Coxeter groups, Coxeter monoids and the Bruhat order

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Abstract Associated with any Coxeter group is a Coxeter monoid, which has the same elements, and the same identity, but a different multiplication. (Some authors call these Coxeter monoids 0-Hecke monoids, because of their relation to the 0-Hecke algebras—the $q = 0$ case of the Hecke algebra of a Coxeter group.) A Coxeter group is defined as a group having a particular presentation, but a pair of isomorphic groups could be obtained via non-isomorphic presentations of this form. We show that when we have both the group and the monoid structure, we can reconstruct the presentation uniquely up to isomorphism and present a characterisation of those finite group and monoid structures that occur as a Coxeter group and its corresponding Coxeter monoid. The Coxeter monoid structure is related to this Bruhat order. More precisely, multiplication in the Coxeter monoid corresponds to element-wise multiplication of principal downsets in the Bruhat order. Using this property and our characterisation of Coxeter groups among structures with a group and monoid operation, we derive a classification of Coxeter groups among all groups admitting a partial order.

Keywords Coxeter Group · Coxeter monoid · 0-Hecke monoid · Bruhat order

1 Introduction

Recall that a Coxeter system (W, S) is a group W with a finite set S of generators, and a presentation of the form

$$W = \langle s \in S \mid s^2 = 1, (st)^{m_{s,t}} = 1 \rangle$$

where for any $s \neq t \in S$ we have $m_{s,t} = m_{t,s}$, is either an integer greater than or equal to 2, or ∞ (in the latter case, we would remove the relation $(st)^\infty = 1$ from the

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set of relations). We represent the Coxeter system (W, S) by a square matrix, called the Coxeter matrix M , with dimension the number of elements of S , whose (s, t) th entry is $m_{s,t}$ for $s \neq t$, and whose diagonal entries are all 1. A Coxeter group W is a group for which there is a set S of generators such that (W, S) is a Coxeter system. The basic properties of Coxeter groups can be found in one of the many books about Coxeter groups, for example, [1, 11], etc.

Usually, for a Coxeter group W , there can be more than one set S such that (W, S) is a Coxeter system. The simplest example of this is the dihedral group of order 12, which is both the group of isometries of a regular hexagon, and the group of isometries of an equilateral triangular prism. That is, the following two Coxeter diagrams produce isomorphic Coxeter groups:

$$I_6 = \begin{array}{c} 6 \\ \cdot \text{---} \cdot \end{array} \quad I_3 \times A_1 = \begin{array}{c} 3 \\ \cdot \text{---} \cdot \end{array} \quad .$$

Another way to state the relation $(s_i s_j)^{m_{ij}} = 1$ is in the form

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}}$$

Using this alternative way, S.V. Tsaranov [18] defines a Coxeter monoid for the Coxeter matrix M , as the monoid

$$\left\langle s_i \in S \mid s_i^2 = s_i, \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}} \right\rangle$$

In fact, Tsaranov defines a Coxeter monoid for a slightly more general class of *generalised Coxeter matrices*, where symmetry does not need to hold exactly, but for this paper we will be interested in the connection between the Coxeter group and the Coxeter monoid for the same Coxeter matrix, so we will only be interested in Coxeter monoids for Coxeter matrices. For such monoids, Tsaranov observes that

Theorem 1 ([18], Theorem 1) *There is a bijection between the Coxeter group and the Coxeter monoid for a given Coxeter matrix.*

More precisely, the reduced words for the Coxeter group and Coxeter monoid are the same, and two reduced words represent the same element of the Coxeter group if and only if they represent the same element of the Coxeter monoid. The special case of this bijection for the symmetric group was important for studying certain relations on planar graphs [12]. Coxeter Monoids have also been studied by J. Dolan and T. Trimble [4] as an alternative way to describe Tits buildings.

Coxeter Monoids are also referred to as *0-Hecke monoids* in the literature, because of the connection with the 0-Hecke algebras. Recall that the (Iwahori-)Hecke algebra of a Coxeter group (W, S) over a field $\mathbb{F}$ for an element $q \in \mathbb{F}$ is the associative algebra $\mathcal{H}_q(W)$ over $\mathbb{F}$ with generators $S_1, \dots, S_n$ corresponding to the generating

set S of W , subject to the relations:

$$S_i^2 = q + (q - 1)S_i$$

$$\underbrace{S_i S_j S_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{S_j S_i S_j \cdots}_{m_{ij} \text{ terms}}$$

where the m_{ij} are the values in the Coxeter matrix of (W, S) . In the particular case, $q = 1$, this is just the group algebra of the group W . The $q \neq 0$ case has been extensively studied, and its representation theory is well understood. However, this theory does not apply when $q = 0$. The $q = 0$ case is called the 0-Hecke algebra, and was first studied by P. Norton [16], and has been further studied by others—see for example [2, 5, 8, 15, 17].

If we instead take the generators, $-S_1, \dots, -S_n$ of the 0-Hecke algebra, we see that, under multiplication, they generate the Coxeter monoid $(W, *)$, so the 0-Hecke algebra is the monoid algebra associated with this monoid. This 0-Hecke monoid is explicitly studied, for example, in [3, 6, 7, 9, 10].

Another important aspect of a Coxeter group is the Bruhat order, which can be defined in a number of ways. The easiest way is just that $a \leq b$ in the Bruhat order if and only if there is a reduced word for a , which is a subword of a reduced word for b . The important properties of the Bruhat order can be found, for example, in [1]. There is a close connection between the multiplication of the Coxeter monoid and the Bruhat order on the Coxeter Group.

Recall the example of two different Coxeter systems that both generate the group of isometries of the regular hexagon. The corresponding Bruhat orders have the following Hasse diagrams:

This also implies that the two Coxeter systems give different Coxeter monoids (see Proposition 6). It is easy to see that the Coxeter group together with either the monoid structure or the Bruhat order entirely determines the Coxeter system. Indeed, either the Bruhat order or the monoid structure uniquely determines the Coxeter system by itself.

In this paper, we will be interested in the Coxeter group as a structure which has both the group and monoid multiplication. That is, we will be considering the struc-

ture

$$(W, *, \cdot, e, (_)^{-1})$$

where W is the set of elements of the Coxeter group or monoid (under the standard bijection between them), $*$ is the Coxeter monoid multiplication, $\cdot$ is the Coxeter group multiplication, e is the identity element (for both the Coxeter group and the Coxeter monoid) and $(_)^{-1}$ is the inverse operation in the Coxeter group.

We provide a characterisation of the finite structures of this form that come from a Coxeter group and the corresponding Coxeter monoid.

We will also consider the Coxeter group as a group with a partial order and characterise the groups with partial orders that occur as Coxeter groups with the Bruhat order.

2 Connection between monoid multiplication and weak and Bruhat orders

In this section, we will discuss how the weak and Bruhat orders relate to the monoid multiplication in the Coxeter monoid in the case when the Coxeter group is finite. We will assume that W is a Coxeter group, with multiplication $\cdot$ (sometimes denoted by concatenation), monoid multiplication $*$, and identity element e .

It will be important to consider when the group and monoid multiplications agree, i.e. when $a * b = a \cdot b$. We will denote this situation by $a \perp b$, and denote the common value by $a \times b$, so the equation $c = a \times b$ is shorthand for $c = a \cdot b = a * b$. We will use $l(x)$ to denote the length of a reduced word for x . Note that $a \perp b$ is equivalent to $l(ab) = l(a) + l(b)$.

The weak order is clearly just the specialisation order of the monoid, namely $a \leq b$ if and only if there exists a c such that $b = a * c$.

For the Bruhat order, we can describe the Bruhat order using the group and monoid operations. Conversely, we can describe the monoid multiplication using the Bruhat order and the group multiplication. We will need the fact that the monoid multiplication preserves the Bruhat order.

Lemma 2 *If $a' \leq a$ and $b' \leq b$ then $a' * b' \leq a * b$ in the Bruhat order.*

Proof By induction on $l(b)$. Clearly, the result holds when $l(b) = 0$. Suppose $b = s \times t$, where $s \in S$. There are two possibilities: $b' = s \cdot t'$ for some $t' \leq t$, or $b' \leq t$. If $b' \leq t$, then since $a' \leq a * s$, we are done. If $b' = s * t'$, for $t' \leq t$, then we know that $a' * b' = a' * s * t'$, and we have that $a' * s \leq a * s$ and $t' \leq t$. Thus, by our induction hypothesis, we are done. $\square$

Using this we describe the Bruhat order in terms of the monoid and group multiplications.

Remark 3 Given two reduced words $a = s_1 \cdots s_n$ and $b = r_1 \cdots r_m$, we can find subwords $s_{i_1} \cdots s_{i_k}$ and $r_{j_1} \cdots r_{j_l}$, with $a' = i_1 < i_2 < \cdots < i_k$ and $b' = j_1 < j_2 < \cdots < j_l$, such that $a * b = a \times b' = a' \times b$.

Lemma 4 For any x and t , if a is a minimal solution to $a * x = t$ in the Bruhat ordering, then $a \perp x$.

Proof We know that $a * x = a' \times x$ for some $a' \leq a$. Therefore, by minimality, we have $a = a'$. Therefore $a * x = a \times x$, i.e. $a \perp x$. $\square$

Lemma 5 For any $x \leq y$ in the Bruhat order, if a is a Bruhat-minimal solution to $a * x = a * y$, then we must have $a \perp x$.

Proof Let $t = a * x$. Suppose that we have $b < a$ in the Bruhat order, such that $b * x = t$. Then since $*$ preserves the Bruhat order, we get $t = b * x \leq b * y$. On the other hand, we have $b * y \leq a * y = t$. Therefore, we get $b * y = t$, so that b is also a solution to $b * x = b * y$. This contradicts minimality of a . Therefore, a must also be a minimal solution to $a * x = t$, and therefore by Lemma 4, we have $a \perp x$. $\square$

Proposition 6 For a finite Coxeter group $(W, \cdot, e, (_)^{-1})$ from a Coxeter matrix M , with corresponding monoid operation $*$, we have $x \leq y$ in the Bruhat order if and only if there is an $a \in W$ such that $x = a^{-1} \cdot (a * y)$.

Proof By Remark 3, we have that $a^{-1} \cdot (a * y) = a^{-1} \cdot (a \times y') = y'$ for some $y' \leq y$ in the Bruhat order.

Conversely, suppose $x \leq y$ in the Bruhat order. If we take the Bruhat-maximal element M , then we know that $M * x = M * y = M$, so the equation $a * x = a * y$ has a solution. Since the Coxeter group is finite, there must therefore be a minimal solution. We let a be this minimal solution, then by Lemma 5, we have that $a \perp x$, that is $a \cdot x = a * x = a * y$. Therefore, $x = a^{-1} \cdot (a * y)$. $\square$

Remark 7 Note that the first half of the proof of Proposition 6 does not require the assumption that W is finite. That is, the inequality $a^{-1} \cdot (a * b) \leq b$ holds for all Coxeter groups.

To describe the monoid multiplication in terms of the Bruhat order and the group multiplication, we first identify elements of the Coxeter group with principal downsets in the Bruhat order, that is, sets of the form $\{x \mid x \leq a\}$ for some a . It is clear that this is a bijection.

We can multiply subsets of a group by multiplying element-wise, i.e. if A and B are subsets of the group G (not necessarily subgroups), then $AB = \{ab \mid a \in A, b \in B\}$.

Proposition 8 The product of the principal Bruhat downsets $\downarrow(a) = \{x \mid x \leq a\}$ and $\downarrow(b) = \{x \mid x \leq b\}$ is the principal Bruhat downset $\downarrow(a * b)$.

Proof If $a' \leq a$ and $b' \leq b$, we want to show that $a'b' \leq a * b$. However, by the definition of $*$, and the deletion principle (that for $s \in S$, if $t_1 \cdots t_n$ is a reduced word, then either $st_1 \cdots t_n$ is a reduced word or $st_1 \cdots t_n = t_1 \cdots t_{i-1}t_{i+1} \cdots t_n$ for some i), it is easy to see that $xy \leq x * y$. Therefore, $a'b' \leq a' * b' \leq a * b$.

Conversely, we can find $b' \leq b$ such that $a.b' = a * b$. This is because $b' = a^{-1}.(a * b)$ is less than or equal to b by Proposition 6 (this is even true in the infinite case—see Remark 7. Now, $a \times b'$ is a reduced word for $a * b$ so anything below $a * b$ corresponds to a subword of a reduced word for a followed by a reduced word for b' . Therefore we get $\downarrow(a * b) \subseteq \downarrow(a) \downarrow(b') \subseteq \downarrow(a) \downarrow(b)$. $\square$

The description of the Coxeter monoid as these subsets of the Coxeter group is one of the motivations for defining the Coxeter monoid in [18], but Tsaranov does not identify these subsets of a Coxeter group as the principal downsets in the Bruhat order. This observation has been made by several people, for example, S. Margolis and B. Steinberg in [14], and a proof is given for the case of the symmetric group in V. Mazorchuk and B. Steinberg [13].

In Sect. 4 we will see that the remarkable property that the product of two principal downsets is another principal downset, is a key component in the characterisation of Coxeter groups with the Bruhat order among all groups with a partial order.

3 Finite Coxeter groups and monoids

In this section, we show how the group and monoid structure, when taken together, can be used to reconstruct the original Coxeter system, and give a characterisation of which finite group and monoid structures can arise from a Coxeter system. We also explain how the characterisation extends to the infinite case.

Recall from Proposition 6 that we can reconstruct the Bruhat order from the group and monoid structure by $x \leq y$ if and only if there is an a such that $x = a^{-1}.(a * y)$. When dealing with abstract group and monoid structures, we will write $x < y$ to mean $(\exists a)(x = a^{-1}.(a * y))$.

Theorem 9 *A finite structure $(W, \cdot, (_)^{-1}, *, e)$ comes from a Coxeter system (W, S) if and only if the following conditions hold:*

1. $(W, \cdot, (_)^{-1}, e)$ is a group.
2. $(W, *, e)$ is a monoid.
3. If $a * b * c = a$ then $a * b = a$.
4. $a * (a^{-1}.(a * b)) = a * b$.
5. If $x * x = x$ then $x = x^{-1}$.
6. If $x < y$ then one of the three following cases applies: $x = e$, $x = y$, or $x = r \times s$, $y = t \times u$, where t and u are not the identity, and $r < t$ and $s < u$.
7. $(a * b)^{-1} = b^{-1} * a^{-1}$.

Remark 10 Note that dual conditions to Conditions (3) and (4) can be obtained using these conditions and Condition (7). That is, we can obtain the condition that if $c * b * a = a$, then $b * a = a$, and the condition $((a * b).b^{-1}) * b = a * b$.

Theorem 11 *If (W, S) is a Coxeter group, with group operation $\cdot$ (sometimes indicated by just concatenation) and corresponding monoid operation $*$, then it satisfies all of the conditions of Theorem 9.*

Proof Conditions (1) and (2) are well-known. Condition (3) is obvious from the definition of $*$. Condition (7) follows because the inverse of an element x in the Coxeter group corresponding to a word $s_1 \cdots s_n$ is just given by the word in reverse order, i.e. $s_n \cdots s_1$. It is easy to see that with the order reversed, the same set of elements could be eliminated by the $*$ operation. To prove Condition (4), we can pick a reduced word for $a * b$ that begins with a reduced word for a . This is obviously possible from the definition of $*$, using induction on $l(b)$. Now, the end of this reduced word must just be $a^{-1} \cdot (a * b)$, so by definition of $*$, we have Condition (4). Condition (6) is immediate from Proposition 6. Condition (5) is because an idempotent element x of the Coxeter monoid satisfies $xs \leq x$ for all $s \leq x$ in the Bruhat order. An idempotent x must therefore be the maximal element of a finite parabolic subgroup, and therefore self-inverse. $\square$

We now aim to prove the converse, namely that these conditions allow us to construct a (unique) Coxeter structure, such that $\cdot$ is multiplication in the Coxeter group and $*$ is multiplication in the Coxeter monoid. For the rest of this section, all results will apply to a finite structure $(W, \cdot, *, e, (_)^{-1})$ satisfying Conditions (1–7). We begin by identifying the set S of Coxeter generators, and showing that they do satisfy the basic properties that we expect. We can identify the Coxeter generators S as the $*$ -irreducible elements of W , i.e. the elements x such that whenever we have $x = a * b$, we get $a = x$ or $b = x$. It is clear that these generate W as a monoid, because W is finite. We want to show that these satisfy the equations for a Coxeter group, and no other equations. We begin with two easy properties of $*$ -irreducible elements.

Proposition 12 *If x is $*$ -irreducible, then it is idempotent (i.e. $x * x = x$).*

Proof If x is $*$ -irreducible, then so is x^{-1} , by Condition (7). Now, by Condition (6), if we define $y = x^{-1} \cdot (x * x^{-1})$, then one of the following cases applies: $y = e$, i.e. $x * x^{-1} = x$, $y = x^{-1}$, i.e. $x * x^{-1} = e$, or $x^{-1} = s \times t$ and $y = u \times v$ with $u < s$ and $v < t$. The third case is impossible, as x^{-1} is $*$ -irreducible. If $x * x^{-1} = e$, then $x * x^{-1} * x = x$, so $x * x^{-1} = x$, by Condition (3), giving $x = e$, so that x is indeed idempotent. Finally, if $x * x^{-1} = x$, we have $x^{-1} = (x * x^{-1})^{-1} = x * x^{-1} = x$, so $x * x = x * x^{-1} = x$. $\square$

Lemma 13 *For $a \in W$ and $s \in S$, $a * s = a \cdot s$ or $a * s = a$.*

Proof In Condition (6), because s is $*$ -irreducible, the third case does not apply, so we have that either $a^{-1} \cdot (a * s) = e$ or $a^{-1} \cdot (a * s) = s$. These give $a * s = a$ or $a * s = a \cdot s$ respectively. $\square$

In the case where W comes from a Coxeter group, we can identify the reduced words easily. A word $x_1 \cdots x_k$ is reduced if for each multiplication in it, $\cdot$ and $*$ both give the same answer. That is, if for any i , we have $x_1 \cdots x_i = (x_1 \cdots x_{i-1}) * x_i$. [Note that this characterisation also works when the x_i are not generators, as long as we choose a reduced word for each x_i .] As in the previous section, we will denote

such a product by $\times$. For our structure, we will define a reduced word to be a word of the form $s_1 \times \cdots \times s_n$, where the s_i are $*$ -irreducible elements. The length of the word will be the number of $*$ -irreducible elements in the product. Since it is clear that elements of S generate W as a monoid, inductively applying Lemma 13 gives that any element of W can also be expressed as some reduced word. We will need the following property of reduced words.

Lemma 14 *If $x = a \times b$, and w_a is a reduced word for a and w_b is a reduced word for b , then $w_a w_b$ is a reduced word for x .*

Proof This is obvious by associativity of $\cdot$ and $*$. $\square$

Since W is finite, for any x there must be some $m \neq n$ such that $x^{*m} = x^{*n}$. W.l.o.g. assume $m < n$. Then clearly, $x^{*m} = x^{*k}$ for any $k > m$, by Condition (3). (This will be the smallest idempotent above x in the Bruhat order.) We will denote it $\bar{x}$. (Note that x^{*m} may not be a reduced word in the Coxeter monoid.) For a finite Coxeter group, $\bar{x}$ is the largest element of the parabolic subgroup generated by all elements below x in the Bruhat order.

We need to show that any two reduced words for an element x of W are equal using identities that involve only two of the generators. We will do this by induction. We will first need to show that if $a \times s = b \times r = x$ are two reduced words for an element $x \in W$, with $r, s \in S$, then there is a reduced word which ends with $\bar{r}\bar{s}$. To do this, we will consider the sets $F_x = \{y \in W \mid x * y = x\}$. For a Coxeter group, F_x is the parabolic subgroup generated by the right descent set of x .

Lemma 15 *If $y * z = y$ and $z * y = z$, then $y = z$ is idempotent.*

Proof We have that $y = y * z = y * z * y = y * y$ is idempotent. Similarly, z is idempotent, so $z = z^{-1}$ by Condition (5), and thus $y = y^{-1} = (y * z^{-1})^{-1} = z * y^{-1} = z$. $\square$

Proposition 16 *For any $x \in W$, F_x is a principal downset in the left ordering ($a \leq_l b$ if and only if $b = a * c$ for some $c \in W$). That is, there is an $\hat{x} \in W$ such that $x * y = x$ if and only if $y \leq_l \hat{x}$. Furthermore, this $\hat{x}$ is idempotent.*

Proof It is clear that F_x is a left downset, because of Condition (3). Let y and z be two left-maximal elements of F_x , i.e. $y \leq_l w \in F_x \Rightarrow y = w$ and $z \leq_l w \in F_x \Rightarrow z = w$. We want to prove that $y = z$. Since W is finite, this will imply that F_x is a principal left downset. Now, we know that $x * y * z = x * z = x$, and that $y \leq_l y * z$. Therefore, we know that $y * z = y$. Similarly, we have that $z * y = z$, so by the previous lemma, $y = z$. The idempotence of $\hat{x}$ is obvious. $\square$

Remark 17 Proposition 16 actually holds for any Coxeter group—the only use we made of the finiteness of W was in showing that F_x is finite. However, for a Coxeter group, we can show that F_x is contained in the Bruhat downset of x , which is finite, so F_x is a principal downset. Indeed, if we require that F_x be finite for every $x \in W$, and that $*$ -irreducible elements generate, then the proof that W is a Coxeter group still applies, even without the assumption that W is finite.

Lemma 18 For any $a, b \in S$, and any $t \in W$, if $a \leq_r t$ and $b \leq_r t$, then $\overline{ab} \leq_r t$. [$\leq_r$ is the right order, given by $x \leq_r y$ if $y = a * x$ for some a .]

Proof Since $a \leq_r t$, and a is idempotent, we have $t * a = t$, and similarly, $t * b = t$, so $t * \overline{ab} = t$, i.e. $\overline{ab} \leq_r t$. $\square$

Lemma 19 For $*$ -irreducible elements s and r , there are two reduced words for $\overline{rs}$ consisting of just the $*$ -irreducible elements r and s . These words have the same length.

Proof Form the sequence of words $v_1 = s, v_2 = s * r, \dots$ inductively by

$$v_n = \begin{cases} v_{n-1} * s & \text{if } n \text{ is odd} \\ v_{n-1} * r & \text{if } n \text{ is even} \end{cases}$$

Since W is finite, it is clear that we must have some $v_k = v_l$ for some $k \neq l$, but by Condition (3), this gives $v_k = v_{k+1} = \dots = \overline{rs}$. Form the sequence of words w_n inductively by $w_1 = r$ and

$$w_n = \begin{cases} w_{n-1} * r & \text{if } n \text{ is odd} \\ w_{n-1} * s & \text{if } n \text{ is even} \end{cases}$$

As for the v_n , we must have $w_l = \overline{rs}$ for some l . We claim that the smallest k such that $v_k = \overline{rs}$ and the smallest l such that $w_l = \overline{rs}$ are equal.

Now, since k is the smallest such that $v_k = \overline{rs}$, for any $m < k$, we must have $v_{m+1} \neq v_m$, so by Lemma 13, we must get $v_{m+1} = v_m * s$ if m is even or $v_{m+1} = v_m * r$ if m is odd. That is, v_k is a reduced word for $\overline{rs}$. Similarly, w_k is a reduced word for $\overline{rs}$. Now, if $k \neq l$, w.l.o.g., $k < l$, then v_k must be a subword of w_l . Let $w_l = r * v_k * t$ for some t . Since w_l is a reduced word, we must have $r \perp v_k = \overline{rs}$, which contradicts the definition of $\overline{rs}$. Therefore, we must have $k = l$, as required. $\square$

Note that from the above proof, one of the words for $\overline{rs}$ ends with r , and the other word ends with s .

Remark 20 If we do not require W to be finite, the assignment $x \mapsto \overline{x}$ is a partial function, since there may be no idempotents above x in the Bruhat order. However, Lemma 18 still holds in the strong sense that if the conditions are satisfied, then $\overline{ab}$ exists and is $\leq_r t$ ($\overline{ab}$ exists because all the elements $(ab)^{*n}$ are in the set F_x , which is finite), and Lemma 19 still holds whenever $\overline{sr}$ exists. This will allow us to deduce conditions under which an infinite W comes from a Coxeter group.

Lemma 21 Any two reduced words for an element x of W have the same length.

Proof The proof is done by induction on the length of the shorter word. If $s_1 \times \dots \times s_n = s \in S$, then because s is $*$ -irreducible, we must have $s_1 = s$, and $n = 1$. Now, suppose the result holds whenever the shorter word has length n or less, and let $t = s_1 \times \dots \times s_n$, where $s_1, \dots, s_n \in S$. Now, suppose $t \times s = r_1 \times \dots \times r_m$. If

$s = r_m$, then we have that $t = r_1 \times \cdots \times r_{m-1}$, so $m - 1 = n$. If not, then by Lemma 18, $t \times s = v \times \overline{sr_m}$. Now, if we choose a reduced word for $\overline{sr_m}$ that ends with s , then we are in our previous situation, so that the combined length of any reduced word for v , and the word for $\overline{sr_m}$ is $n + 1$. Now, if we instead choose a reduced word for $\overline{sr_m}$ that ends with r_m , we have a reduced word of length $n + 1$ that ends with r_m . Now, by the induction hypothesis, we deduce that $m = n + 1$. $\square$

Lemma 22 *If two reduced words made up from elements in S are equal, then they can be made equal using only equations of the form $ab \cdots ba = ba \cdots ab$ or $ab \cdots ab = ba \cdots ba$, for elements a and b in S .*

Proof The proof is done by induction on the word length. Suppose that for words of length at most $n - 1$, the result holds. Let t be the common value of the two reduced words $s_1 \cdots s_n$ and $r_1 \cdots r_n$. If $s_n = r_n$, then by removing the last element from each word, we get a shorter equality, so we can apply our inductive hypothesis. Therefore, it is sufficient to consider the case $s_n \neq r_n$. However, by Lemma 18 we have that $t = v \times \overline{s_n r_n}$, as a reduced word, for some v . We can choose a reduced word for $\overline{s_n r_n}$ that ends with s_n , and so by our inductive hypothesis, we can identify this with $s_1 \cdots s_n$, using only equations of the prescribed form. Similarly, there is a reduced word for $\overline{s_n r_n}$ that ends with r_n , and so by our inductive hypothesis, we can identify this with $r_1 \cdots r_n$, using only equations of the prescribed form. The two reduced words for $\overline{s_n r_n}$ can also be identified using an equation of the prescribed form. $\square$

We have shown that: every element of W can be expressed as a reduced word; any two reduced words for an element of w can be made equivalent using the equations of the form $ab \cdots ba = ba \cdots ab$ or $ab \cdots ab = ba \cdots ba$, for elements a and b in S , where the words being identified have the same length; and the only equations relating words of different length are derived from $s.s = e$ and $s*s = s$, which hold for all elements of S . We therefore see that (W, S) is a Coxeter structure; $\cdot$ is the resulting Coxeter group multiplication; and $*$ is the resulting Coxeter monoid multiplication.

Remark 23 The condition that W must be finite was used in the proof of Proposition 16, but as we remarked then, this proposition holds even if W is infinite, provided that F_x is finite for each x . The finiteness of F_x is also sufficient for Lemma 18 to hold. It was also used to ensure that the $*$ -irreducible elements generate W .

For a general Coxeter group, all the conditions clearly hold, with the possible exception of (6), which is based on Proposition 6. The use of this condition was for proving Lemmas 13 and 12.

Therefore, we can modify the above proof to show that

Theorem 24 *If W satisfies Conditions (1–5) and (7), and also satisfies:*

1. $*$ -irreducible elements generate W under $\times$;
2. Lemmas 13 and 12 hold;
3. the sets F_x are finite for each x ,

then W is a Coxeter group, with multiplication $\cdot$ and monoid multiplication $$.*

We will use this in the next section to characterise Coxeter groups as groups with a partial order.

4 Characterisation of Coxeter groups with Bruhat order

We prove the following characterisation of Coxeter groups among all groups with a partial order.

Theorem 25 *Let $(G, \cdot, 1, (_)^{-1})$ be a group, and $\leq$ a partial order on the set of elements of G , such that:*

1. *1 is the smallest element (i.e. $(\forall x) 1 \leq x$).*
2. *The elements that cover 1 in the partial order (i.e. the elements x such that any $1 \leq a \leq x$ must satisfy either $a = 1$ or $a = x$) generate G .*
3. *$(_)^{-1}$ preserves $\leq$. That is, if $a \leq b$, then $a^{-1} \leq b^{-1}$.*
4. *The element-wise product of two principal downsets is another principal downset. That is, for any two elements a and b , there is an element $a * b$ such that*

$$\{x \in G \mid x \leq a * b\} = \{yz \mid y \leq a, z \leq b\}$$

Then G is a Coxeter group, and $\leq$ is the Bruhat order for the Coxeter system on G given by the elements which cover 1 in the order $\leq$.

Remark 26 The converse to this theorem, namely that the Bruhat order on a Coxeter group satisfies Conditions 1–4, is also true—Conditions 1–3 are easy to verify, and Condition 4 follows from Proposition 8.

We will begin with some lemmas about a group satisfying Conditions 1–4 of Theorem 25. In the following lemmas, G is a group with identity 1, and $\leq$ is a partial order on G satisfying Conditions 1–4 of Theorem 25. We will denote the set of elements that cover 1 by S .

Lemma 27 *If s covers 1, then $s^2 = 1$.*

Proof Since s covers 1, we have $\downarrow(s) = \{1, s\}$. We also have that s^{-1} covers 1, since if $a \leq s^{-1}$, then $a^{-1} \leq s$. Therefore, $\downarrow(s * s^{-1}) = \downarrow(s) \downarrow(s^{-1}) = \{1, s, s^{-1}\}$. This must be a principal downset, but the principal downsets of the three elements of this set are $\{1\}$, $\{1, s\}$ and $\{1, s^{-1}\}$, respectively. Therefore, we must have $s = s^{-1}$, i.e. $s^2 = 1$. $\square$

Lemma 28 *If s covers 1, and a is any element of G , then either $a * s = a$ or $a * s = a.s$.*

Proof We know that $\downarrow(a * s) = \downarrow(a) \downarrow(s) = \downarrow(a) \cup (\downarrow(a)\{s\})$. Therefore we must either have $a * s \in \downarrow(a)$ or $a * s = a's$ for some $a' \leq a$. In the first case, clearly $a * s = a$. Suppose $a * s = a's$ for some $a' \leq a$. This means that $a' * s = a * s$, so

we have that $a = a''s'$ for some $a'' \leq a'$ and $s' \leq s$. Since s covers 1, we have either $s' = 1$, in which case $a'' = a' = a$, so that $a * s = a.s$, or $s' = s$, so that $a = a''s$ for some $a'' \leq a'$. This means that $a \in \downarrow(a'') \downarrow(s)$, so since $\downarrow(a'') \downarrow(s)$ is a principal downset, we get that $a' \in \downarrow(a'') \downarrow(s)$, so that either $a' = a''$, in which case $a * s = a's = a''s = a$, or $a' = a'''s$ for some $a''' \leq a''$. However, this would mean that $a's = a'''s^2 = a''' \leq a$, so that $a * s \leq a$, and since we automatically have $a \leq a * s$, this gives $a * s = a$. $\square$

Lemma 29 *For any $a, b \in G$, there is some $b' \leq b \in G$ such that $a * b = ab' = a * b'$.*

Proof We know that G is generated by the set S of elements which cover 1, so we can prove the result by induction on b . When $b = 1$, the result is obvious. When $b \in S$, this is just the previous lemma. Suppose the result holds for b' , and that $b = b's = b' * s$ for some $s \in S$ (so $b' \leq b$). Then we have $a * b = a * b' * s = a * b'$ or $a * b = a * b' * s = (a * b')s$. In the first case, the result follows by induction. In the second case, the inductive hypothesis gives us that $a * b' = a * b'' = ab''$ for some $b'' \leq b'$. Now we have that $a * b = a * b'' * s = (ab'') * s = ab''s$, so we just need to show that $b''s = b'' * s$. By the previous lemma, either $b'' * s = b''s$ or $b'' * s = b''$. In the second case, since $a * b' = a * b''$, we would have $a * b'' * s = a * b''$, so in either case, the result holds. $\square$

Lemma 30 *For any $x \in G$, $\downarrow(x)$ is finite.*

Proof This is obvious if x covers 1. However, since the covers of 1 generate, x can be expressed as a product $x = s_1 \cdots s_n$ of covers of 1. Therefore, $x \in \downarrow(s_1) \cdots \downarrow(s_n)$. Since the product of principal downsets is another principal downset, we have that $\downarrow(s_1) \cdots \downarrow(s_n)$ is a finite downset. Since it contains x , it must also contain $\downarrow(x)$, which must therefore be finite. $\square$

Proof of Theorem 25 We can use Theorem 24. We have that $*$ -irreducible elements generate, since covers of 1 are clearly $*$ -irreducible. We have already shown that Lemmas 13 and 12 hold (Lemmas 27 and 28 and the definition of $*$). Therefore, we need to show that the set $F_x = \{y \mid x * y = x\}$ is finite for each x , and that Conditions (1–5) and (7) of Theorem 9 hold for $.$ and $*$.

To show that F_x is finite, we first observe that $D_x = \{(x * y)y^{-1} \mid y \in W\}$ is finite for each x . This is because, by the dual to Lemma 29, we have that $D_x \subseteq \downarrow(x)$, so it is finite by Lemma 30. Now, however, we have the function $F_x \rightarrow D_x$ given by $y \mapsto (x * y)y^{-1} = xy^{-1}$. It is clear that this function is injective, so F_x is finite.

Now we need to show that Conditions (1–5) and (7) of Theorem 9 hold. Condition (1) is given. For Condition (2), it is easy to see that $\downarrow(1) = \{1\}$ is indeed an identity for $*$, so we need to prove associativity of $*$. However, this is obvious because

$$\downarrow((a * b) * c) = \downarrow(a * b) \downarrow(c) = \downarrow(a) \downarrow(b) \downarrow(c) = \downarrow(a) \downarrow(b * c) = \downarrow(a * (b * c))$$

Condition (3) is also clear, because by definition, we have $x \leq x * y$ for any $x, y \in G$. Condition (4) follows from Lemma 29, because we know that $a * b = a \times b'$ for some $b' \leq b$. Therefore, $a * (a^{-1} \cdot (a * b)) = a * (a^{-1} \cdot (a \times b')) = a * b' = a * b$.

Condition (5) follows because from $x * x = x$, we derive $x^n \leq x$ for every $n = 1, 2, \dots$, and since $\downarrow(x)$ is finite by Lemma 30, we deduce that $x^n = 1$ for some n . Therefore, $x^{-1} = x^{n-1} \leq x$. Because $(_)^{-1}$ preserves $\leq$, we also get $x \leq x^{-1}$, and so $x = x^{-1}$. Condition (7) is obvious because $(_)^{-1}$ preserves $\leq$. $\square$

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