of the construction of J

black dot and a later white dot. Define $J(w) = w$ for any w with $y(w) = 0$ (which holds if and only if $a(w) = d(w)$). The properties of heights and colors of dots show that J interchanges a and d . Moreover, J does not depend on the initial choice of w_0 , since choosing a different $w'_0 \in I \cap C$ maximizing $a(w) - d(w)$ will produce a cyclically shifted version of the original drawing with the same matchings of black dots to white dots. In this sense, J is canonically determined from the given maps f and h .

Example 1 Let $W = \{1, 2, \dots, 15\}$, and define a, d, f, h as shown in this table:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(w)$	7	6	5	6	5	4	3	2	1	3	2	7	6	2	1
$d(w)$	1	2	3	2	3	4	5	6	7	5	6	1	2	6	7
$a(w) - d(w)$	6	4	2	4	2	0	-2	-4	-6	-2	-4	6	4	-4	-6
$f(w)$	2	3	-	5	6	7	8	9	-	11	-	13	-	15	-
$h(w)$	-	-	10	-	-	-	-	-	1	-	4	-	14	-	12

Note $I = \{1, 4, 10, 12, 14\}$ and $T = \{3, 9, 11, 13, 15\}$. The left side of Fig. 1 shows the f -chains drawn vertically, with each $w \in W$ drawn at height $a(w) - d(w)$. On the right of the figure, we draw the two cycles of $f \cup h$ in the first quadrant as described above. The horizontal arrows indicate the action of J , namely:

$$J : 1 \leftrightarrow 9, 2 \leftrightarrow 11, 3 \leftrightarrow 10, 4 \leftrightarrow 8, 5 \leftrightarrow 7, 6 \leftrightarrow 6, 12 \leftrightarrow 15, 13 \leftrightarrow 14.$$

Example 2 Figure 2 gives another example of the construction where $a(w) - d(w)$ is odd for all w . Here $W = \{1, 2, \dots, 18\}$, the f -chains are shown on the left of the figure, $I = \{1, 5, 10, 15\}$, $T = \{4, 9, 14, 18\}$, and $h : T \rightarrow I$ sends 4 to 15, 9 to 1, 14 to 5, and 18 to 10. We picked $w_0 = 1$, but choosing $w_0 = 10$ instead would lead to the same J .

Fig. 2 Another example of the construction of J

Remark 5 In the situation where $a(w) \geq d(w)$ for all $w \in I$, one can use the f -chains to give a simpler bijective proof of joint symmetry. For, in this situation, every f -chain must touch the “midline” where $a(w) - d(w) = 0$ (shown as a dotted line on the left in Figs. 1 and 2). We can break all the f -chains at the midline and rearrange the bottom halves (using the bijection $h : T \rightarrow I$) to create new chains, each of which is symmetric about the midline. More precisely, the top half of the f -chain starting at $h(w)$ is reattached at the midline to the bottom half of the f -chain ending at w , for all $w \in T$. For instance, in Fig. 2 the new chains would be:

$$\begin{aligned} 1 \rightarrow 2 \rightarrow 3 \rightarrow 7 \rightarrow 8 \rightarrow 9, & \quad 15 \rightarrow 4, \\ 10 \rightarrow 11 \rightarrow 12 \rightarrow 16 \rightarrow 17 \rightarrow 18, & \quad 5 \rightarrow 6 \rightarrow 13 \rightarrow 14. \end{aligned}$$

For these new symmetric chains, we can define $J : W \rightarrow W$ by sending each $w \in W$ to the unique object w' on the same chain as w that has $a(w') = d(w)$ and $d(w') = a(w)$.

Remark 6 Mikhail Mazin gives another combinatorial proof of Theorem 2 and we briefly describe it here. Note that it does not give a canonical bijection as above.

First of all, one can assume that W is homogeneous, i.e. the total degree $a(w) + d(w)$ is fixed for $w \in W$. Furthermore, we can think of W as a multiset of integers $d(w) - a(w)$. Then it suffices to prove the following assertion.

Assertion: If W is a multiset of integers with a partition into chains where the chain map adds 2, and $I = -T$ where I is the multiset of initial numbers of these chains and T is the multiset of the terminal numbers of them, then we assert that $W = -W$, i.e. W is symmetric.

We say that two chains are overlapping if they have common elements and neither one contains the other. It is easy to adjust the chains into non-overlapping ones. We claim that all chains are either symmetric or come in couples: $i \rightarrow \dots \rightarrow j$ and $-j \rightarrow \dots \rightarrow -i$, therefore $W = -W$. To prove the claim, take a chain $i \rightarrow \dots \rightarrow j$ of minimal length. Then there are no initial and no terminal numbers between i and j . Note that $-i$ is terminal and $-j$ is initial by the symmetry $I = -T$. Therefore, there

are two possibilities: (a) if $i < 0 < j$, then $i = -j$ and the chain is symmetric; (b) if not, then there exists a chain which starts at $-j$ and ends at $-i$, otherwise the chain starting at $-j$ and the one ending at $-i$ are overlapping. The claim easily follows from here.

3 Chain maps for higher q, t -Catalan polynomials

3.1 The chain conjecture for m -Dyck words

Throughout this section, we fix $m, n \in \mathbb{N}^+$ and let $W = W_n^{(m)}$. We propose to prove the joint symmetry of $C_n^{(m)}(q, t)$ using the methods of Sect. 2. More precisely, we make the following conjecture.

Conjecture 7 *Let $T = T_n^{(m)}$ be the set of m -Dyck words of length n with $\gamma_1 = 0$, namely, $T = \{(0, 0, \gamma_2, \dots, \gamma_{n-1}) \in W\}$. There exists a set $I = I_n^{(m)} \subseteq W$ and a bijection $f : W \setminus T \rightarrow W \setminus I$ such that*

$$\text{area}(f(w)) = \text{area}(w) - 1 \quad \text{and} \quad \text{dinv}_m(f(w)) = \text{dinv}_m(w) + 1 \quad \text{for all } w \in W \setminus T, \quad (2)$$

and $C_T(q, t) = C_I(t, q)$ (using the statistics area and dinv_m).

We will prove this conjecture for all $m \geq 1$ and all $n \leq 4$. Using Theorem 2, we can then conclude that $C_n^{(m)}(q, t) = C_n^{(m)}(t, q)$ for these choices of m and n . There are two main difficulties in proving the conjecture for larger n . First, even if we can guess what the map f should be (as we can for $n = 5$), it may be hard to characterize the set I of initial objects for the f -chains and hence to prove $C_T(q, t) = C_I(t, q)$. Second, it does not seem possible to give a single unified formula for the map f . Even for $n = 4$, we will need to build f by pasting together “partial chain maps” f_0 (defined in Sect. 3.2) and f_1 (defined in Sect. 4.3) defined on domains smaller than $W \setminus T$.

3.2 The partial chain map f_0

This subsection defines, for any fixed n and m , a set $A_0 \subseteq W \setminus T$, a set $B_0 \subseteq W$, and a bijection $f_0 : A_0 \rightarrow B_0$ that satisfies the formulas in (2) for all $w \in A_0$. Informally speaking, f_0 is the “default version” of the chain map f , but it can only be applied to the m -Dyck words in A_0 . For $n < 4$, A_0 is the whole set $W \setminus T$, so f_0 is already sufficient to build all the chains needed to prove joint symmetry (see Sects. 4.1 and 4.2). For $n = 4$, however, we will need to glue f_0 with another map f_1 as discussed in Sect. 4.3.

Definition 8 For any $\gamma \in W$, let $r = r(\gamma)$ be the minimum index $i \in \{2, 3, \dots, n-1\}$ with $\gamma_i - \gamma_{i-2} \leq m$, or n if no such index exists. Let A_0 be the set of $\gamma \in W$ such that $\gamma_{r-1} - 1 \leq \gamma_{n-1} + m$ and $\gamma_1 > 0$ (note $A_0 \subseteq W \setminus T$). Define a map f_0 with domain A_0 by

$$f_0(\gamma) = (\gamma_0, \gamma_1, \dots, \gamma_{r-2}, \gamma_r, \gamma_{r+1}, \dots, \gamma_{n-1}, \gamma_{r-1} - 1) \in W, \quad \forall \gamma \in A_0.$$

We assert that $\gamma_{r-1} > 0$ if and only if $\gamma \in W \setminus T$ (i.e., $\gamma_1 > 0$). Indeed, the equivalence is obvious in the case $r = 2$ since $\gamma_{r-1} = \gamma_1$. For the case $r \geq 3$, we have both $\gamma_{r-1} > 0$ and $\gamma_1 > 0$: indeed, by the definition of r we have $\gamma_{r-1} > \gamma_{r-3} + m > 0$, while $\gamma_1 \geq \gamma_2 - m > \gamma_0 + m - m = 0$.

Lemma 9 For all $\gamma \in A_0$,

$$\text{area}(f_0(\gamma)) = \text{area}(\gamma) - 1 \quad \text{and} \quad \text{dinv}_m(f_0(\gamma)) = \text{dinv}_m(\gamma) + 1.$$

Proof The first equality is immediate. By definition of dinv_m ,

$$\begin{aligned} \text{dinv}_m(\gamma) &= \sum_{\substack{i < j \\ i \neq r-1 \neq j}} \text{sc}_m(\gamma_i - \gamma_j) + \sum_{i < r-1} \text{sc}_m(\gamma_i - \gamma_{r-1}) + \sum_{r-1 < i} \text{sc}_m(\gamma_{r-1} - \gamma_i), \\ \text{dinv}_m(f_0(\gamma)) &= \sum_{\substack{i < j \\ i \neq r-1 \neq j}} \text{sc}_m(\gamma_i - \gamma_j) + \sum_{i \neq r-1} \text{sc}_m(\gamma_i - \gamma_{r-1} + 1). \end{aligned}$$

Let us simplify the second summation in the second equation. Since $\text{sc}_m(x) = \text{sc}_m(1 - x)$ for any $x \in \mathbb{R}$, this summation equals $\sum_{i \neq r-1} \text{sc}_m(\gamma_{r-1} - \gamma_i)$. Thus to show $\text{dinv}_m(f_0(\gamma)) = \text{dinv}_m(\gamma) + 1$, we need only show $\sum_{i < r-1} \text{sc}_m(\gamma_{r-1} - \gamma_i) = 1 + \sum_{i < r-1} \text{sc}_m(\gamma_i - \gamma_{r-1})$. Since $\gamma_{r-2} - \gamma_{r-3} \leq m$ and $\gamma_{r-1} - \gamma_{r-3} > m$, we have $\gamma_{r-2} < \gamma_{r-1}$ (note that the inequality holds if $r = 2$ since $\gamma \notin T$). For every $i < r - 2$,

$$\gamma_i + m < \gamma_{i+2} < \gamma_{i+4} < \cdots < (\gamma_{r-2} \text{ or } \gamma_{r-1}) \leq \gamma_{r-1},$$

therefore $\gamma_{r-1} - \gamma_i > m$, and

$$\text{sc}_m(\gamma_{r-1} - \gamma_i) = 0 = \text{sc}_m(\gamma_i - \gamma_{r-1}).$$

For $i = r - 2$, $0 < \gamma_{r-1} - \gamma_{r-2} \leq m$, hence

$$\text{sc}_m(\gamma_{r-1} - \gamma_{r-2}) = \text{sc}_m(\gamma_{r-1} - \gamma_{r-2} + 1) + 1 = \text{sc}_m(\gamma_{r-2} - \gamma_{r-1}) + 1.$$

Summing up, we conclude that

$$\text{dinv}_m(f_0(\gamma)) = 1 + \sum_{i < j} \text{sc}_m(\gamma_i - \gamma_j) = \text{dinv}_m(\gamma) + 1. \quad \square$$

Definition 10 For any $\gamma \in W$, let $r' = r'(\gamma)$ be the minimum index $i \in \{2, 3, \dots, n\}$ with $\gamma_{i-2} \geq \gamma_{n-1} + 1 - m$, or 0 if no such index exists. Let B_0 be the set of $\gamma \in W$ such that $r'(\gamma) > 0$, $\gamma_{r'-1} \leq \gamma_{n-1} + 1 + m$, and $\gamma_i - \gamma_{i-2} > m$ for $2 \leq i \leq r' - 2$. Define a map g_0 with domain B_0 by setting

$$g_0(\gamma) = (\gamma_0, \dots, \gamma_{r'-2}, \gamma_{n-1} + 1, \gamma_{r'-1}, \dots, \gamma_{n-2}) \in W, \quad \forall \gamma \in B_0.$$

Lemma 11 The map f_0 is a bijection from A_0 to B_0 with inverse g_0 .

Proof Take any $\gamma \in A_0$. In the proof of Lemma 9, we showed that $\gamma_{r-1} - \gamma_i > m$ for $i < r - 2$. On the other hand, $\gamma_{r-1} - \gamma_{r-2} \leq m$. Let $\gamma' = f_0(\gamma)$; then r is the smallest integer such that $\gamma'_{r-2} \geq \gamma'_{n-1} + 1 - m$ and $\gamma'_i - \gamma'_{i-2} > m$ for $2 \leq i \leq r - 2$. So $f_0(\gamma) \in B_0$, $r = r'(f_0(\gamma))$, and hence Definition 10 gives

$$\begin{aligned} g_0 \circ f_0(\gamma) &= g_0((\gamma_0, \dots, \gamma_{r-2}, \gamma_r, \dots, \gamma_{n-1}, \gamma_{r-1} - 1)) \\ &= (\gamma_0, \dots, \gamma_{r-2}, (\gamma_{r-1} - 1) + 1, \gamma_r, \dots, \gamma_{n-1}) = \gamma. \end{aligned}$$

Similarly, we can show that g_0 maps B_0 into A_0 and $f_0 \circ g_0(\gamma) = \gamma$ for any $\gamma \in B_0$, completing the proof that f_0 and g_0 are mutually inverse bijections. $\square$

4 Proof of joint symmetry for $n \leq 4$

In this section, we prove the chain conjecture 7, and hence the joint symmetry of $C_n^{(m)}(q, t)$, for all positive integers m, n with $n \leq 4$. First note that these conjectures hold when $n = 1$ because $C_1^{(m)}(q, t) = 1$ for every m . In the next three subsections we shall settle the cases $n = 2, 3, 4$, respectively.

4.1 The case $n = 2$

We have $W = W_2^{(m)} = \{(0, i) : 0 \leq i \leq m\}$ and $C_2^{(m)}(q, t) = \sum_{i=0}^m q^i t^{m-i}$, which is evidently jointly symmetric. Nevertheless, let us see what our approach does when $n = 2$. We have $T = \{(0, 0)\}$, $r(\gamma) = 2$ for all $\gamma \in W$, $A_0 = \{(0, i) : 1 \leq i \leq m\} = W \setminus T$, $B_0 = \{(0, i) : 0 \leq i < m\}$, and $I = W \setminus B_0 = \{(0, m)\}$. We take the map $f : W \setminus T \rightarrow W \setminus I$ to be the map $f_0 : A_0 \rightarrow B_0$ given by $f_0((0, i)) = (0, i - 1)$ for $1 \leq i \leq m$, which has the correct effect on area and dinv_m . The unique map $h : T \rightarrow I$ gives a bijective proof of $C_T(q, t) = C_I(t, q)$ since

$$C_T(q, t) = q^{\text{area}((0,0))} t^{\text{dinv}_m((0,0))} = t^m = t^{\text{area}((0,m))} q^{\text{dinv}_m((0,m))} = C_I(t, q).$$

So $C_2^{(m)}(q, t) = C_2^{(m)}(t, q)$ by Theorem 2.

4.2 The case $n = 3$

Let $W = W_3^{(m)}$. We have $T = \{(0, 0, i) : 0 \leq i \leq m\}$ and $A_0 = \{\gamma \in W : \gamma_1 > 0\} = W \setminus T$, so we can take $f = f_0 : A_0 \rightarrow B_0$ and $I = W \setminus B_0 = \{(0, i, i + m) : 0 \leq i \leq m\}$. For $n = 3$, the definition of f can be rephrased as follows:

$$f(\gamma) = \begin{cases} (\gamma_0, \gamma_2, \gamma_1 - 1), & \text{if } \gamma_2 \leq m; \\ (\gamma_0, \gamma_1, \gamma_2 - 1), & \text{if } \gamma_2 > m. \end{cases}$$

We already proved that f is a bijection (Lemma 11) having the correct effect on area and dinv_m (Lemma 9). We need to prove that $C_T(q, t) = C_I(t, q)$. Define

$$\begin{aligned} g : I \setminus \{(0, 0, m)\} &\longrightarrow I \setminus \{(0, m, 2m)\}, \quad \gamma \mapsto (\gamma_0, \gamma_1 - 1, \gamma_2 - 1); \\ g' : T \setminus \{(0, 0, m)\} &\longrightarrow T \setminus \{(0, 0, 0)\}, \quad \gamma \mapsto (0, 0, \gamma_2 + 1). \end{aligned}$$

Fig. 3 Left: the triangle that contains all (j, k) for monomials $q^j t^k$ in $C_3^{(m)}(q, t)$. Right: the special case when $m = 4$. The lattice points on the line marked I (green line) give monomials in $C_I(q, t)$; the lattice points on the line marked T (red line) give monomials in $C_T(q, t)$. The other lines (of slope -1) illustrate f -chains (Color figure online)

Then g increases div_m by 1 and decreases area by 2, whereas g' increases area by 1 and decreases div_m by 2. The set I consists of a single g -chain of $m + 1$ objects starting at $(0, m, 2m)$ and ending at $(0, 0, m)$. So

$$\begin{aligned} C_I(q, t) &= q^{\text{area}((0, m, 2m))} t^{\text{div}_m((0, m, 2m))} (1 + q^{-2}t + q^{-4}t^2 + \cdots + q^{-2m}t^m) \\ &= q^{3m}t^0 + q^{3m-2}t^1 + q^{3m-4}t^2 + \cdots + q^m t^m. \end{aligned}$$

Similarly, $C_T(q, t) = q^0 t^{3m} + q^1 t^{3m-2} + q^2 t^{3m-4} + \cdots + q^m t^m$. Hence $C_T(q, t) = C_I(t, q)$. Note that every monomial in $C_I(q, t)$ has a different total degree. It follows that

$$C_3^{(m)}(q, t) = \sum_{j=0}^m (q^{3m-2j} t^j + q^{3m-2j-1} t^{j+1} + \cdots + q^j t^{3m-2j}),$$

so $C_3^{(m)}(q, t)$ is the sum of all monomials $q^j t^k$ where (j, k) runs through all lattice points that are either inside or on the boundary of the triangle with vertices $(0, 3m)$, (m, m) , $(3m, 0)$ (Fig. 3). In particular, for $n = 3$, there is a unique involution on W giving a bijective proof of joint symmetry.

4.3 The case $n = 4$

Let $W = W_4^{(m)}$. We define a partial chain map f_1 and glue it with f_0 to obtain a bijective map f that satisfies Conjecture 7.

Definition 12 Let $A_1 = \{(0, \gamma_1, \gamma_2, \gamma_3) \in W : \gamma_2 - \gamma_3 > m + 1\}$. (Note that $A_1 \subseteq W \setminus T$ and $A_0 \cap A_1 = \emptyset$.) Define a map f_1 with domain A_1 by

$$f_1((0, \gamma_1, \gamma_2, \gamma_3)) = (0, \gamma_3 + 1, \gamma_1 - 1, \gamma_2 - 1) \in W.$$

Let $B_1 = f_1(A_1)$. Define $A = A_0 \cup A_1$, $B = B_0 \cup B_1$, and $f : A \rightarrow B$ by $f(\gamma) = f_0(\gamma)$ for $\gamma \in A_0$ and $f(\gamma) = f_1(\gamma)$ for $\gamma \in A_1$.

Lemma 13 $W \setminus A = \{(0, 0, \gamma_2, \gamma_3) \in W\} = T$.

Proof First, observe that $W \setminus A_0$ contains the following three classes of elements (which correspond to $r = 2, 3, 4$, respectively):

- (a) $\gamma_2 \leq m, \gamma_1 = 0$;
- (b) $\gamma_2 > m, \gamma_3 - \gamma_1 \leq m$, and $\gamma_2 - 1 > \gamma_3 + m$;
- (c) $\gamma_2 > m, \gamma_3 - \gamma_1 > m$, and $\gamma_3 = 0$.

(Note that for $r = 3$ we do not need to include the case $\gamma_2 > m, \gamma_3 - \gamma_1 \leq m$, and $\gamma_1 = 0$, since $\gamma_1 \geq \gamma_2 - m > 0$.) Elements satisfying (b) are exactly those in A_1 . No elements satisfy (c). Thus $W \setminus A$ contains elements only in (a), i.e., those of the form $(0, 0, \gamma_2, \gamma_3)$. $\square$

Lemma 14 $B_0 \cap B_1 = \emptyset$, so $f : A \rightarrow B$ is a bijection.

Proof Assume, to the contrary, that there is a $\gamma \in A_1$ such that $f(\gamma) = \gamma' \in B_0$. Thus $\gamma'_1 = \gamma_3 + 1, \gamma'_2 = \gamma_1 - 1, \gamma'_3 = \gamma_2 - 1$. Let $r' = r'(\gamma')$ be defined as in Definition 10. Note r' can be 2, 3, or 4. Below we show that all three cases give a contradiction.

- (i) If $r' = 2$, then $0 \geq \gamma'_3 + 1 - m$ by the definition of r' , hence $\gamma_2 \leq m$. But $\gamma_2 - \gamma_3 > m + 1$ by the definition of A_1 . Thus $\gamma_2 > m$, a contradiction.
- (ii) If $r' = 3$, then $\gamma'_1 \geq \gamma'_3 + 1 - m$ by the definition of r' , which implies $\gamma_2 - \gamma_3 \leq m + 1$ and again contradicts the definition of A_1 .
- (iii) If $r' = 4$, then $\gamma'_2 = \gamma'_2 - \gamma'_0 > m$ by the definition of B_0 . Thus $\gamma_1 > m + 1$ which contradicts the condition that $\gamma_1 \leq m$.

Since f_1 is evidently an injection, we conclude that f is a bijection, whose inverse can be defined by $f^{-1}(\gamma) = f_0^{-1}(\gamma)$ for $\gamma \in B_0$ and $f^{-1}(\gamma) = (0, \gamma_2 + 1, \gamma_3 + 1, \gamma_1 - 1)$ for $\gamma \in B_1$. $\square$

Lemma 15 For every $\gamma \in A$,

$$\text{area}(f(\gamma)) = \text{area}(\gamma) - 1 \quad \text{and} \quad \text{dinv}_m(f(\gamma)) = \text{dinv}_m(\gamma) + 1.$$

Proof Because of Lemma 9, we need only to consider $\gamma \in A_1$. The first equality is evident. The left-hand side of the second equality is

$$\begin{aligned} & \text{sc}_m(-\gamma_1 + 1) + \text{sc}_m(-\gamma_2 + 1) + \text{sc}_m(-\gamma_3 - 1) + \text{sc}_m(\gamma_1 - \gamma_2) \\ & + \text{sc}_m(\gamma_3 - \gamma_1 + 2) + \text{sc}_m(\gamma_3 - \gamma_2 + 2). \end{aligned}$$

Write each summand in terms of $\text{sc}_m(\gamma_i - \gamma_j)$ for $i < j$ (and recall that $\gamma_0 = 0$):

- (i) $0 < \gamma_1 \leq m \Rightarrow \text{sc}_m(-\gamma_1 + 1) = \text{sc}_m(-\gamma_1) + 1$.
- (ii) $\gamma_2 - \gamma_3 > m + 1 \Rightarrow \gamma_2 \geq m + 2 \Rightarrow \text{sc}_m(-\gamma_2 + 1) = \text{sc}_m(-\gamma_2) = 0$.

(iii) $(\gamma_2 - \gamma_3 > m + 1)$ and $(\gamma_2 \leq 2m) \Rightarrow 0 \leq \gamma_3 < m - 1 \Rightarrow \text{sc}_m(-\gamma_3 - 1) = \text{sc}_m(-\gamma_3) - 1$.

(iv) $\text{sc}_m(\gamma_1 - \gamma_2)$ is unchanged.

(v) $m \geq \gamma_1 \geq \gamma_2 - m > \gamma_3 + 1 \Rightarrow 1 < \gamma_1 - \gamma_3 \leq m \Rightarrow \text{sc}_m(\gamma_3 - \gamma_1 + 2) = \text{sc}_m(\gamma_1 - \gamma_3 - 1) = \text{sc}_m(\gamma_1 - \gamma_3) + 1$.

(vi) $\gamma_2 - \gamma_3 > m + 1 \Rightarrow \text{sc}_m(\gamma_3 - \gamma_2 + 2) = \text{sc}_m(\gamma_2 - \gamma_3 - 1) = \text{sc}_m(\gamma_2 - \gamma_3) = 0$.

Summing up (i)–(vi), we obtain the second equality. $\square$

Lemma 16 (1) $B_1 = \{\gamma \in W : \gamma_1 \geq 1, \gamma_2 \leq m - 1, \gamma_3 - \gamma_1 \geq m\}$;

(2) $I = W \setminus B$ is the disjoint union $D_1 \cup D_2 \cup D_3$, where:

$$D_1 = \{(0, \gamma_1, \gamma_2, \gamma_2 + m) : 0 < \gamma_1 \leq m < \gamma_2 \leq \gamma_1 + m\},$$

$$D_2 = \{(0, \gamma_1, m, \gamma_3) : 0 \leq \gamma_1 \leq m, \gamma_1 + m \leq \gamma_3 \leq 2m\},$$

$$D_3 = \{(0, 0, \gamma_2, \gamma_3) : 0 \leq \gamma_2 < m \leq \gamma_3 \leq \gamma_2 + m\}.$$

Proof The proof of (1) is routine. For (2), we first find out $W \setminus B_0$. By a case-by-case study for $r' = 2, 3, 4$, it is straightforward to check that B_0 contains those γ with $\gamma_3 \leq m - 1$ (in the case $r'(\gamma) = 2$), those with $m \leq \gamma_3 \leq \gamma_1 + m - 1$ (in the case $r'(\gamma) = 3$), and those with $\gamma_1 + m \leq \gamma_3 \leq \gamma_2 + m - 1$ and $\gamma_2 > m$ (in the case $r'(\gamma) = 4$). Hence $W \setminus B_0$ consists of $\gamma \in W$ that satisfy

$$\gamma_1 + m \leq \gamma_3 \leq \gamma_2 + m - 1 \quad \text{and} \quad \gamma_2 \leq m \quad (3)$$

or

$$\gamma_3 \geq \max(\gamma_1, \gamma_2) + m. \quad (4)$$

Note that (4) is equivalent to $\gamma_1 \leq \gamma_2$ and $\gamma_3 = \gamma_2 + m$, because $\gamma_3 \leq \gamma_2 + m$. Therefore $W \setminus B$ consists of three classes of elements:

$(0, \gamma_1, \gamma_2, \gamma_2 + m)$ where $\gamma_1 \leq m < \gamma_2$,

$(0, 0, \gamma_2, \gamma_3)$ where $\gamma_2 < m \leq \gamma_3$, and

$(0, \gamma_1, m, \gamma_3)$ where $\gamma_1 + m \leq \gamma_3$,

which gives (2). $\square$

Remark 17 One can check that f can be defined piecewise as follows:

if $\gamma_2 \leq m$ then $f(\gamma) = (\gamma_0, \gamma_2, \gamma_3, \gamma_1 - 1)$;

if $\gamma_2 > m$ then

$$\left| \begin{array}{l} \text{if } \gamma_3 - \gamma_1 > m \text{ then } f(\gamma) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3 - 1); \\ \text{if } \gamma_3 - \gamma_1 \leq m \text{ then} \\ \quad \left| \begin{array}{l} \text{if } \gamma_2 - \gamma_3 > m + 1 \text{ then } f(\gamma) = (\gamma_0, \gamma_3 + 1, \gamma_1 - 1, \gamma_2 - 1); \\ \text{if } \gamma_2 - \gamma_3 \leq m + 1 \text{ then } f(\gamma) = (\gamma_0, \gamma_1, \gamma_3, \gamma_2 - 1). \end{array} \right. \end{array} \right.$$

Now to finish the proof of Conjecture 7 for $n = 4$, we shall show the following:

$$\sum_{\gamma \in W \setminus A} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} = \sum_{\gamma \in W \setminus B} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)}. \quad (5)$$

By Lemma 13, $W \setminus A$ consists of $\gamma = (0, 0, \gamma_2, \gamma_3) \in W$. Since $\gamma_2 \leq m$, we can write $W \setminus A$ as the disjoint union $D'_1 \cup D'_2 \cup D'_3$, where

$$\begin{aligned} D'_1 &= \{\gamma \in W \setminus A : \gamma_2 > \gamma_3\}, \\ D'_2 &= \{\gamma \in W \setminus A : \gamma_2 \leq \gamma_3 \leq m\}, \\ D'_3 &= \{\gamma \in W \setminus A : \gamma_3 > m\}. \end{aligned}$$

Then $\text{dinv}_m(\gamma) = \text{sc}_m(0) + 2\text{sc}_m(-\gamma_2) + 2\text{sc}_m(-\gamma_3) + \text{sc}_m(\gamma_2 - \gamma_3)$ is equal to

$$\begin{cases} m + 2(m - \gamma_2) + 2(m - \gamma_3) + (m + 1 - \gamma_2 + \gamma_3) = 6m + 1 - 3\gamma_2 - \gamma_3, & \text{if } \gamma \in D'_1; \\ m + 2(m - \gamma_2) + 2(m - \gamma_3) + (m + \gamma_2 - \gamma_3) = 6m - \gamma_2 - 3\gamma_3, & \text{if } \gamma \in D'_2; \\ m + 2(m - \gamma_2) + 0 + (m + \gamma_2 - \gamma_3) = 4m - \gamma_2 - \gamma_3, & \text{if } \gamma \in D'_3. \end{cases} \quad (6)$$

On the other hand, thanks to Lemma 16, $\text{dinv}_m(\gamma)$ is equal to

$$\begin{cases} (m - \gamma_1) + 0 + 0 + (m + \gamma_1 - \gamma_2) + 0 + 0 = 2m - \gamma_2, & \text{if } \gamma \in D_1; \\ (m - \gamma_1) + 0 + 0 + (m + \gamma_1 - m) + 0 + (m + m - \gamma_3) = 3m - \gamma_3, & \text{if } \gamma \in D_2; \\ m + 2(m - \gamma_2) + 0 + (m + \gamma_2 - \gamma_3) = 4m - \gamma_2 - \gamma_3, & \text{if } \gamma \in D_3. \end{cases} \quad (7)$$

Therefore, the assertion (5) follows from the following lemma.

Lemma 18

$$\sum_{\gamma \in D_1 \cup D_2} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} = \sum_{\gamma \in D'_1 \cup D'_2} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)}, \quad (8)$$

$$\sum_{\gamma \in D_3} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} = \sum_{\gamma \in D'_3} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)}. \quad (9)$$

Proof The equality (9) follows immediately from the one-to-one correspondence from D'_3 to D_3 that sends $(0, 0, \gamma_2, \gamma_3)$ to $(0, 0, 2m - \gamma_3, 2m - \gamma_2)$. Alternatively, both sides can be shown to be equal to

$$\sum_{u=0}^{m-1} \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v}. \quad (10)$$

Next we prove (8). Define

$$\begin{aligned} g : (D_1 \cup D_2) \setminus \{(0, 0, m, \gamma_3) \in W : \gamma_3 \geq m\} \\ \longrightarrow (D_1 \cup D_2) \setminus \{(0, \gamma_1, \gamma_1 + m, \gamma_1 + 2m) \in W\} \end{aligned}$$

by

$$g(\gamma) = \begin{cases} (0, \gamma_1, \gamma_2 - 1, \gamma_3 - 1), & \text{if } \gamma \in D_1; \\ (0, \gamma_1 - 1, \gamma_2, \gamma_3 - 1), & \text{if } \gamma \in D_2. \end{cases}$$

Note that g is a bijection with inverse defined as

$$g^{-1}(\gamma) = \begin{cases} (0, \gamma_1, \gamma_2 + 1, \gamma_3 + 1), & \text{if } m \leq \gamma_2 < \gamma_1 + m, \gamma_3 = \gamma_2 + m; \\ (0, \gamma_1 + 1, \gamma_2, \gamma_3 + 1), & \text{if } \gamma_1 < m, \gamma_1 + m \leq \gamma_3 < 2m. \end{cases}$$

Using the formula (7) to compute dinv_m , it is straightforward to verify that

$$\begin{aligned} \sum_{\gamma \in \{(0,0,m,\gamma_3) \in W : \gamma_3 \geq m\}} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} &= q^{2m} t^{2m} + q^{2m+1} t^{2m-1} + \dots + q^{3m} t^m, \\ \sum_{\gamma \in \{(0,\gamma_1,\gamma_1+m,\gamma_1+2m) \in W\}} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} &= q^{6m} t^0 + q^{6m-3} t^1 + \dots + q^{3m} t^m, \end{aligned} \quad (11)$$

and that g increases dinv_m by 1 and decreases area by 2. By iterating the map g we obtain maximal sequences $(\gamma, g(\gamma), \dots, g^{(\gamma_1)}(\gamma))$ that start from elements in $\{(0, \gamma_1, \gamma_1 + m, \gamma_1 + 2m) \in W\}$ and end at elements in $\{(0, 0, m, \gamma_3) \in W : \gamma_3 \geq m\}$. Moreover, if we define the $(1, 2)$ -weight of γ to be the integer $\text{area}(\gamma) + 2\text{dinv}_m(\gamma)$, then each sequence contains elements of the same $(1, 2)$ -weight. Since the $(1, 2)$ -weights of those γ appearing in the first (resp. second) equation of (11) are distinct, the last element of a sequence is determined by the first element. As a result, the left-hand side of (8) must equal

$$\begin{aligned} \sum_{u=0}^m \sum_{v=u}^{2m-u} q^{6m-u-2v} t^v &= (q^{6m} t^0 + q^{6m-2} t^1 + q^{6m-4} t^2 + q^{6m-6} t^3 + \dots + q^{2m} t^{2m}) \\ &\quad + (q^{6m-3} t^1 + q^{6m-5} t^2 + q^{6m-7} t^3 + \dots + q^{2m+1} t^{2m-1}) \\ &\quad + (q^{6m-6} t^2 + q^{6m-8} t^3 + \dots + q^{2m+2} t^{2m-2}) \\ &\quad \vdots \\ &\quad + (q^{3m} t^m). \end{aligned} \quad (12)$$

On the other hand, define

$$g' : (D'_1 \cup D'_2) \setminus \{(0, 0, \gamma_2, m) \in W\} \longrightarrow (D'_1 \cup D'_2) \setminus \{(0, 0, 0, \gamma_3) \in W\}$$

by $g'(\gamma) = (0, 0, \gamma_3 + 1, \gamma_2)$. Note that g' is a bijection with inverse $g'^{-1}(\gamma) = (0, 0, \gamma_3, \gamma_2 - 1)$. Using (6), we can verify that

$$\begin{aligned} \sum_{\gamma \in \{(0,0,\gamma_2,m) \in W\}} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)} &= q^{2m} t^{2m} + q^{2m+1} t^{2m-1} + q^{2m+2} t^{2m-2} + \dots + q^{3m} t^m, \\ \sum_{\gamma \in \{(0,0,0,\gamma_3) \in W\}} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)} &= q^{6m} t^0 + q^{6m-3} t^1 + q^{6m-6} t^2 + \dots + q^{3m} t^m. \end{aligned}$$

Fig. 4 Left: the triangle that contains Δ defined in Remark 19. Right: the support of $C_T(q, t)$ in the case $n = 4, m = 2$ as discussed in Example 3, where $\Delta = D'_1 \cup D'_2$ corresponds to the 9 lattice points to the northwest of 022 (inclusive), and D'_3 corresponds to the lowest three lattice points labeled with 013, 023, 024. Note that 022 and 013 correspond to the same lattice point as they have the same weight $q^4 t^4$

and that g' increases area by 1 and decreases dinv_m by 2. For each $\gamma = (0, 0, 0, \gamma_3) \in W$, iterating the map g' produces the sequence $\gamma, g'(\gamma), \dots, g'^{(m-\gamma_3)}(\gamma)$ where $g'^{(m-\gamma_3)}(\gamma)$ is in the set $\{(0, 0, \gamma_2, m) \in W\}$. By a similar argument as above, the right-hand side of (8) is also equal to (12). This finishes the proof of (8) and therefore completes the proof of Conjecture 7 for $n = 4$. $\square$

Remark 19 We can also prove (8) directly by showing that

$$\sum_{\gamma \in D'_1 \cup D'_2} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)} = \sum_{(x, y) \in \Delta} q^x t^y = \sum_{\gamma \in D_1 \cup D_2} t^{\text{area}(\gamma)} q^{\text{dinv}_m(\gamma)}, \quad (13)$$

where Δ is the set of lattice points that are either inside or on the boundary of the triangle with vertices $(0, 6m), (m, 3m), (2m, 2m)$ (Fig. 4 Left).

Indeed, note that the set of all points inside or on the boundary of the triangle is given by $\{(u + v, 6m - u - 3v) : u, v \in \mathbb{R}, 0 \leq u \leq v \leq m\}$. To be a lattice point in this triangle, either u or $u - 1/2$ is an integer. The lattice points with u being an integer are in one-to-one correspondence with pairs $(\text{area}(\gamma), \text{dinv}_m(\gamma))$ for $\gamma \in D'_2$ (by letting $\gamma_2 = u, \gamma_3 = v$); the lattice points with $u - 1/2$ being an integer are in one-to-one correspondence with pairs $(\text{area}(\gamma), \text{dinv}_m(\gamma))$ for $\gamma \in D'_1$ (by letting $\gamma_2 = v + 1/2, \gamma_3 = u - 1/2$). Thus the pairs $(\text{area}(\gamma), \text{dinv}_m(\gamma))$ for γ in D'_2 (resp. D'_1) form the set $\{(x, y) \in \Delta | x + y \text{ is even}\}$ (resp. $\{(x, y) \in \Delta | x + y \text{ is odd}\}$).

On the other hand, the set of all points inside or on the boundary of the triangle can also be defined as $\{(3m - v, m + u + v) : u, v \in \mathbb{R}, 0 \leq u \leq 2m, u + m \leq v \leq u/2 + 2m\}$. The lattice points therein with $v \leq 2m$ are in one-to-one correspondence with pairs $(\text{dinv}_m(\gamma), \text{area}(\gamma))$ for $\gamma \in D_1$ (by letting $\gamma_1 = u, \gamma_3 = v$); the lattice points therein with $v > 2m$ are in one-to-one correspondence with pairs $(\text{dinv}_m(\gamma), \text{area}(\gamma))$ for $\gamma \in D_2$ (by letting $\gamma_1 = u - v + 2m, \gamma_2 = v - m$). Thus the pairs $(\text{dinv}_m(\gamma), \text{area}(\gamma))$ for γ in D_2 (resp. D_1) form the set $\{(x, y) \in \Delta | x \geq m\}$ (resp. $\{(x, y) \in \Delta | x < m\}$).

Fig. 5 The f -chains of nonzero lengths in Example 3

Therefore both equalities in (13) hold.

Example 3 Let $n = 4$ and $m = 2$. There are fifty-five 2-Dyck words in $W = W_4^{(2)}$. Since γ_0 is always zero, we use $\gamma_1\gamma_2\gamma_3(q^at^d)$ to denote the Dyck word $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ with $(\text{area}, \text{dinv}_m) = (a, d)$. The f -chains for this set are listed below.

- (1) an f -chain from $246(q^{12}t^0)$ to $000(q^0t^{12})$ of length 12,
- (2) an f -chain from $235(q^{10}t^1)$ to $010(q^1t^{10})$ of length 9,
- (3) an f -chain from $124(q^7t^2)$ to $020(q^2t^7)$ of length 5,
- (4) an f -chain from $123(q^6t^3)$ to $021(q^3t^6)$ of length 3,
- (5) an f -chain from $224(q^8t^2)$ to $001(q^1t^9)$ of length 7,
- (6) an f -chain from $135(q^9t^1)$ to $011(q^2t^8)$ of length 7,
- (7) six chains of length zero: $024(q^6t^2)$, $023(q^5t^3)$, $022(q^4t^4)$, $012(q^3t^5)$, $002(q^2t^6)$, $013(q^4t^4)$.

Figure 5 illustrates the six f -chains with nonzero lengths, where the length of a chain is defined as the number of arrows.

Moreover,

$$\begin{aligned} D_1 &= \{246, 235, 135\}, & D_2 &= \{224, 124, 123, 024, 023, 022\}, \\ D_3 &= \{012, 002, 013\}, & D'_1 &= \{010, 020, 021\}, \\ D'_2 &= \{000, 001, 011, 022, 012, 002\}, & D'_3 &= \{024, 023, 013\}. \end{aligned}$$

The initial points of chains have generating function

$$C_I(q, t) = q^{12}t^0 + q^{10}t^1 + q^7t^2 + q^6t^3 + q^8t^2 + q^9t^1 + q^6t^2 \\ + q^5t^3 + q^4t^4 + q^3t^5 + q^2t^6 + q^4t^4.$$

The terminal points of chains have generating function (see Fig. 4 Right)

$$C_T(q, t) = q^0t^{12} + q^1t^{10} + q^2t^7 + q^3t^6 + q^1t^9 + q^2t^8 + q^6t^2 \\ + q^5t^3 + q^4t^4 + q^3t^5 + q^2t^6 + q^4t^4.$$

We see that $C_T(q, t) = C_I(t, q)$, so that $C_W(q, t) = C_W(t, q)$.

Theorem 20 For $j, k \in \mathbb{N}$, let $c(j, k)$ be the coefficient of $q^j t^k$ in $C_4^{(m)}(q, t)$. For $a \in \mathbb{R}$, define $\lfloor a \rfloor^+ = \max(\lfloor a \rfloor, 0)$.

(a) If $j + k > 4m$, then

$$c(j, k) = \min \left(\left\lfloor \frac{-6m + 2 + 3j + k}{2} \right\rfloor^+, \left\lfloor \frac{6m + 2 - j - k}{2} \right\rfloor^+, \right. \\ \left. \left\lfloor \frac{-6m + 2 + j + 3k}{2} \right\rfloor^+ \right).$$

(b) If $j + k = 4m$, then

$$c(j, k) = \min \left(\left\lfloor \frac{-m + 2 + j}{2} \right\rfloor^+, \left\lfloor \frac{-m + 2 + k}{2} \right\rfloor^+ \right).$$

(c) If $j + k < 4m$, then $c(j, k) = 0$.

As a consequence, the sequence $(c(d, 0), c(d-1, 1), c(d-2, 2), \dots, c(1, d-1), c(0, d))$ is unimodal for every positive integer d .

Proof We may assume $j \geq k$ because of the q, t -joint symmetry. Using Remark 4,

$$c(j, k) = |C_I(q, t)|_{q\text{-deg} \geq j}^{\deg=j+k} - |C_I(q, t)|_{t\text{-deg} > j}^{\deg=j+k}. \quad (14)$$

First, observe that all monomials in $C_I(q, t)$ have degree at least $4m$, so $c(j, k) = 0$ if $j + k < 4m$. This proves (c).

Then we consider the case $j + k > 4m$. Because of (12), $|C_I(q, t)|_{q\text{-deg} \geq j}^{\deg=j+k}$ is equal to the cardinality of the set

$$\{(u, v) : 0 \leq u \leq m, u \leq v \leq 2m - u, 6m - u - v = j + k, j \leq 6m - u - 2v\}.$$

In this set, $\max(6m - j - 2k, 0) \leq u \leq (6m - j - k)/2$, and v is determined by u . Thus the cardinality is $\max((6m - j - k)/2 - \max(6m - j - 2k, 0) + 1, 0)$, that is,

$$\min \left(\left\lfloor \frac{-6m + 2 + j + 3k}{2} \right\rfloor^+, \left\lfloor \frac{6m + 2 - j - k}{2} \right\rfloor^+ \right).$$

On the other hand, $|C_I(q, t)|_{\deg=j+k}^{\deg=j+k}$ is the cardinality of the set

$$\{(u, v) : 0 \leq u \leq m, u \leq v \leq 2m - u, 6m - u - v = j + k, j < v\}.$$

The conditions imply $0 \leq u \leq 6m - 2j - k$, but $6m - 2j - k < 2m - j < 0$, so the set is empty. This implies (a).

Finally we consider the case $j + k = 4m$. The degree- $4m$ part of $C_I(q, t)$ is equal to

$$\sum_{u=0}^{m-1} \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} + \sum_{u=0}^m q^{2m+u} t^{2m-u} = \sum_{u=0}^m \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} \quad (15)$$

which is q, t -symmetric thanks to the bijection $(u, v) \mapsto (2m - v, 2m - u)$. Therefore all f -chains have length 0, and $c(j, k)$ is equal to the coefficient of $q^j t^k$ in (15). This implies (b). $\square$

5 Comparison to the Garsia–Haiman formula

This section proves Conjecture 1 for all $m \geq 1$ and all $n \leq 4$. Fix $m \geq 1$. For $n = 1$, we have

$$AC_1^{(m)}(q, t) = \frac{(1-q)(1-t)}{(1-t)(1-q)} = 1 = C_1^{(m)}(q, t).$$

For $n = 2$, we compute:

$$\begin{aligned} AC_2^{(m)}(q, t) &= \frac{(q^1 t^0)^{m+1} (1-q)(1-t)(1+q)(1-q)}{(q-t)(1-t)(1-q^2)(1-q)} \\ &\quad + \frac{(q^0 t^1)^{m+1} (1-q)(1-t)(1+t)(1-t)}{(1-t^2)(1-t)(t-q)(1-q)} \\ &= \frac{q^m}{1-t/q} + \frac{t^m}{1-q/t} = \frac{q^{m+1} - t^{m+1}}{q-t} \\ &= q^m + q^{m-1}t + q^{m-2}t^2 + \cdots + t^m = C_2^{(m)}(q, t). \end{aligned}$$

5.1 The case $n = 3$

The partitions of $n = 3$ are (3), (1, 1, 1) and (2, 1). Using the Garsia–Haiman formula (1) gives

$$\begin{aligned} AC_3^{(m)}(q, t) &= \frac{(q^3 t^0)^{m+1} (1-q)(1-t)(1+q+q^2)(1-q)(1-q^2)}{(q^2-t)(q-t)(1-t)(1-q^3)(1-q^2)(1-q)} \\ &\quad + \frac{(q^0 t^3)^{m+1} (1-q)(1-t)(1+t+t^2)(1-t)(1-t^2)}{(1-t^3)(1-t^2)(1-t)(t^2-q)(t-q)(1-q)} \\ &\quad + \frac{(q^1 t^1)^{m+1} (1-q)(1-t)(1+q+t)(1-q)(1-t)}{(q-t^2)(1-t)(1-t)(t-q^2)(1-q)(1-q)}. \end{aligned}$$

By canceling common factors and using the map σ on $\mathbb{Q}(q, t)$ that sends $F(q, t)$ to $F(q, t) + F(t, q)$, we can rewrite this as

$$AC_3^{(m)}(q, t) = \sigma\left(\frac{q^{3m}}{(1-t/q^2)(1-t/q)}\right) + \frac{q^m t^m (1+q+t)}{(1-t^2/q)(1-q^2/t)}.$$

On the other hand, recall from Sect. 4.2 that for $n = 3$, $I = \{(0, i, i+m) : 0 \leq i \leq m\}$ and $C_I(q, t) = \sum_{v=0}^m q^{3m-2v} t^v$. As $3m-2v \geq v$ for $0 \leq v \leq m$, we can apply Lemma 3(b). Before doing so, we rewrite $C_I(q, t)$ as follows:

$$\begin{aligned} C_I(q, t) &= \sum_{v=0}^m q^{3m-2v} t^v = q^{3m} \sum_{v=0}^m (t/q^2)^v \\ &= q^{3m} \left(\frac{1 - (t/q^2)^{m+1}}{1 - t/q^2} \right) \\ &= \frac{q^{3m}}{1 - t/q^2} + \frac{q^m t^m}{1 - q^2/t}. \end{aligned}$$

By the lemma,

$$\begin{aligned} C_W(q, t) &= \sigma\left(\frac{C_I(q, t)}{1 - t/q}\right) \\ &= \sigma\left(\frac{q^{3m}}{(1-t/q^2)(1-t/q)}\right) \\ &\quad + q^m t^m \left[\frac{1}{(1-q^2/t)(1-t/q)} + \frac{1}{(1-t^2/q)(1-q/t)} \right] \\ &= \sigma\left(\frac{q^{3m}}{(1-t/q^2)(1-t/q)}\right) + \frac{q^m t^m (1+q+t)}{(1-t^2/q)(1-q^2/t)}, \end{aligned}$$

where the last equality follows by routine algebra. We now see that $C_3^{(m)}(q, t) = AC_3^{(m)}(q, t)$.

5.2 The case $n = 4$

The partitions of $n = 4$ are (4) , $(1, 1, 1, 1)$, $(3, 1)$, $(2, 1, 1)$ and $(2, 2)$. Writing out the Garsia–Haiman formula (1) and canceling common factors gives

$$\begin{aligned} AC_4^{(m)}(q, t) &= \frac{(q^6 t^0)^{m+1}}{(q^3 - t)(q^2 - t)(q - t)} + \frac{(q^0 t^6)^{m+1}}{(t^3 - q)(t^2 - q)(t - q)} \\ &\quad + \frac{(q^3 t^1)^{m+1}(1+q+q^2+t)}{(q^2 - t^2)(q - t)(t - q^3)} + \frac{(q^1 t^3)^{m+1}(1+t+t^2+q)}{(q - t^3)(t^2 - q^2)(t - q)} \\ &\quad + \frac{(q^2 t^2)^{m+1}(1 - qt)}{(q - t^2)(q - t)(t - q^2)(t - q)}. \end{aligned}$$

We can rewrite this as

$$\begin{aligned} AC_4^{(m)}(q, t) &= \sigma \left(\frac{q^{6m}}{(1-t/q)(1-t/q^2)(1-t/q^3)} \right) \\ &\quad - \sigma \left(q^{3m} t^m \frac{t/q + t/q^2 + t/q^3 + t^2/q^3}{(1-t^2/q^2)(1-t/q)(1-t/q^3)} \right) \\ &\quad + q^{2m} t^{2m} \frac{q^2 t^2 (1-qt)}{(q-t^2)(t-q^2)(q-t)(t-q)}. \end{aligned}$$

Recall from Lemma 16 that $I = I_4^{(m)}$ is the disjoint union $I = D_1 \cup D_2 \cup D_3$. We derived formulas for $C_{D_3}(q, t)$ and $C_{D_1 \cup D_2}(q, t)$ in (10) and (12). Adding these formulas gives

$$\begin{aligned} C_I(q, t) &= \sum_{u=0}^{m-1} \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} + \sum_{u=0}^m \sum_{v=u}^{2m-u} q^{6m-u-2v} t^v \\ &= \sum_{u=0}^{m-1} \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} + \sum_{u=0}^m q^{2m+u} t^{2m-u} + \sum_{u=0}^{m-1} \sum_{v=u}^{2m-u-1} q^{6m-u-2v} t^v \\ &= \sum_{u=0}^m \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} + \sum_{u=0}^{m-1} \sum_{v=u}^{2m-u-1} q^{6m-u-2v} t^v. \end{aligned} \quad (16)$$

To continue, write $I = I' \cup I''$, where $I' = \{w \in I : \text{area}(w) + \text{dinv}_m(w) = 4m\}$ and $I'' = I \setminus I'$. Note that the first double sum in (16) is $C_{I'}(q, t)$, and the second double summation is $C_{I''}(q, t)$. Also write $T = T' \cup T''$ and $W = W' \cup W''$, where T' (resp. W') consists of the objects w in T (resp. W) with $\text{area}(w) + \text{dinv}_m(w) = 4m$. The map f restricts to decompose W' (resp. W'') into f -chains that go from I' to T' (resp. I'' to T''), and we have $C_{T'}(q, t) = C_{I'}(t, q)$ and $C_{T''}(q, t) = C_{I''}(t, q)$.

It is routine to check that $C_{I'}(q, t) = C_{I'}(t, q)$. Hence, Lemma 3(a) applies to give

$$\begin{aligned} C_{W'}(q, t) &= C_{I'}(q, t) \\ &= \sum_{u=0}^m \sum_{v=m}^{u+m} q^{u+v} t^{4m-u-v} \\ &= \sum_{u=0}^m q^u t^{4m-u} \sum_{v=m}^{u+m} (q/t)^v \\ &= \sum_{u=0}^m q^u t^{4m-u} \frac{(q/t)^m - (q/t)^{m+u+1}}{1 - q/t} \\ &= \frac{q^m t^{3m}}{1 - q/t} \sum_{u=0}^m (q/t)^u - \frac{q^{m+1} t^{3m-1}}{1 - q/t} \sum_{u=0}^m (q^2/t^2)^u \end{aligned}$$

$$\begin{aligned}
&= \frac{q^m t^{3m}}{(1-q/t)^2} - \frac{q^{2m} t^{2m} \cdot q/t}{(1-q/t)^2} - \frac{q^m t^{3m} \cdot q/t}{(1-q/t)(1-q^2/t^2)} \\
&\quad + \frac{q^{3m} t^m \cdot q^3/t^3}{(1-q/t)(1-q^2/t^2)} \\
&= \frac{q^m t^{3m}}{(1-q/t)(1-q^2/t^2)} + \frac{q^{3m} t^m}{(1-t/q)(1-t^2/q^2)} \\
&\quad - \frac{q^{2m} t^{2m} \cdot q/t}{(1-q/t)^2} \\
&= \sigma \left(\frac{q^{3m} t^m}{(1-t/q)(1-t^2/q^2)} \right) - \frac{q^{2m} t^{2m} \cdot q/t}{(1-q/t)^2}. \tag{17}
\end{aligned}$$

On the other hand, since $6m - u - 2v \geq v$ for all u, v appearing in the second sum in (16), we can compute $C_{W''}(q, t)$ using Lemma 3(b). First we calculate $C_{I''}(q, t)$ to be

$$\begin{aligned}
\sum_{u=0}^{m-1} \sum_{v=u}^{2m-u-1} q^{6m-u-2v} t^v &= \sum_{u=0}^{m-1} q^{6m-u} \sum_{v=u}^{2m-u-1} (t/q^2)^v \\
&= \sum_{u=0}^{m-1} q^{6m-u} \frac{(t/q^2)^u - (t/q^2)^{2m-u}}{1-t/q^2} \\
&= \frac{q^{6m}}{1-t/q^2} \sum_{u=0}^{m-1} (t/q^3)^u - \frac{q^{2m} t^{2m}}{1-t/q^2} \sum_{u=0}^{m-1} (q/t)^u \\
&= \frac{q^{6m}(1-(t/q^3)^m)}{(1-t/q^2)(1-t/q^3)} - \frac{q^{2m} t^{2m}(1-(q/t)^m)}{(1-t/q^2)(1-q/t)} \\
&= \frac{q^{6m}}{(1-t/q^2)(1-t/q^3)} - \frac{q^{2m} t^{2m}}{(1-t/q^2)(1-q/t)} \\
&\quad + \frac{q^{3m} t^m (q/t - t/q^3)}{(1-t/q^2)(1-t/q^3)(1-q/t)} \\
&= \frac{q^{6m}}{(1-t/q^2)(1-t/q^3)} - \frac{q^{2m} t^{2m}}{(1-t/q^2)(1-q/t)} \\
&\quad + \frac{q^{3m} t^m (q/t + 1/q)}{(1-t/q^3)(1-q/t)}. \tag{18}
\end{aligned}$$

Dividing by $(1-t/q)$ and applying σ , the first term here becomes

$$\sigma \left(\frac{q^{6m}}{(1-t/q)(1-t/q^2)(1-t/q^3)} \right).$$

The second term here, when combined with the second term in (17), becomes

$$\begin{aligned} & q^{2m} t^{2m} \left[\frac{-q/t}{(1-q/t)^2} - \sigma \left(\frac{1}{(1-t/q)(1-t/q^2)(1-q/t)} \right) \right] \\ &= q^{2m} t^{2m} \frac{q^2 t^2 (1-qt)}{(q-t^2)(t-q^2)(q-t)(t-q)}. \end{aligned}$$

Finally, the third term in (18), when combined with the first term in (17), becomes

$$\begin{aligned} & \sigma \left(q^{3m} t^m \left[\frac{1}{(1-t/q)(1-t^2/q^2)} + \frac{q/t + 1/q}{(1-t/q)(1-t/q^3)(1-q/t)} \right] \right) \\ &= -\sigma \left(q^{3m} t^m \frac{t/q + t/q^2 + t/q^3 + t^2/q^3}{(1-t^2/q^2)(1-t/q)(1-t/q^3)} \right). \end{aligned}$$

Adding up all the pieces, we get $C_4^{(m)}(q, t) = C_W(q, t) = AC_4^{(m)}(q, t)$.

6 Rational-slope q, t -Catalan polynomials

As mentioned in the Introduction, the combinatorial formulas for higher q, t -Catalan polynomials can be interpreted as generating functions for “ m -Dyck paths,” which are lattice paths contained within the triangle with vertices $(0, 0)$, (mn, n) , and $(0, n)$. More generally, one can also define versions of the q, t -Catalan polynomials counting lattice paths staying within other triangles. We shall focus on proving joint symmetry of rational-slope q, t -Catalan polynomials for triangles of height 4. At the end of the section, we briefly discuss Gorsky and Mazin’s proof of joint symmetry [4] for triangles of height 3.

We begin by reviewing some definitions and results from [11]. For $r, s \in \mathbb{N}^+$, define an $r \times s$ Dyck path to be a lattice path from $(0, 0)$ to (r, s) that lies above the diagonal line segment joining $(0, 0)$ to (r, s) . Let $L_{r,s}^+$ be the set of $r \times s$ Dyck paths. For a path $\pi \in L_{r,s}^+$, let $\text{area}(\pi)$ be the number of lattice squares that lie entirely above the diagonal and below π . The lattice squares above π in the triangle with vertices $(0, 0)$, $(0, s)$ and (r, s) form a partition diagram denoted $D(\pi)$. Define

$$\begin{aligned} h_{r/s}^+(\pi) &= \sum_{c \in D(\pi)} \chi \left(\frac{a(c)}{l(c) + 1} \leq r/s < \frac{a(c) + 1}{l(c)} \right); \\ h_{r/s}^-(\pi) &= \sum_{c \in D(\pi)} \chi \left(\frac{a(c)}{l(c) + 1} < r/s \leq \frac{a(c) + 1}{l(c)} \right). \end{aligned}$$

The rational q, t -Catalan number for $r \times s$ Dyck paths is defined as

$$C_{r',s',n'}(q, t) = \sum_{\pi \in L_{r,s}^+} q^{\text{area}(\pi)} t^{h_{r/s}^+(\pi)}$$

where $n' = \gcd(r, s)$, $r' = r/n'$, and $s' = s/n'$. (This indexing convention is used to match the notation in [11].) In [11], an involution $I : L_{r,s}^+ \rightarrow L_{r,s}^+$ was defined that preserves area and interchanges $h_{r/s}^+$ and $h_{r/s}^-$. It follows that we could replace $h_{r/s}^+$ by $h_{r/s}^-$ in the definition of $C_{r',s',n'}(q, t)$.

The rational q, t -Catalan number $C_{nm+1,n,1}(q, t)$ is exactly the same as $C_{m,1,n}(q, t)$ because the natural inclusion ι of $L_{nm,n}^+$ into $L_{nm+1,n}^+$ is bijective, and $h_m^+(\pi) = h_{(mn+1)/n}^+(\iota(\pi))$ for $\pi \in L_{nm,n}^+$. Furthermore, it was shown in [7, Lemma 6.3.3] that $h_m^+(\pi) = \text{div}_m(\pi)$, which implies $C_{m,1,n}(q, t) = C_n^{(m)}(q, t)$. Thus we only need to discuss the two cases of $(4m+2) \times 4$ and $(4m-1) \times 4$ Dyck paths. In the following two subsections, we identify $r \times 4$ Dyck paths π with “ $r \times 4$ Dyck words” $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ as follows: number the rows of the triangle zero to three, from bottom to top. Then for $i = 0, 1, 2, 3$, the number of cells in the i -th row that lie above π is $mi - \gamma_i$.

6.1 $(4m+2) \times 4$ Dyck paths

Assume $m \geq 1$. The analog of $W_n^{(m)}$ in Sect. 3.1 in the $(4m+2) \times 4$ case is

$$W = \{\gamma = (0, \gamma_1, \gamma_2, \gamma_3) : \gamma_1 \geq 0, \gamma_2 \geq -1, \gamma_3 \geq -1, \gamma_{i+1} \leq \gamma_i + m \text{ for } i = 0, 1, 2\}.$$

Lemma 21 For any $\pi \in L_{4m+2,4}^+$, we have $h_m^+(\pi) = h_{(4m+2)/4}^-(\pi)$.

Proof For each π , it suffices to show that for any cell $c \in D(\pi)$, the two conditions $a(c)/(l(c)+1) \leq m < (a(c)+1)/l(c)$ and $a(c)/(l(c)+1) < (4m+2)/4 \leq (a(c)+1)/l(c)$, which appeared in the definition of $h_m^+(\pi)$ and $h_{(4m+2)/4}^-(\pi)$, are equivalent. Indeed, since $0 \leq l(c) \leq 2$, the fraction $a(c)/(l(c)+1)$ is in the open interval $(m, (4m+2)/4)$ only if $a(c) = 3m+1$ and $l(c) = 2$; but the latter is impossible since $a(c) \leq 3m$. Thus $a(c)/(l(c)+1) \notin (m, (4m+2)/4)$. Similarly, $(a(c)+1)/l(c) \notin (m, (4m+2)/4)$. $\square$

Proposition 22 $C_{2m+1,2,2}(q, t) = C_{2m+1,2,2}(t, q)$.

Proof We can replace $h_{(4m+2)/4}^+$ by $h_{(4m+2)/4}^-$ using the involution at $(4m+2)/4$ defined in [11] and then replace the latter by h_m^+ , thanks to Lemma 21. Thus we need to prove

$$\sum_{\pi \in L_{4m+2,4}^+} q^{\text{area}(\pi)} t^{h_m^+(\pi)} = \sum_{\pi \in L_{4m+2,4}^+} t^{\text{area}(\pi)} q^{h_m^+(\pi)}. \quad (19)$$

The proof of (19) is similar to that for the $4m \times 4$ case (Sect. 4.3). The bijection f_0 is defined the same way as Definition 8 except that the domain A_0 consists of $\gamma \in W$ that satisfy the following condition: let $q \geq 2$ be the smallest integer such that $\gamma_q - \gamma_{q-2} \leq m$, or let $q = n (= 4)$ if there is no such integer, then $\gamma_{q-1} - 1 \leq \gamma_{n-1} + m$ and $\gamma_{q-1} \geq 0$; moreover, $\gamma_2 \neq -1$ (otherwise $f(\gamma) = (0, -1, *, *)$ is not in W). A case-by-case study of $q = 2, 3, 4$ shows

$$W \setminus A_0 = \{(0, \gamma_1, -1, \gamma_3) \in W\} \cup \{\gamma \in W : \gamma_2 > m, \gamma_3 < \gamma_2 - m - 1\},$$

and $B_0 = f_0(A_0)$ consists of $\gamma \in W$ such that one of the following holds:

- (a) $\gamma_3 \leq m - 1$, or
- (b) $\gamma_3 > m - 1$, $\gamma_3 \leq \gamma_1 + m - 1$, $\gamma_2 \leq \gamma_3 + m + 1$, or
- (c) $\gamma_2 > m$, $\gamma_3 - \gamma_1 > m - 1$, $\gamma_3 \leq \gamma_2 + m - 1$.

Further computation shows

$$W \setminus B_0 = \{\gamma \in W : \gamma_1 + m \leq \gamma_3 \leq \gamma_2 + m - 1, \gamma_2 \leq m\} \\ \cup \{\gamma \in W : \gamma_1 \leq \gamma_2, \gamma_3 = \gamma_2 + m\}.$$

Next, A_1, B_1, f_1, A, B , and f are defined by the same formula as Definition 12 (but the meaning of W is different). Lemma 13 should be revised to say:

$$W \setminus A = \{(0, \gamma_1, -1, \gamma_3) \in W\}.$$

The map $f : A \rightarrow B$ is also a bijection and changes (area, h_m^+) to $(\text{area} - 1, h_m^+ + 1)$. Lemma 16 should be revised to say:

- (1) $B_1 = \{\gamma \in W : \gamma_2 \leq m - 1, \gamma_3 - \gamma_1 \geq m\}$.
- (2) $I = W \setminus B$ is equal to the disjoint union $D_1 \cup D_2$, where

$$D_1 = \{(0, \gamma_1, \gamma_2, \gamma_2 + m) : 0 \leq \gamma_1 \leq m \leq \gamma_2 \leq \gamma_1 + m\}, \\ D_2 = \{(0, \gamma_1, m, \gamma_3) : 0 \leq \gamma_1 \leq m - 1, \gamma_1 + m \leq \gamma_3 \leq 2m - 1\}.$$

We claim that the analog of (5) still holds. Note that $W \setminus A = D'_1 \cup D'_2$, where $D'_1 = \{(0, \gamma_1, -1, \gamma_3) \in W : \gamma_1 > \gamma_3\}$, and $D'_2 = \{(0, \gamma_1, -1, \gamma_3) \in W : \gamma_1 \leq \gamma_3\}$. Moreover,

$$(\text{area}(\gamma), h_m^+(\gamma)) = \begin{cases} (\gamma_1 + \gamma_3 + 1, 6m + 1 - 3\gamma_1 - \gamma_3), & \text{if } \gamma \in D'_1; \\ (\gamma_1 + \gamma_3 + 1, 6m - \gamma_1 - 3\gamma_3), & \text{if } \gamma \in D'_2; \\ (\gamma_1 + 2\gamma_2 + 2, 2m - \gamma_2), & \text{if } \gamma \in D_1; \\ (\gamma_1 + \gamma_3 + m + 2, 3m - \gamma_3), & \text{if } \gamma \in D_2. \end{cases}$$

The analog of (13) holds where Δ is the set of lattice points that are either inside or on the boundary of the triangle with vertices $(0, 6m + 2)$, $(m, 3m + 2)$, $(2m, 2m + 2)$. Indeed, the (area, h_m^+) pairs for γ in D'_1 (resp. D'_2) form the set $\{(x, y) \in \Delta : x + y \text{ is even}\}$ (resp. $\{(x, y) \in \Delta : x + y \text{ is odd}\}$), and the (h_m^+, area) pairs for γ in D_1 (resp. D_2) form the set $\{(x, y) \in \Delta : x \leq m\}$ (resp. $\{(x, y) \in \Delta : x > m\}$). $\square$

We can express $C_{2m+1,2,2}$ in the form similar to Garsia–Haiman formula as

$$\sigma \left(q^{6m} \frac{q^8}{(q-t)(q^2-t)(q^3-t)} - q^{3m} t^m \frac{q^4 t(1+q)}{(q-t)^2(q^3-t)} \right) \\ + q^{2m} t^{2m} \frac{q^2 t^2 (q^2 t + q t^2 - q^2 - t^2)}{(q-t)^2 (q^2-t)(t^2-q)}.$$

Moreover, we have the following analog of Theorem 20 and the proof is omitted.

Theorem 23 For $j, k \in \mathbb{N}$, let $c(j, k)$ be the coefficient of $q^j t^k$ in $C_{2m+1,2,2}(q, t)$. For $a \in \mathbb{R}$, define $\lfloor a \rfloor^+ = \max(\lfloor a \rfloor, 0)$. Then for $4m + 2 \leq j + k \leq 6m + 2$,

$$c(j, k) = \min\left(\left\lfloor \frac{-6m + 3j + k}{2} \right\rfloor^+, \left\lfloor \frac{6m + 4 - j - k}{2} \right\rfloor^+, \left\lfloor \frac{-6m + j + 3k}{2} \right\rfloor^+\right).$$

Otherwise $c(j, k) = 0$. As a consequence, the sequence $(c(d, 0), c(d - 1, 1), c(d - 2, 2), \dots, c(1, d - 1), c(0, d))$ is unimodal for every positive integer d .

6.2 $(4m - 1) \times 4$ Dyck paths

Assume $m \geq 1$. The analog of $W_n^{(m)}$ in Sect. 3.1 in the $(4m - 1) \times 4$ case is

$$W = \{\gamma = (0, \gamma_1, \gamma_2, \gamma_3) : \gamma_1, \gamma_2, \gamma_3 \geq 1, \gamma_{i+1} \leq \gamma_i + m \text{ for } i = 0, 1, 2\}.$$

Lemma 24 For any $\pi \in L_{4m-1,4}^+$, $h_m^-(\pi) = h_{(4m-1)/4}^+(\pi)$.

Proof Similar to the proof of Lemma 21. Use the fact that there is no fraction in the open interval $((4m - 1)/4, m)$ with denominator at most 3. $\square$

We need the following property of the involution $I_{4m/4}$ defined in [11]: the restriction of $I_{4m/4}$ to $L_{4m-1,4}^+$, denoted by I , is an involution since it does not change the number of arrows heading to the vertex 0 in the multigraph. This involution I exchanges h_m^+ with h_m^- and keeps area unchanged.

Proposition 25 $C_{4m-1,4,1}(q, t) = C_{4m-1,4,1}(t, q)$.

Proof Since $h_{(4m-1)/4}^+(\pi) = h_m^-(\pi) = h_m^+(I(\pi))$ and $\text{area}(\pi) = \text{area}(I(\pi))$, Proposition 25 is equivalent to the following after the substitution of π by $I(\pi)$:

$$\sum_{\pi \in L_{4m-1,4}^+} q^{\text{area}(\pi)} t^{h_m^+(\pi)} = \sum_{\pi \in L_{4m-1,4}^+} t^{\text{area}(\pi)} q^{h_m^+(\pi)}. \quad (20)$$

The proof of (20) is again similar to the $4m \times 4$ case (Sect. 4.3). The bijection f_0 is defined the same way as in Definition 8 except that the domain A_0 is the subset of W consisting of $\gamma = (\gamma_0, \dots, \gamma_{n-1})$ (where $n = 4$) that satisfies the following condition: let $q \geq 2$ be the smallest integer such that $\gamma_q - \gamma_{q-2} \leq m$, or let $q = n$ if there is no such integer, then $\gamma_{q-1} - 1 \leq \gamma_{n-1} + m$ and $\gamma_{q-1} > 1$. Then

$$W \setminus A_0 = \{(0, 1, \gamma_2, \gamma_3) \in W : \gamma_2 \leq m\} \cup \{\gamma \in W : \gamma_2 - 1 > \gamma_3 + m\}.$$

Next, A_1, B_1, f_1, A, B , and f are defined by the same formula as Definition 12 (although the meaning of W is different). Lemma 13 should be revised to say:

$$W \setminus A = \{(0, 1, \gamma_2, \gamma_3) \in W : \gamma_2 \leq m\}.$$

The map $f : A \rightarrow B$ is a bijection and changes (area, h_m^+) to $(\text{area} - 1, h_m^+ + 1)$. Lemma 16 should be revised to say:

- (1) $B_1 = \{\gamma \in W : \gamma_1 \geq 2, \gamma_2 \leq m-1, \gamma_3 - \gamma_1 \geq m\}$.
- (2) $W \setminus B$ is equal to the disjoint union $D_1 \cup D_2 \cup D_3$, where

$$\begin{aligned} D_1 &= \{(0, \gamma_1, \gamma_2, \gamma_2 + m) : 2 \leq \gamma_1 \leq m \leq \gamma_2 \leq \gamma_1 + m\} \\ &\quad \cup \{(0, 1, m+1, 2m+1)\}, \\ D_2 &= \{(0, \gamma_1, m, \gamma_3) : \gamma_1 \geq 2, \gamma_1 + m \leq \gamma_3 \leq 2m-1\}, \\ D_3 &= \{(0, 1, \gamma_2, \gamma_3) : 1 \leq \gamma_2 \leq m, 1+m \leq \gamma_3 \leq \gamma_2 + m\}. \end{aligned}$$

We claim that the analog of (5) still holds. Indeed, $W \setminus A = D'_1 \cup D'_2 \cup D'_3$, where $D'_1 = \{(0, 1, \gamma_2, \gamma_3) \in W : \gamma_3 < \gamma_2 \leq m\}$, $D'_2 = \{(0, 1, \gamma_2, \gamma_3) \in W : \gamma_2 \leq \gamma_3 \leq m\}$, $D'_3 = \{(0, 1, \gamma_2, \gamma_3) \in W : \gamma_2 \leq m, \gamma_3 > m\}$. Moreover,

$$(\text{area}(\gamma), h_m^+(\gamma)) = \begin{cases} (\gamma_2 + \gamma_3 - 2, 6m + 2 - 3\gamma_2 - \gamma_3), & \text{if } \gamma \in D'_1; \\ (\gamma_2 + \gamma_3 - 2, 6m + 1 - \gamma_2 - 3\gamma_3), & \text{if } \gamma \in D'_2; \\ (\gamma_2 + \gamma_3 - 2, 4m - \gamma_2 - \gamma_3), & \text{if } \gamma \in D'_3; \\ (\gamma_1 + 2\gamma_2 + m - 3, 2m - \gamma_2), & \text{if } \gamma \in D_1; \\ (\gamma_1 + \gamma_3 + m - 3, 3m - \gamma_3), & \text{if } \gamma \in D_2; \\ (\gamma_2 + \gamma_3 - 2, 4m - \gamma_2 - \gamma_3), & \text{if } \gamma \in D_3. \end{cases}$$

The analog of (13) holds where Δ is the set of lattice points that are either inside or on the boundary of the triangle with vertices $(0, 6m-3)$, $(m-1, 3m)$, $(2m-2, 2m+1)$. The symmetry follows from the observation that the (area, h_m^+) pairs for γ in D'_1 (resp. D'_2) form the set $\{(x, y) \in \Delta : x+y \text{ is even}\}$ (resp. $\{(x, y) \in \Delta : x+y \text{ is odd}\}$), while the (h_m^+, area) pairs for γ in D_1 (resp. D_2) form the set $\{(x, y) \in \Delta : x \leq m\}$ (resp. $\{(x, y) \in \Delta : x > m\}$).

The analog of (9) can be proved by the one-to-one correspondence from D'_3 to D_3 that sends (γ_2, γ_3) to $(2m+1-\gamma_3, 2m+1-\gamma_2)$. $\square$

Example 4 Consider the $(4m-1) \times 4$ case for $m=2$. There are thirty objects in $W_{4m-1,4}$, which produce f -chains of nonzero lengths listed in Fig. 6 as well as three chains of length zero: $124(q^4t^2)$, $113(q^2t^4)$, $123(q^3t^3)$. These are subchains of the chains in Example 3. Note that $\gamma_1, \gamma_2, \gamma_3 > 0$.

Moreover, $D_1 = \{246, 235, 224, 135\}$, $D_2 = \emptyset$, $D_3 = \{124, 113, 123\}$, $D'_1 = \{121\}$, $D'_2 = \{111, 112, 122\}$, $D'_3 = \{124, 113, 123\}$. The set of initial points I has generating function

$$C_I(q, t) = q^9t^0 + q^7t^1 + q^5t^2 + q^6t^1 + q^4t^2 + q^2t^4 + q^3t^3.$$

The set of terminal points T has generating function

$$C_T(q, t) = q^0t^9 + q^1t^7 + q^1t^6 + q^2t^5 + q^4t^2 + q^2t^4 + q^3t^3.$$

Since $C_T(q, t) = C_I(t, q)$, we get $C_W(q, t) = C_W(t, q)$.

Fig. 6 The f -chains of nonzero lengths in Example 4

Remark 26 For $n \leq 3$, the only rational q, t -Catalan polynomials for $r \times n$ Dyck paths that are not equal to some $C_n^{(m)}(q, t)$ occur in the case when $r = 3m - 1$ and $n = 3$. This case is similar to Sect. 6.2: $W = \{(0, \gamma_1, \gamma_2) : 1 \leq \gamma_1 \leq m, 1 \leq \gamma_2 \leq \gamma_1 + m\}$, $W \setminus A_0 = D'_1 \cup D'_2$ where $D'_1 = \{(0, 1, \gamma_2) : 1 \leq \gamma_2 \leq m\}$ and $D'_2 = \{(0, 1, m + 1)\}$; $W \setminus B_0 = D_1 \cup D_2$ where $D_1 = \{(0, \gamma_1, \gamma_1 + m) : 1 \leq \gamma_1 \leq m\}$ and $D_2 = \{(0, 1, m)\}$, and

$$(\text{area}(\gamma), h_m^+(\gamma)) = \begin{cases} (\gamma_2 - 1, 3m - 2\gamma_2), & \text{if } \gamma \in D'_1; \\ (m, m - 1), & \text{if } \gamma \in D'_2; \\ (2\gamma_1 + m - 2, m - \gamma_1), & \text{if } \gamma \in D_1; \\ (m - 1, m), & \text{if } \gamma \in D_2. \end{cases}$$

Thus

$$\begin{aligned} \sum_{\gamma \in D'_1} q^{\text{area}(\gamma)} t^{h_m^+(\gamma)} &= q^0 t^{3m-2} + q^1 t^{3m-4} + q^2 t^{3m-6} + \dots + q^{m-1} t^m \\ &= \sum_{\gamma \in D_1} t^{\text{area}(\gamma)} q^{h_m^+(\gamma)}, \\ \sum_{\gamma \in D'_2} q^{\text{area}(\gamma)} t^{h_m^+(\gamma)} &= q^m t^{m-1} = \sum_{\gamma \in D_2} t^{\text{area}(\gamma)} q^{h_m^+(\gamma)}, \end{aligned}$$

and therefore $\sum_{\gamma \in W \setminus A_0} q^{\text{area}(\gamma)} t^{h_m^+(\gamma)} = \sum_{\gamma \in W \setminus B_0} t^{\text{area}(\gamma)} q^{h_m^+(\gamma)}$. By a similar argument as in Proposition 25, this leads to a proof of the joint symmetry of the rational q, t -Catalan polynomial $C_{3m-1,3,1}(q, t)$.

We can express $C_{4m-1,4,1}$ in the form similar to Garsia–Haiman formula as

$$\sigma \left(q^{6m} \frac{q^3}{(q-t)(q^2-t)(q^3-t)} - q^{3m} t^m \frac{q(t+qt+q^2+q^2t)}{(q-t)(q^2-t^2)(q^3-t)} \right) \\ + q^{2m} t^{2m} \frac{qt(qt-1)}{(q-t)^2(q^2-t)(t^2-q)}.$$

Moreover, we have the following analog of Theorem 20 and the proof is omitted.

Theorem 27 For $j, k \in \mathbb{N}$, let $c(j, k)$ be the coefficient of $q^j t^k$ in $C_{4m-1,4,1}(q, t)$. For $a \in \mathbb{R}$, define $\lfloor a \rfloor^+ = \max(\lfloor a \rfloor, 0)$.

(a) If $4m - 1 \leq j + k \leq 6m - 3$, then

$$c(j, k) = \min \left(\left\lfloor \frac{-6m + 5 + 3j + k}{2} \right\rfloor^+, \left\lfloor \frac{6m - 1 - j - k}{2} \right\rfloor^+, \left\lfloor \frac{-6m + 5 + j + 3k}{2} \right\rfloor^+ \right).$$

(b) If $j + k = 4m - 2$, then

$$c(j, k) = \min \left(\left\lfloor \frac{-m + 2 + j}{2} \right\rfloor^+, \left\lfloor \frac{-m + 2 + k}{2} \right\rfloor^+ \right).$$

(c) Otherwise $c(j, k) = 0$.

As a consequence, the sequence $(c(d, 0), c(d - 1, 1), c(d - 2, 2), \dots, c(1, d - 1), c(0, d))$ is unimodal for every positive integer d .

6.3 Gorsky and Mazin's approach

The reader may find it helpful to compare our method with the following combinatorial formulation of Gorsky and Mazin's approach in [4]. Let $r > 3$ be an integer with $\gcd(r, 3) = 1$, and set $k = \lfloor r/3 \rfloor$. Define

$$X = \{(c, d) \in \mathbb{N}^2 : 0 \leq c \leq d, c \leq k, d \leq \lfloor 2r/3 \rfloor\},$$

$$Y = \{(a, b) \in \mathbb{N}^2 : a + 3b \leq m - 1\}.$$

One sees that X is the disjoint union $X_1 \cup X_2 \cup X_3$, and Y is the disjoint union $Y_1 \cup Y_2 \cup Y_3$, where

$$X_1 = \{(c, d) \in X : d \leq k\};$$

$$X_2 = \{(c, d) \in X : d > k \text{ and } d - c \leq k\};$$

$$X_3 = \{(c, d) \in X : d - c > k\};$$

$$Y_1 = \{(a, b) \in Y : a + b \leq k\};$$

$$Y_2 = \{(a, b) \in Y : a + b > k \text{ and } a + b + k \text{ is even}\};$$

$$Y_3 = \{(a, b) \in Y : a + b > k \text{ and } a + b + k \text{ is odd}\}.$$

if $\gamma_2 \leq m$ then $f(\gamma) = (\gamma_0, \gamma_2, \gamma_3, \gamma_4, \gamma_1 - 1)$;
 if $\gamma_2 > m$ then
 if $\gamma_4 \geq m$ then
 if $\gamma_3 - \gamma_1 \leq m$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_2 - 1)$;
 if $\gamma_3 - \gamma_1 > m$ then
 if $\gamma_4 - \gamma_2 \leq m$ then
 if $\gamma_3 - \gamma_4 \leq m + 1$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_3 - 1)$;
 if $\gamma_3 - \gamma_4 > m + 1$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_4 + 1, \gamma_2 - 1, \gamma_3 - 1)$;
 if $\gamma_4 - \gamma_2 > m$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4 - 1)$;
 if $\gamma_4 < m$ then
 if $\gamma_3 - \gamma_1 \leq m$ then
 if $\gamma_2 - \gamma_4 \leq m + 1$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_3, \gamma_4, \gamma_2 - 1)$;
 if $\gamma_2 - \gamma_4 > m + 1$ then
 if $\gamma_3 - \gamma_4 \leq m + 1$ then
 if $\gamma_2 - \gamma_3 > m$ or $\gamma_3 \leq m$ then $f(\gamma) = (\gamma_0, \gamma_4 + 1, \gamma_3, \gamma_1 - 1, \gamma_2 - 1)$;
 if $\gamma_2 - \gamma_3 \leq m$ and $\gamma_3 > m$ then $f(\gamma) = (\gamma_0, \gamma_4 + 1, \gamma_1, \gamma_3 - 1, \gamma_2 - 1)$;
 if $\gamma_3 - \gamma_4 > m + 1$ then $f(\gamma) = (\gamma_0, \gamma_4 + 1, \gamma_1, \gamma_2 - 1, \gamma_3 - 1)$;
 if $\gamma_3 - \gamma_1 > m$ then
 if $\gamma_2 - \gamma_4 \leq m + 1$ then
 if $\gamma_3 - \gamma_4 \leq m + 1$ then $f(\gamma) = (\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_3 - 1)$;
 if $\gamma_3 - \gamma_4 > m + 1$ then $f(\gamma) = (\gamma_0, \gamma_4 + 1, \gamma_1, \gamma_2 - 1, \gamma_3 - 1)$
 if $\gamma_2 - \gamma_4 > m + 1$ then
 if $\gamma_3 - \gamma_4 > m + 2$ and $\gamma_2 - \gamma_4 > m + 2$ then
 $f(\gamma) = (\gamma_0, \gamma_4 + 2, \gamma_1 - 1, \gamma_2 - 1, \gamma_3 - 1)$;
 if $\gamma_3 - \gamma_4 \leq m + 2$ or $\gamma_2 - \gamma_4 = m + 2$ then
 $f(\gamma) = (\gamma_0, \gamma_1, \gamma_4 + 1, \gamma_2 - 1, \gamma_3 - 1)$.

Fig. 7 The conjectured chain map for $n = 5$

For $(c, d) \in X$, define $\text{area}(c, d) = m - 1 - (c + d)$ and $h^+(c, d)$ = the number of cells x in the partition diagram with $3a(x) - ml(x) \in \{-2, -1, 0, \dots, m\}$. For $(a, b) \in Y$, define $\text{wt}_1(a, b) = m - 1 - (a + 2b)$ and $\text{wt}_2(a, b) = a + b$. Define $f : X \rightarrow Y$ by

$$f(c, d) = \begin{cases} (d - c, c), & \text{if } (c, d) \in X_1; \\ (3d - 2k - c, c - d + k), & \text{if } (c, d) \in X_2; \\ (3c - d + 2k + 2, d - c - k - 1), & \text{if } (c, d) \in X_3. \end{cases}$$

Then f is a bijection that sends area to wt_1 and h^+ to wt_2 . Define $I : Y \rightarrow Y$ via $I(a, b) = (m - 1 - a - 3b, b)$. Then I is an involution that interchanges wt_1 and wt_2 . Finally, let $g = f^{-1} \circ I \circ f$ be an involution on X . Then g interchanges area and h^+ . This implies joint symmetry of $C_{r,3,1}(q, t)$.

7 The joint symmetry of $C_5^{(m)}(q, t)$

For $n = 5$, we define a conjectural chain map f with domain $W \setminus T$ in Fig. 7. A routine but lengthy case-by-case study shows that f decreases area by 1 and increases dinv_m by 1, and f is one-to-one. Our main obstacle to proving Conjecture 7 for $n = 5$ is that we do not know how to describe the set I of initial objects of the f -chains, which is the complement of the image of f in W . We leave it as an open problem to characterize I and the image of f explicitly, and to prove $C_T(q, t) = C_I(t, q)$. We wrote Macaulay 2 code verifying that $C_T(q, t) = C_I(t, q)$ for $m \leq 10$. So, the chain conjecture 7 and joint symmetry holds for these m when $n = 5$.

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