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Kazhdan–Lusztig polynomials of boolean elements

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Abstract We give closed combinatorial product formulas for Kazhdan–Lusztig polynomials and their parabolic analogue of type q in the case of boolean elements, introduced in (Marietti in J. Algebra 295:1–26, 2006), in Coxeter groups whose Coxeter graph is a tree. Such formulas involve Catalan numbers and use a combinatorial interpretation of the Coxeter graph of the group. In the case of classical Weyl groups, this combinatorial interpretation can be restated in terms of statistics of (signed) permutations. As an application of the formulas, we compute the intersection homology Poincaré polynomials of the Schubert varieties of boolean elements.

Keywords Coxeter groups · Kazhdan–Lusztig polynomials · Boolean elements · Poincaré polynomials

1 Introduction

In their fundamental paper [16] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan–Lusztig polynomials of W (see, e.g., [13, Chap. 7] or [2, Chap. 5]). These polynomials play an important role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [2, Chap. 5], and the references therein). In particular, their coefficients give the dimensions of the intersection cohomology modules for Schubert varieties (see, e.g., [17]).

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In order to find a method for the computation of the dimensions of the intersection cohomology modules corresponding to Schubert varieties in G/P, where P is a parabolic subgroup of the Kac–Moody group G, in 1987 Deodhar [9] introduced two parabolic analogues of these polynomials which correspond to the roots $x=q$ and $x=-1$ of the equation $x^2=q+(q-1)x$. These parabolic Kazhdan–Lusztig polynomials reduce to the ordinary ones for the trivial parabolic subgroup and are also related to them in other ways (see, e.g., Proposition 2.3 below). Besides these connections, the parabolic polynomials also play a direct role in several areas including the theories of generalized Verma modules [7], tilting modules [24, 25] and Macdonald polynomials [11, 12].

The purpose of this work is to give explicit combinatorial product formulas for all (parabolic and ordinary) Kazhdan–Lusztig polynomials indexed by pairs of boolean elements (see Sect. 2 for the definition) in all Coxeter groups whose Coxeter graph is a tree. Our results show that all such polynomials have nonnegative coefficients, conjectured by Kazhdan and Lusztig [16] and recently proved by Elias and Williamson [10], and give a combinatorial interpretation of them in terms of Catalan numbers and the Coxeter graph of the group.

In the case of classical Weyl groups, this combinatorial interpretation can be restated in terms of exceedances and other statistics of (signed) permutations. Our results also confirm, only for boolean elements, a conjecture of Brenti on the parabolic Kazhdan–Lusztig polynomials of type q: the Kazhdan–Lusztig polynomials computed on two elements of a parabolic quotient are greater (in coefficient-wise comparison) than the polynomial computed on the same elements in a quotient included in the first (as set).

In literature there are many works which give methods to compute Kazhdan–Lusztig polynomials in special contests: for example, there are formulas for ordinary Kazhdan–Lusztig polynomials for covexillary permutations of type A due to Lascoux [18], for cominuscule elements in types B and D due to Boe [3] and for Hermitian symmetric pairs due to Brenti [4, 6]. The previous works give a combinatorial formula for all Kazhdan–Lusztig polynomials in special Coxeter groups; in this work, instead, we give a formula that could be applied to a special class of elements in a greater family of Coxeter groups. In some cases, the results overlap but the formula are obviously different.

The organization of the paper is as follows. In the next section we recall definitions, notation and results that are used in the rest of this work. In Sect. 3 we give some lemmas about the computation of parabolic Kazhdan–Lusztig polynomials indexed by boolean elements and introduce and illustrate some properties of "Catalan triangle" which will appear in the main result. In Sect. 4 we state and prove our main result, namely, an explicit closed combinatorial formula for all (parabolic and ordinary) Kazhdan–Lusztig polynomials of boolean elements of Coxeter group whose Coxeter graph is a tree. In Sect. 5 we restate the formulas using statistics associated with (signed) permutations for the classical Weyl groups. Finally, in Sect. 7 we use our main result to compute the intersection homology Poincaré polynomials indexed by boolean elements in all Coxeter groups whose Coxeter graphs have at most one vertex with more than two adjacent vertices.

2 Definitions, notation and preliminaries

We let $\mathbb{P} := \{1, 2, 3, \ldots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$, $\mathbb{Z} := \mathbb{N} \cup \{-1, -2, \ldots\}$. For all $m, n \in \mathbb{Z}$, $m \leq n$, we set $[m, n] := \{m, m + 1, \ldots, n\}$ and $[n] := [1, n]$. Given a set A we denote by $\#A$ its cardinality.

We follow [26, Chap. 3] for poset notation and terminology. In particular, given a poset $(P, \leq)$ and $u, v \in P$ we let $[u, v] := \{w \in P | u \leq w \leq v\}$ and call this an *interval* of P. We say that v *covers* u, denoted $u \lhd v$ (or, equivalently, that u is *covered* by v) if $\#[u, v] = 2$.

We follow [13] for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $u \in W$ we denote by $l(u)$ the length of u in W, with respect to S, i.e. the minimal length of words $s_{i_1} \cdots s_{i_k} = u$ whose alphabet is S (such minimal words are called reduced). Given $u, v \in W$ we denote by $l(u, v) = l(v) - l(u)$. We let $D_R(u) := \{s \in S | l(us) < l(u)\}$ the set of the right descents of u, $D_L(u) := \{s \in S | l(su) < l(u)\}$ the set of the left descents of u and we denote by ϵ the identity of W.

Let $s, s' \in S$ and define $\alpha_{s,s'} := ss'ss'ss' \cdots$ the alternating word of length $m(s, s')$. Given a word w in the alphabet S let us call a *nil-move* the deletion of a subword of the form ss, and a *braid-move* the replacement of a factor $\alpha_{s,s'}$ by $\alpha_{s',s}$. The following result can be found in [2, Theorem 3.3.1].

Theorem 2.1 (Word property) *Let (W, S) be a Coxeter system and $w \in W$.*

- *Any expression $s_1 s_2 \cdots s_q$ for w can be transformed into a reduced expression for w by a sequence of nil-moves and braid-moves*;
- *every two reduced expressions for w can be connected via a sequence of braid-moves.*

Given $J \subseteq S$ we let W_J the parabolic subgroup generated by J and

$$W^J := \{u \in W | l(su) > l(u) \text{ for all } s \in J\}. \tag{1}$$

Note that $W^{\emptyset} = W$ (here we use a different notation from that given in [2], in which the same set is denoted by ${}^J W$). If W_J is finite, then we denote by $w_0(J)$ its longest element. We will always assume that W^J is partially ordered by *Bruhat order*. Recall (see e.g. [13, Chaps. 5.9 and 5.10]) that this means that $x \leq y$ if and only if for one reduced word of y (equivalently for all) there exists a subword that is a reduced word of x. Given $u, v \in W^J$, $u \leq v$, we let

$$[u, v]^J := \{w \in W^J | u \leq w \leq v\},$$

and $[u, v] := [u, v]^{\emptyset}$.

For $J \subseteq S$, $x \in \{-1, q\}$, and $u, v \in W^J$ we denote by $P^{J,x}_{u,v}(q)$ the parabolic Kazhdan–Lusztig polynomials in W^J of type x (we refer the reader to [9] for the definitions of these polynomials; see also Proposition 2.3 below). We denote by $P_{u,v}(q)$ the ordinary Kazhdan–Lusztig polynomials.

For $u, v \in W^J$ let $\mu_{J,q}(u, v)$ be the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P^{J,q}_{u,v}(q)$ (so $\mu_{J,q}(u, v) = 0$ when $l(v) - l(u)$ is even). It is well known that if $u, v \in W^J$ then

$\mu_{J,q}(u,v)=\mu(u,v)$, the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P_{u,v}(q)$ (see Corollary 2.4 below). The following result is due to Deodhar, and we refer the reader to [9] for its proof.

Proposition 2.2 *Let (W,S) be a Coxeter system, $J\subseteq S$, and $u,v\in W^J$, $u\leq v$. Then for each $s\in D_R(v)$ we have that*

$$P^{J,q}_{u,v}(q)=\widetilde{P}_{u,v}-\widetilde{M}_{u,v}, \tag{2}$$

where

$$\widetilde{P}_{u,v}=\begin{cases}P^{J,q}_{us,vs}+qP^{J,q}_{u,vs} & \text{if } us<u;\\ qP^{J,q}_{us,vs}+P^{J,q}_{u,vs} & \text{if } u<us\in W^J;\\ 0 & \text{if } u<us\notin W^J,\end{cases}$$

and

$$\widetilde{M}_{u,v}=\sum_{u\leq w<vs\mid ws<w}\mu(w,vs)q^{\frac{l(w,v)}{2}}P^{J,q}_{u,w}(q).$$

The parabolic Kazhdan–Lusztig polynomials are related to their ordinary counterparts in several ways, including the following one, which may be taken as their definition in most cases.

Proposition 2.3 *Let (W,S) be a Coxeter system, $J\subseteq S$ and $u,v\in W^J$. Then we have that*

$$P^{J,q}_{u,v}(q)=\sum_{w\in W_J}(-1)^{l(w)}P_{wu,v}(q).$$

Moreover, if W_J is finite, then

$$P^{J,-1}_{u,v}(q)=P_{w_0(J)u,w_0(J)v}(q).$$

A proof of this result can be found in [9] (see Proposition 3.4, and Remark 3.8). Since for all $w\in W_J$ and $u\in W^J$ we have $l(wu)=l(w)+l(u)$ by [2, Proposition 2.4.4], then the degree of $P_{wu,v}(q)$ in Proposition 2.3 is less than $\frac{1}{2}(l(u,v)-1)$ except when $w=\epsilon$. Therefore we have

Corollary 2.4 *For any $J\subseteq S$ and $u,v\in W^J$ we have*

$$\mu_{J,q}(u,v)=\mu(u,v).$$

A proof of the following result can be found in [22, Corollary 2.9 and the previous remark].

Proposition 2.5 *Let (W,S) a Coxeter system and $J\subseteq S$. Let $u,v\in W^J$ and $s\in D_R(v)$.*

(a) *If* $us \notin W^J$ *then* $P^{J,q}_{u,v}(q) = 0$;
(b) *if* $us \in W^J$ *then* $P^{J,q}_{us,v}(q) = P^{J,q}_{u,v}(q)$;
(c) *if* $\mu(u,v) \neq 0$ *then* $D_R(v) \subseteq D_R(u)$ *and* $D_L(v) \subseteq D_L(u)$.

In the rest of the paper we will consider parabolic Kazhdan–Lusztig polynomials of type q. Therefore we will write $P^J_{u,v}$ instead of $P^{J,q}_{u,v}$.

Let (W,S) be any Coxeter system and t be a reflection in W. Following Marietti [20–22], we say that t is a *boolean reflection* if it admits a *boolean expression*, which is, by definition, a reduced expression of the form $s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$ with $s_k \in S$, for all $k \in \{1,\dots,n\}$ and $s_i \neq s_j$ if $i \neq j$. We say that $u \in W$ is a *boolean element* if u is smaller than a boolean reflection in the Bruhat order. Let $\overline{v}$ be a reduced word of a boolean element and $s \in S$; we denote by $\overline{v}(s)$ the number of occurrences of s in $\overline{v}$.

Given a Coxeter system (W,S), the Coxeter graph of W is a graph whose vertex set is S and two vertices s, s' are joined by an edge if $ss' \neq s's$. We label this edge with $m(s,s')$, the smallest positive integer such that $(ss')^{m(s,s')} = \epsilon$ ($m(s,s') = \infty$ if there is no such integer). We say that W is a *tree-Coxeter group* if its Coxeter graph is a tree.

3 Preliminary results

In this section we give some preliminary lemmas which are needed to prove the main theorem in the next section. For any generator $s_i \in S$ we set $S^i := S \setminus \{s_i\}$ and we denote by $C(s_i)$ the subset of S^i of all elements commuting with s_i.

Lemma 3.1 *Let* $u, v \in W^J$ *such that* $s_i u, s_i v \in W^J_{S^i}$ *(in particular, there exist reduced words for* u, v *starting with* s_i *and with no other occurrences of* s_i*). Then*

$$P^J_{u,v} = P^{J \cap C(s_i)}_{s_i u, s_i v}.$$

Proof The statement is trivial if $l(v) = 1$. Suppose that $l(v) > 1$. Then there exists $s_j \in D_R(v)$, $j \neq i$. Note that for any $w \in W$ with $s_i w \in W_{S^i}$ we have that $D_L(w) \subseteq \{s_i\} \cup C(s_i)$, more precisely $D_L(w) = \{s_i\} \cup (D_L(s_i w) \cap C(s_i))$. Therefore $us_j \in W^J$ if and only if $s_i u s_j \in W^{J \cap C(s_i)}$. In this case, by Proposition 2.2 we have

$$\begin{aligned} P^J_{u,v} &= q^c P^J_{us_j, vs_j} + q^{1-c} P^J_{u, vs_j} - \sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w,vs_j)}{2}} P^J_{u,w} \\ &= q^c P^{J\cap C(s_i)}_{s_i u s_j, s_i v s_j} + q^{1-c} P^{J\cap C(s_i)}_{s_i u, s_i v s_j} \\ &\quad - \sum_{\substack{s_i u \leq s_i w \leq s_i v s_j \\ s_i w s_j < s_i w}} \mu(s_i w, s_i v s_j) q^{\frac{l(s_i w, s_i v s_j)}{2}} P^{J\cap C(s_i)}_{s_i u, s_i w} \\ &= P^{J\cap C(s_i)}_{s_i u, s_i v} \end{aligned}$$

by induction, where c is 0 or 1. The equalities hold since the map from $[u,v]^J$ to $[s_i u, s_i v]^{J\cap C(s_i)}$ given by left-multiplication by s_i is an isomorphism of posets. □

Lemma 3.2 *Let $u, v \in W^J$ be such that $u, s_i v \in W_{S^i}$ (in particular, there are no occurrences of s_i in any reduced expression of u and $s_i v$). Then*

$$P^J_{u,v} = \begin{cases} P^J_{u,s_i v} & \textit{if } s_i v \in W^J, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof If $l(v) = 1$, there is nothing to prove. Let we suppose $l(v) > 1$ and let $s_j \in D_R(v)$, $s_j \neq s_i$. If $us_j \notin W^J$ the claim is trivial by Proposition 2.5. Therefore we may assume $us_j \in W^J$.

Suppose that $s_i v \in W^J$ and proceed by induction on $l(v)$. Then by Proposition 2.2 we get

$$\begin{aligned} P^J_{u,v} &= q^c P^J_{us_j,vs_j} + q^{1-c} P^J_{u,vs_j} - \sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w,vs_j)}{2}} P^J_{u,w} \\ &= q^c P^J_{us_j,s_i vs_j} + q^{1-c} P^J_{u,s_i vs_j} - \sum_{\substack{u \leq s_i w \leq s_i vs_j \\ s_i ws_j < s_i w}} \mu(s_i w, s_i vs_j) q^{\frac{l(s_i w,s_i vs_j)}{2}} P^J_{u,s_i w} \\ &= P^J_{u,s_i v}, \end{aligned}$$

where c is 0 or 1. The equalities hold by induction on $l(vs_j)$: if $w \in W_{S^i}$ then $\mu(w, vs_j)$ is 0 since by induction either $P^J_{w,vs_j} = 0$ or $P^J_{w,vs_j} = P^J_{w,s_i vs_j}$ and therefore P^J_{w,vs_j} does not have the maximum degree. Otherwise, if $s_i w \notin W^J$ then $P^J_{u,w} = 0$ by induction, else $P^J_{u,w} = P^J_{u,s_i w}$ and $\mu(w, vs_j) = \mu(s_i w, s_i vs_j)$ by Lemma 3.1 and Corollary 2.4.

Finally, if $s_i v \notin W^J$ we may assume that $s_i vs_j \notin W^J$ (for this, choose a suitable right descent s_j) except in the case $v = s_i s_j$ and $u = \epsilon$ which is trivial. Then by induction

$$P^J_{u,v} = -\sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w,vs_j)}{2}} P^J_{u,w}.$$

Fix $w \in W^J$ with $u \leq w \leq vs_j$ and $ws_j < w$. We prove that $\mu(w, vs_j) P^J_{u,w} = 0$. If $w \in W_{S^i}$ then $\mu(w, vs_j) = 0$ by induction. Otherwise, if $s_i w \in W_{S^i}$ then by Lemma 3.1 we have $\mu(w, vs_j) = \mu(s_i w, s_i vs_j)$. Now, if $s_i w \notin W^J$ then by induction $P^J_{u,w} = 0$, else both $s_i vs_j \notin W^J$ and $s_i w \in W^J$ imply that $D_L(s_i vs_j) \nsubseteq D_L(s_i w)$ and by (c) of Proposition 2.5 we have $\mu(s_i w, s_i vs_j) = 0$. □

We now introduce a family of numbers which are used in the next section. The *Catalan triangle* is a triangle of numbers formed in the same manner as Pascal's triangle, except that no number may appear on the left of the first element (see [23, sequence A008313]).

$$\begin{array}{llllllllll}
1 & & & & & & & & & \\
 & 1 & & & & & & & & \\
1 & & 1 & & & & & & & \\
 & 2 & & 1 & & & & & & \\
2 & & 3 & & 1 & & & & & \\
 & 5 & & 4 & & 1 & & & & \\
5 & & 9 & & 5 & & 1 & & & \\
 & 14 & & 14 & & 6 & & 1 & & \\
14 & & 28 & & 20 & & 7 & & 1 & \\
 & 42 & & 48 & & 27 & & 8 & & 1
\end{array}$$

Let $h \geq 1$. We set

$$f_h(q) = \sum_{i=0}^{[\frac{h}{2}]} C(h, i) q^{[\frac{h}{2}]-i},$$

where $[h]$ denotes the integer part of h and $C(h, i)$ is the ith number in the hth row (here we start the enumeration from 0). For example, $f_4(q) = 2q^2 + 3q + 1$; $f_7(q) = 14q^3 + 14q^2 + 6q + 1$. We denote by $\mu(f_h(q))$ the coefficient of $q^{\frac{h}{2}}$ in $f_h(q)$. Therefore $\mu(f_h(q)) = 0$ if h is odd. Then we have the following easy result, whose proof we omit.

Lemma 3.3 *For all $h \geq 0$,*

$$f_h(q)(1 + q) - \mu\big(f_h(q)\big)q^{\frac{h}{2}+1} = f_{h+1}(q).$$

Note that in the first column we find the classical Catalan numbers (see [23, sequence A008313] for details).

4 Parabolic Kazhdan–Lusztig polynomials

Let (W, S) be a tree-Coxeter group. Let $t = s_{i_1} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ be a boolean reflection, $i_j \neq i_h$ for $j \neq h$. Consider the Coxeter graph G and represent it as a rooted tree with root the vertex corresponding to the generator s_n. In this paper all the roots will be depicted on the right of their graphs. In Fig. 1 we give the Coxeter graph of the affine Weyl group $\widetilde{D}_{11}$.

According to such rooted graph we say that s_j is on the right (respectively on the left) of s_i if and only if there exists an edge joining both corresponding vertices and, in addition, the only path joining s_i (respectively s_j) to s_n crosses the node s_j (respectively s_i).

Let $\overline{w}$ be a word in the alphabet $s_{i_1}, \ldots, s_{i_k}$. In the following we denote by $\overline{w}(s_{i_j})$ the number of all occurrences of the element s_{i_j} in the word $\overline{w}$. Let $u, v \in W$ be such that $u, v \leq t$. As defined in [22] for linear Coxeter group, we denote by $\overline{u}, \overline{v}$ the only reduced expressions of u, v satisfying the following properties:

– $\overline{v}$ is a subword of $s_{i_1}\cdots s_{i_{k-1}}s_{i_k}s_{i_{k-1}}\cdots s_{i_1}$ and if i_j is such that $\overline{v}(s_{i_j})=1$ and $\overline{v}(s_{i_h})=0$, where s_{i_h} is the only element on the right of s_{i_j}, then $\overline{v}$ has s_{i_j} in the leftmost admissible position;
– $\overline{u}$ is a subword of $\overline{v}$ and if i_j is such that $\overline{u}(s_{i_j})=1$ and $\overline{u}(s_{i_h})=0$, we apply the same above rule.

Here we give an example. Let $t=s_1s_2\cdots s_5s_{10}s_{11}s_9s_8s_7s_6s_7s_8s_9s_{11}s_{10}s_5\cdots s_2s_1$ in $\widetilde{D}_{11}$, see Fig. 1. Let $v=s_4s_5s_{10}s_{11}s_6s_7s_8s_9s_5s_4s_2s_1$ and $u=s_8s_6s_1$ then $\overline{v}=s_1s_2s_4s_5s_{10}s_{11}s_6s_7s_8s_9s_5s_4$ and $\overline{u}=s_1s_6s_8$.

Now we give a graphical representation of the pair $(\overline{u},\overline{v})$. We start from the rooted tree of the Coxeter graph and we substitute for each vertex a table with one column and two rows. In the first row we write $\overline{v}(s_j)$ (s_j is the element associated with the vertex); in the second row we write $\overline{u}(s_j)$. In the case $\overline{v}(s_j)=1$, it is possible that s_j is on the left or on the right of s_n (the root) as subword of t. We distinguish the two cases by writing 1_l if s_j is on the left of s_n, and 1_r otherwise. By convention we write 1_l in the root s_n if $\overline{v}(s_n)\neq 0$. We apply the same rule to the second row. Moreover, in

Fig. 1 The Coxeter graph of $\widetilde{D}_{11}$ with root s_6, corresponding to the reflection $t=s_1s_2\cdots s_5s_{10}s_{11}s_9s_8s_7$ $s_6s_7s_8s_9s_{11}s_{10}s_5\ldots s_2s_1$

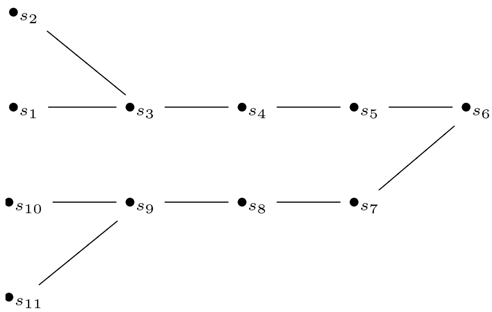
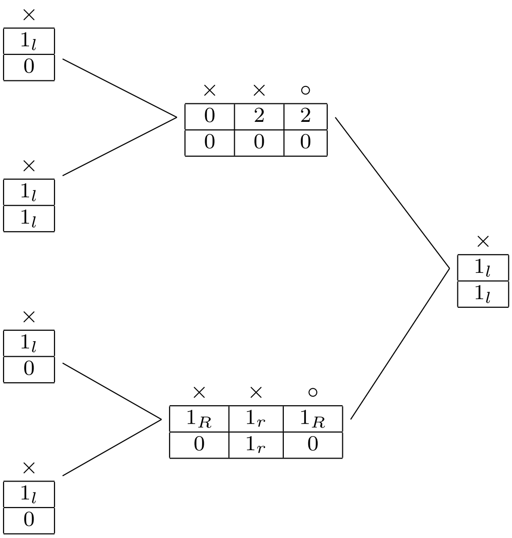


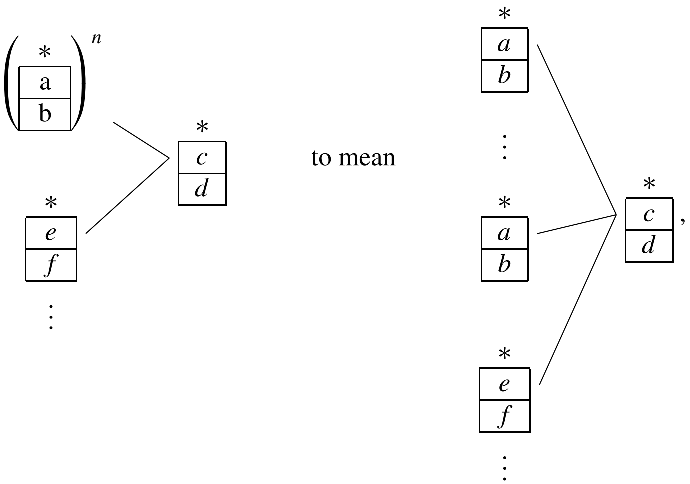
Fig. 2 Diagram of $(\overline{u}=s_1s_6s_8,\overline{v}=s_1s_2s_4s_5s_{11}$ $s_{10}s_6s_7s_8s_9s_5s_4)$ in $\widetilde{D}_{11}$



the first row, we use capital letter R instead of r if the second row of the column to the right does not contain 0.

We mark the column corresponding to s_j with $\circ$ if $j \in J$ and with $\times$ if $j \notin J$. Finally, if a vertex s_j has only one vertex on the left then we write the two corresponding columns in the same table. In Fig. 2 we give the graphical representation of the pair $(\overline{u}, \overline{v})$ in $\widetilde{D}_{11}$, with $J = \{s_5, s_7\}$.

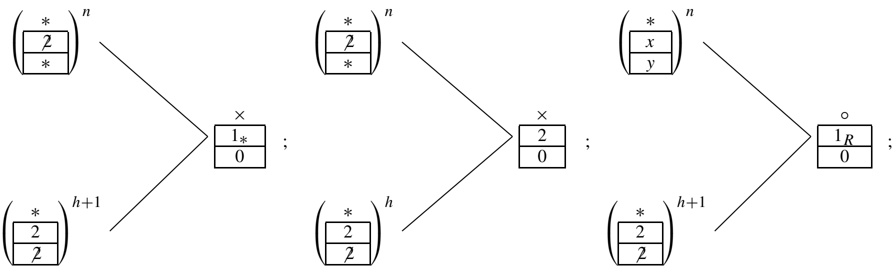
In the sequel a symbol $*$ denotes the possibility to have arbitrary entries in the cell. A symbol such as $\not{1}_l$, $\not{0}$, etc. means that the value in the cell is not 1_l, 0, etc. Moreover we will be interested in subdiagrams of such representations, i.e. diagrams obtained by deleting one or more columns. Since the order of the tables from top to bottom is not important (while the order from left to right is fundamental), we use the following notation:

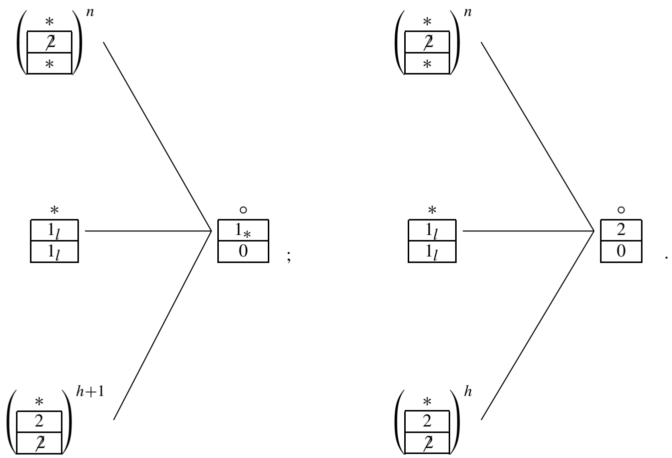


(3)

where the column with entries a, b is repeated n times. Now we give all the definitions necessary to Theorem 4.1.

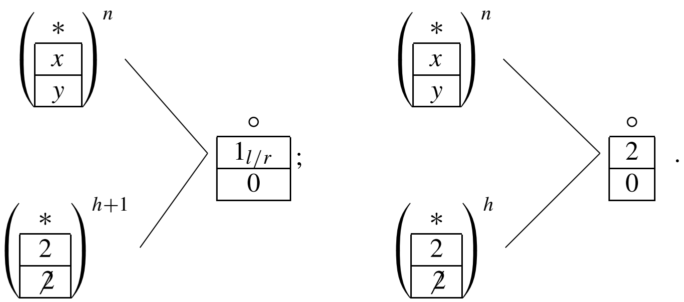
Given a pair $(\overline{u}, \overline{v})$ in W, we let $a_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following types:



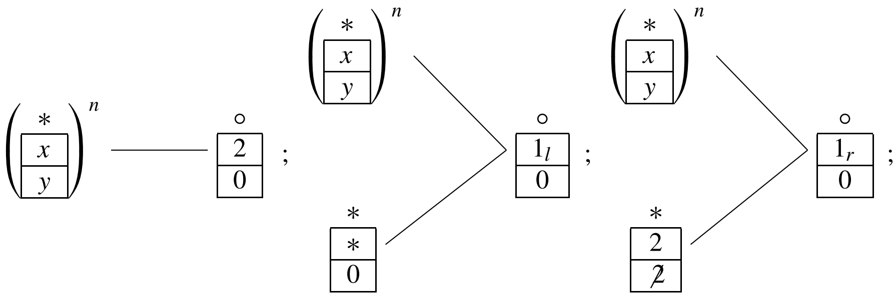


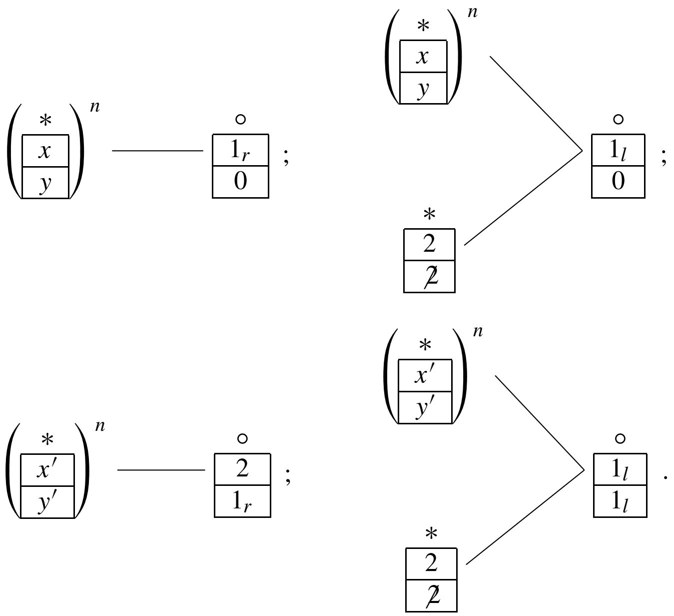
In the previous diagrams (and the same is true for the next diagrams) we consider only the subdiagram with n and h taken as biggest as possible, i.e. we have to consider all left neighbor columns of the column on the right of each diagram.

We define $b_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following types:

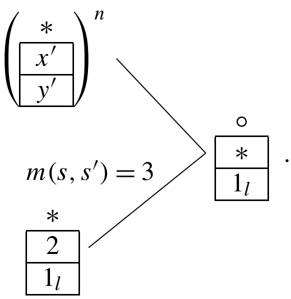


We set $c(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following types:





Finally, we set $c'(\overline{u},\overline{v})$ be the number of subdiagrams of the diagram of $(\overline{u},\overline{v})$ of the following type:



In all previous diagrams $(x, y) \in P_1$, $(x', y') \in P_1 \cup P_2$ with $P_1 = \{(1_l, 0), (1_r, 0), (1_r, 1_r), (2, 1_r)\}$, $P_2 = \{(1_R, 0), (1_R, 1_r), (2, 0)\}$. In each diagram (x, y), (x', y'), $(\not 2, *)$ or $(2, \not 2)$ are not necessarily the same pair for all $n \geq 0$ (or $h \geq 0$) columns. We can now state the main result of this work.

Theorem 4.1 *Let $J \subseteq S$, $u, v \in W^J$ and set $\overline{c}(\overline{u},\overline{v}) = c(\overline{u},\overline{v}) + c'(\overline{u},\overline{v})$. We have*

$$P^J_{u,v}(q) = \begin{cases} \prod_{h\geq 1} f_{h+1}^{a_h}(f_{h+1}-1)^{b_h} & \text{if } \overline{c}(\overline{u},\overline{v}) = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 4.2 *Let $J \subseteq S$, $u, v \in W^J$ with $l(v) - l(u) \geq 3$ odd. Then $\mu(u, v) \neq 0$ if and only if the entries in each column of the diagram of $(\overline{u},\overline{v})$ are equal except for* *exactly one subdiagram which is*

$$\left(\begin{array}{c} * \\ \hline \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1_l} \\ \hline \end{array}\right)^{h+1} \text{———} \begin{array}{|c|} \multicolumn{1}{c}{*} \\ \hline \emptyset \\ \hline 0 \\ \hline \end{array} \quad \textit{or} \quad \left(\begin{array}{c} * \\ \hline \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1_l} \\ \hline \end{array}\right)^{h} \text{———} \begin{array}{|c|c|c|} \multicolumn{1}{c}{*} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{*} \\ \hline 2 & \dots & 2 \\ \hline 0 & \dots & 0 \\ \hline \end{array}$$

In this case $\mu(u, v) = C([\frac{h+1}{2}])$, *the* $[\frac{h+1}{2}]$*-th Catalan number.*

Proof If in the diagram of $(\overline{u}, \overline{v})$ there are more than one subdiagram with the properties in the statement, then by Theorem 4.1, $P^J_{u,v}$ is the product of at least two factors. Since $l(v) - l(u)$ is equal to the sum of the differences between top row and bottom row entries, we have that the degree of $P^J_{u,v}$ is at most $\frac{l(v)-l(u)-2}{2}$. The last part of the statement follows by properties of $f_h(q)$. □

In the case of the classical Kazhdan–Lusztig polynomials, Theorem 4.1 becomes much simpler.

Corollary 4.3 *Let W be a tree-Coxeter group and $u, v \in W$ be boolean elements. Then $P_{u,v}(q) = \prod_{h\geq 1} f^{a_h}_{h+1}$, where a_h is defined before Theorem 4.1.*

Proof Just note that if $J = \emptyset$ then $b_h(\overline{u}, \overline{v}) = c(\overline{u}, \overline{v}) = 0$, $h \geq 1$. □

For example, the Kazhdan–Lusztig polynomial of the pair (u, v) depicted in Fig. 2 is $P^J_{u,v} = f_2(q) - 1 = q$, since $a_h = 0$ for all $h \geq 0$, $b_1 = 1$ and $b_h = 0$ for all $h \neq 1$.

Remark 4.4 Theorem 4.1 implies result in [22, Theorem 5.2].

We give the following easy consequence of Theorem 4.1 which proves, in the case of boolean elements, a conjecture of Brenti (private communication).

Corollary 4.5 *Let $I \subseteq J$ and $u, v \in W^J$. Then*

$$P^J_{u,v}(q) \leq P^I_{u,v}(q)$$

in the coefficient-wise comparison.

Proof Let $s \in J \setminus I$. The corresponding column of the diagram of $P^J_{u,v}$ is marked by $\circ$ and the same column in the diagram associated with $P^I_{u,v}$ is marked by $\times$. Consider a subdiagram of type a_h: by replacing a $\times$ with a $\circ$, it is possible that we get a diagram of type b_h or c. The vice versa is not possible. The claim follows by Theorem 4.1. □

We now prove Theorem 4.1.

Proof We argue the proof by induction on $l(v)$. The main idea is to consider one leftmost column of the diagram and compute its contribution in the Kazhdan–Lusztig polynomial; then delete such column and change the column on its right if necessary.

Apply then the induction on such diagram (the length of the element v is equal to the sum of the elements in the top place of all columns).

If $l(v) = 1$ then $P^J_{u,v} = 1$, since $u \leq v$ and the result is trivial. Now suppose $l(v) \geq 2$. Let C be one of the leftmost columns in the diagram. The entries of C can be filled by several values.

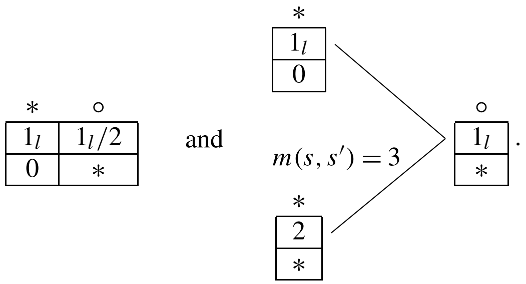
We first consider the case that C contains the pair $(1_r, 0)$ or $(1_R, 0)$. Let $s \in S$ be the element corresponding to C. Then $s \in D_R(v)$ and $us \not\leq vs$ because $s \leq us$ but $s \not\leq vs$. Moreover, since $s \not\leq vs$ we have that $w \not\leq vs$ and $P^J_{w,vs} = 0$ for any w such that $ws < w$. By Proposition 2.2 we have that $\widetilde{M}_{u,v} = 0$ and $P^J_{u,v} = P^J_{u,vs}$. The statement follows because in all the subdiagrams of type a_h (respectively b_h, c, c') we can delete the column C and have again a subdiagram of type a_h (respectively b_h, c, c').

If C contains the pair $(1_R, 1_r)$ or $(1_r, 1_r)$ then $u \not\leq vs$ and therefore $P^J_{u,v} = P^J_{us,vs}$ by Proposition 2.2. The statement follows.

Now suppose that C contains $(1_l, 1_l)$. By Lemma 3.1, $P^J_{u,v} = P^{J \cap C(s)}_{su,sv}$. Since C is a column on the left, $|C(s)| = 1$. Therefore the Kazhdan–Lusztig polynomial associated with the diagram is the same of that associated with the diagram without the column C and with the column on the right marked with $\times$. Apply induction hypotheses and note that for any subdiagram of type a_h $(h \geq 0)$ it is possible to remove one leftmost column with entries $(1_l, 1_l)$ having again a diagram of type a_h. Moreover, note that this agrees with the assumption $(1_l, 1_l) \notin P_1 \cup P_2$.

If C contains $(2, 2)$ then $u \not\leq vs$ and by Proposition 2.2 $P^J_{u,v} = P^J_{us,vs}$. We are in the case $(1_l, 1_l)$. As before, it is possible to remove a column with entries $(2, 2)$ from a diagram of type a_h without change its type and the assumption $(2, 2) \notin P_1 \cup P_2$ ensures that such entries are not in any subdiagram of type b_h, c or c'. The claim follows by induction.

If C contains $(1_l, 0)$ then by Lemma 3.2, $P^J_{u,v} = P^J_{u,sv}$ except in the case $sv \notin W^J$. Then we have to exclude



These diagrams are included in $c'(\overline{u}, \overline{v})$, and in the elements 1, 2, 5 and 6 of $c(\overline{u}, \overline{v})$. If C contains $(2, 1_r)$ use the same arguments above to have $P^J_{u,v} = P^J_{us,vs}$ and come back in the case $(1_l, 0)$.

Now suppose that C contains $(2, 0)$ and the bottom entry in the column on the right is non-zero. By Theorem 2.1, this assumption implies $s \notin D_L(us)$. Therefore $us \not\leq vs$. Moreover, there is no $w \in W^J$ with $u \leq w < vs$ and $ws < w$: in fact $ws < w \leq vs$ implies that the only occurrence of s in the word $\overline{w}$ is on the first place (since the same is for the word $\overline{vs}$); therefore $s \in D_L(w) \cap D_R(w)$ and thus if we denote by

t the element on the right of s then $w(t) = 0$ but this implies that $u \not\leq w$, impossible. By Proposition 2.2 we have $P^J_{u,v} = P^J_{u,vs}$. Then, by previous arguments, we are in the case $(1_l, 0)$ and this agrees with the assumption $(2, 0) \in P_2$.

If C contains $(2, 1_l)$ and the second entry in the column on the right is non-zero, then $us \notin W^J$ if and only if the diagram is such as in $c'(\overline{u}, \overline{v})$ (it is an easy consequence of Theorem 2.1). Otherwise $P^J_{u,v} = P^J_{u,vs}$ since, as before, there is no $w \in W^J$ with $u \leq w < vs$ and $ws < w$. Then we come back to the case $(1_l, 1_l)$.

Finally we have to consider the cases $(2, 1_l)$ or $(2, 0)$ with the second entry in the column on the right equal to 0. By Proposition 2.5, they can be treated as the same case. Note that in the definition of diagrams of type a_h, b_h, c or c' there is no difference in both cases. Therefore we assume that C contains $(2, 1_l)$.

For the diagram

$$\left(\begin{array}{c} * \\ \boxed{2} \\ \boxed{1_l} \end{array}\right)^h \;\text{———}\; \begin{array}{c} * \\ \boxed{1_*} \\ \boxed{0} \end{array} \tag{4}$$

the corresponding Kazhdan–Lusztig polynomial is $P^J_{u,v} = f_h - \alpha$, where $\alpha = 1$ when there are $\circ$ and 1_l on the rightmost column and $\alpha = 0$ otherwise. To show this, note that $P^J_{u,vs}$ is represented by a diagram with a leftmost column having entries equal to $(1_l, 1_l)$. By induction, the polynomial is equal to $P^J_{su,svs}$, whose diagram is as in (4) but with $h - 1$ instead of h. The polynomial $P^J_{us,vs}$ is represented by a diagram with a leftmost column with $(1_l, 0)$ and by induction $P^J_{us,vs} = P^J_{us,svs}$. Finally, by Corollary 4.2 and by induction $\mu(w, vs) \neq 0$ only if the diagram of w coincides with the diagram of v in all other columns not depicted in (4). Apply Proposition 2.2 and have $P^J_{u,v} = f_{h-1}(q) - \alpha + qf_{h-1}(q) - \mu(f_{h-1}(q))q^{\frac{h-1}{2}}$. By Lemma 3.3 we get $P^J_{u,v} = f_h - \alpha$ (note that if $h = 1$, $f_1 - 1 = 0$ and this agrees with the 3rd and 5th elements in $c(\overline{u}, \overline{v})$).

For the last subcase,

$$\left(\begin{array}{c} * \\ \boxed{2} \\ \boxed{1_l} \end{array}\right)^h \;\text{———}\; \begin{array}{c} * \\ \boxed{2} \\ \boxed{0} \end{array}, \tag{5}$$

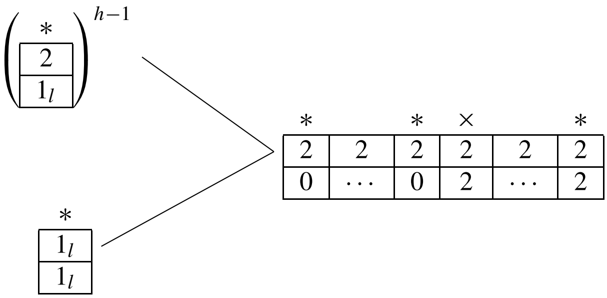
the analysis is a bit harder. Let us assume that on the right of this diagram there is a sequence of m columns

$$\begin{array}{|c|c|c|c|} \multicolumn{1}{c}{*} & \multicolumn{1}{c}{*} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{*} \\ \hline 2 & 2 & \cdots & 2 \\ \hline 0 & 0 & \cdots & 0 \\ \hline \end{array} \tag{6}$$

ending with a column whose entries are not $(2, 0)$ or with a column corresponding to a vertex of degree greater than 2. Suppose that exactly k of these columns have a $\circ$ and the other $m - k$ have a $\times$. Let $\overline{P}^J_{u,v}$ be the Kazhdan–Lusztig polynomial corresponding to the diagram of $(\overline{u}, \overline{v})$ after deleting the subdiagrams depicted in (5) and (6). Using the same techniques as above and by induction we have

$$P^J_{u,v} = \big(f_h(q) - \alpha\big) q^k (1+q)^{m-k} \overline{P}^J_{u,v} + f_h(q) q^{k+1} (1+q)^{m-k} \overline{P}^J_{u,v} - \widetilde{M}_{u,v},$$

where $\alpha = 1$ if there is a $\circ$ on the rightmost column of (5) and $\alpha = 0$ otherwise, and $\widetilde{M}_{u,v}$ is the sum in Proposition 2.2. Note that by induction and Corollary 4.2, $\mu(w, vs) \neq 0$ only if the diagram of w coincides with that of v in all the columns not depicted in (5) and (6). More precisely, for any such w, the diagram of $(\overline{w}, \overline{vs})$ is of the form



and in all other columns the top entries are equal to the bottom entries. Therefore $\widetilde{M}_{u,v}$ is

$$\overline{P}^J_{u,v} \mu\big(f_h(q)\big)\big(q^{\frac{h-2}{2}+k}(q+1)^{m-k-1} + q^{\frac{h-2}{2}+k+1}(q+1)^{m-k-2} + \cdots + q^{\frac{h-2}{2}+m-1} + q^{\frac{h-2}{2}+m}\big)$$

if h is even and 0 if h is odd. In this formula the powers of q include both the contributions of $q^{\frac{l(w,vs)}{2}}$ and of $P^J_{u,w}$. In the case h even, $h \geq 4$, we have

$$\begin{aligned}\widetilde{M}_{u,v} &= \overline{P}^J_{u,v} \mu\big(f_h(q)\big)\big(q^{\frac{h-2}{2}+k}\big((q+1)^{m-k} - q^{m-k}\big) + q^{\frac{h-2}{2}+m}\big) \\ &= \overline{P}^J_{u,v} \mu\big(f_h(q)\big)(q+1)^{m-k} q^{\frac{h-2}{2}+k}\end{aligned}$$

and therefore

$$\begin{aligned}P^J_{u,v} &= \overline{P}^J_{u,v} q^k (1+q)^{m-k}\big(f_h(q) - \alpha + q f_h(q) - \mu\big(f_h(q)\big) q^{\frac{h-2}{2}}\big) \\ &= P^J_{u,v} q^k (1+q)^{m-k}\big(f_{h+1}(q) - \alpha\big)\end{aligned}$$

by Lemma 3.3. Analogously, if h is odd, $h \geq 3$, we have

$$P^J_{u,v} = \overline{P}_{u,v} q^k (1+q)^{m-k}\big(f_h(q) - \alpha + q f_h(q)\big) = \overline{P}_{u,v} q^k (1+q)^{m-k}\big(f_{h+1}(q) - \alpha\big).$$

The cases $h = 1$ and $h = 2$ are similar (note that $f_1(q) - \alpha = 0$ if $\alpha = 1$). Thus the proof is completed. □

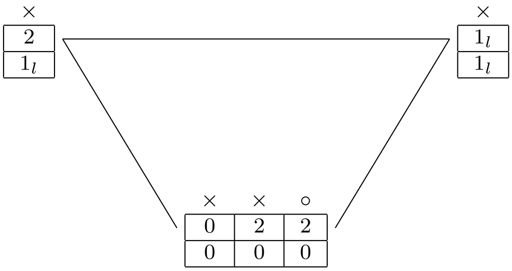
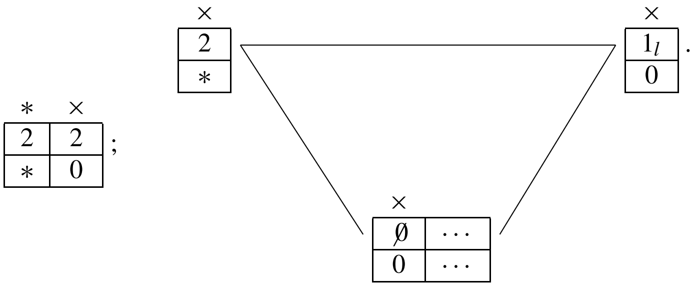


Fig. 3 Diagram of $(\overline{u} = s_0s_4, \overline{v} = s_0s_2s_3s_4s_3s_2s_0)$ in $\widetilde{A}_4$, with boolean reflection $t = s_0s_1s_2s_3s_4s_3s_2s_1s_0$ and $J = \{s_3\}$

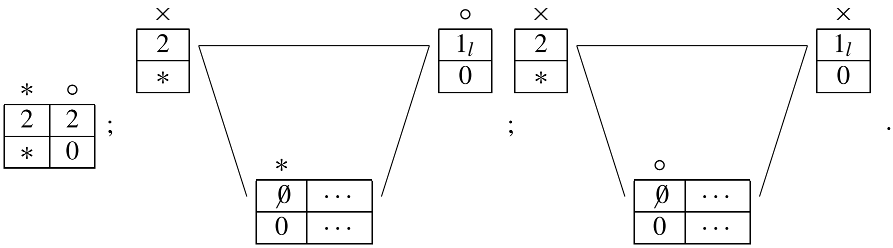
Now we consider the case of $\widetilde{A}_n$ for $n \geq 2$ ($\widetilde{A}_1$ is a tree-Coxeter group). The Coxeter diagram of $\widetilde{A}_n$ is a cycle, therefore we cannot apply Theorem 4.1. However, we use the same arguments of its proof to have an analogue result. Consider a boolean reflection t in $\widetilde{A}_n$ of length $2n+1$. Then it is easy to check that $t = s_{i+1}s_{i+2}\cdots s_ns_0\cdots s_{i-1}s_is_{i-1}\cdots s_0s_n\cdots s_{i+2}s_{i+1}$ for some $i \in [0,n]$ (the indices are modulo $n+1$). For any pair $(u,v) \in W^2$, $u \leq v \leq t$ we depict a diagram whose rightmost column contains $(\overline{u}(s_i), \overline{v}(s_i))$. The leftmost column contains $(\overline{u}(s_{i+1}), \overline{v}(s_{i+1}))$ and the other columns are defined by following the cyclic Coxeter diagram of $\widetilde{A}_n$. See Fig. 3 for an example.

In what follows we assume that $\overline{v}(s_j) \neq 0$ for all $j = 0, \ldots, n$. In fact, otherwise v can be identified as an element in A_n and we can apply Theorem 4.1.

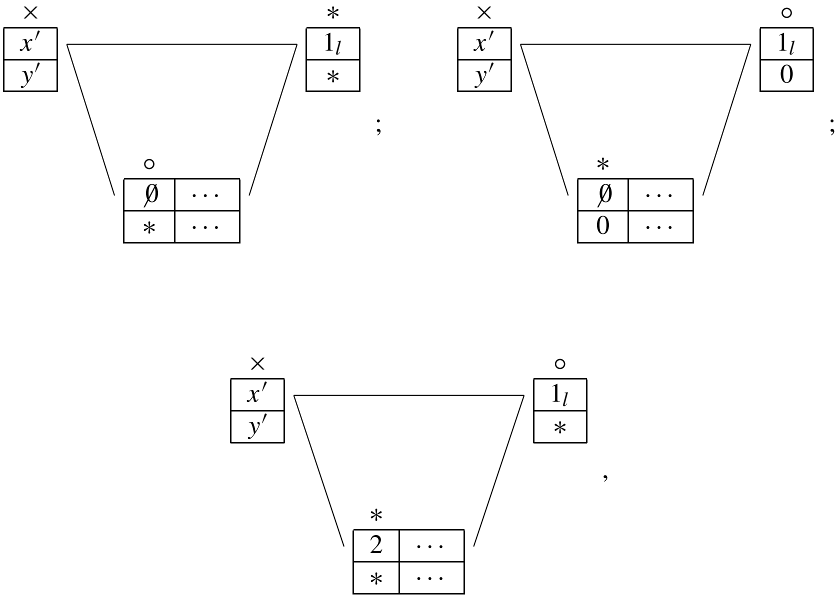
We define $a(\overline{u}, \overline{v})$ to be the number of subdiagrams of the type



We set $b(\overline{u}, \overline{v})$ to be the number of subdiagrams of the type



Finally, we set $c''(\overline{u},\overline{v})$ to be the number of the subdiagrams of the type



where $(x,y),(x',y')\in\{(1_l,0),(1_r,0),(1_r,1_r),(2,1_r)\}$. Moreover, (x',y') could be $(2,0)$ (respectively $(2,1_l)$) if there were a non-zero entry (resp. exactly one non-zero entry with a $\circ$) in the second row of one of the two columns on the right of the first column.

Theorem 4.6 *Let $u,v\in\widetilde{A}_n$ boolean elements. Then*

$$P^J_{u,v}=\begin{cases} q^{b(\overline{u},\overline{v})}(1+q)^{a(\overline{u},\overline{v})} & \text{if } c(\overline{u},\overline{v})+c''(\overline{u},\overline{v})=0,\\ 0 & \text{otherwise.}\end{cases}$$

The proof is the same as in Theorem 4.1. Delete the leftmost column if it contains $(1_*,0),(1_*,1_*),(2,2)$ by using Lemmas 3.2 and 3.1. If it contains the pair $(2,\not 2)$ then consider the cases with the second entries of both column on the right to be zero and non-zero. In the first case apply Proposition 2.2 and note that $\widetilde{M}_{u,v}=0$. We left to the reader all the details. Note that $c'(\overline{u},\overline{v})$ does not appear in the statement.

Remark 4.7 For the classical Kazhdan–Lusztig polynomials, Theorem 4.6 reduces to [21, Theorem 4.4].

Corollary 4.3 has another combinatorial reformulation. Let W be a Coxeter group and $v\in W$. Define the *maximal singular locus* for v to be the set

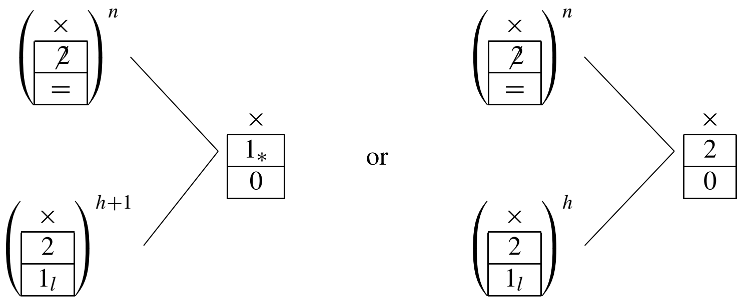
$$\mathrm{RS}(v)=\big\{u\in W\,|\,P_{u,v}(q)>1,\ P_{w,v}(q)=1\ \forall w,\ u<w<v\big\}. \qquad (7)$$

Equivalently, it is the set of u which are maximal in Bruhat order among permutations with $P_{u,v}(q) > 1$. For type A the set $\mathrm{RS}(v)$ has a combinatorial interpretation due to Billey–Warrington [1], Cortez [8], Kassel–Lascoux–Reutenauer [15] and Manivel [19].

Corollary 4.8 *Let W be a tree-Coxeter group and let $v \in W$ be a boolean element. Then*

$$P_{u,v}(q) = \prod_{w \geq u, w \in \mathrm{RS}(v)} P_{w,v}(q).$$

Proof By Corollary 4.3 it is not hard to see that for any boolean element $v \in W$ the set $\mathrm{RS}(v)$ contains all elements $u \leq v$ such that the diagram of (u, v) has exactly one occurrence of



and all entries in each other column of the diagram are equal (in the previous subdiagram the symbol $=$ denotes the same value of the above entry). The claim follows. □

Note that the previous result is not true for parabolic case. Let $u, v \in A_4^J$ be given by the following diagram

×	×	○	×
2	2	2	2
0	0	0	0

By Theorem 4.1, $P^J_{u,v}(q) = q(1+q)$. Let $\mathrm{RS}^J(v)$ be given with the same rules in (7). Then it is easy to check that $\mathrm{RS}^J(v) = \{w\}$ where w is the element such that the diagram of (w, v) is

×	×	○	×
2	2	2	2
2	1	0	1

But $P^J_{w,v}(q) = q \neq P^J_{u,v}(q)$.

5 Kazhdan–Lusztig polynomials of boolean signed permutation

In this section we consider the combinatorial interpretation of the finite Coxeter groups A_n, B_n and D_n as (signed) permutations and restate Theorem 4.1 by using statistics of such permutations. We recall (see e.g. [2, Chaps. 1, 8]) that A_n is the group of permutations of the set $\{1,\dots,n+1\}$, B_n is the set of permutations π of $\{-n,-n+1,\dots,-1,1,\dots,n-1,n\}$ such that $\pi(-i)=-\pi(i)$ for all $i\le n$, and D_n is the subset of permutations $\pi\in B_n$ such that the cardinality $\#(\{\pi(1),\dots,\pi(n)\}\cap\{-1,\dots,-n\})$ is even. Note that each permutation π of A_n, B_n and D_n is uniquely determined by $[\pi(1),\dots,\pi(n)]$. We call this sequence the *window notation* of π.

Given a (signed) permutation π, if $\pi(i)>i$ we say that $\pi(i)$ is a *top exceedance* and i is a *bottom exceedance* of π.

It is well known that the set of all reflections in A_n is given by transpositions (i,j), with $i<j\le n+1$. Any such transposition admits $s_is_{i+1}\cdots s_{j-2}s_{j-1}s_{j-2}\cdots s_{i+1}s_i$ as reduced expression. So every reflection in the symmetric group is boolean and an element π is boolean if and only if it is smaller than the top transposition $(1,n+1)$, i.e. π admits a reduced expression which is a subword of $s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_1$.

Lemma 5.1 *Let $\pi\in A_n$. Then π is a boolean element if and only if $\#(\pi(\{1,\dots,i\})\cap\{1,\dots,i\})\ge i-1$ for all $i\le n$.*

Moreover, if $\overline{\pi}$ is the reduced expression of π (as defined at the beginning of Sect. 4), subword of $s_1\cdots s_n\cdots s_1$, then $\overline{\pi}(s_i)=1_l$ if $i+1$ is a top exceedance of π; $\overline{\pi}(s_i)=1_r$ if $i+1$ is a top exceedance of π^{-1}; $\overline{\pi}(s_i)=2$ if and only if $\pi(i+1)=i+1$ and $\pi(\{1,\dots,i\})\ne\{1,\dots,i\}$; $\overline{\pi}(s_i)=0$ if and only if $\pi(\{1,\dots,i\})=\{1,\dots,i\}$.

Proof The first part is an immediate consequence of the Tableau Criterion for the Bruhat order (see e.g. [2, Theorem 2.6.3]).

We prove the second part. Fix an index $i\le n$. Let $\overline{\pi}'$ be the subword of $\overline{\pi}$ with only letters $s_{i+1},\dots,s_n$. Then $s_i\overline{\pi}'(i)=i+1$. If we multiply $s_i\overline{\pi}'$ by s_j, $j<i$, on the left or on the right, then the element $i+1$ may be moved on the left in the windows notation. Therefore $i+1$ is a top exceedance of π if $\overline{\pi}(s_i)=1_l$. Since for any element whose expression is $s_{j_1},\dots,s_{j_n}$ the inverse is given by $s_{j_n},\dots,s_{j_1}$, then it follows that $i+1$ is a top exceedance of π^{-1}, if $\overline{\pi}(s_i)=1_r$. The third case is similar since $s_i\overline{\pi}'s_i(i+1)=i+1$. The last case is trivial. □

Given $\pi\in A_n$, we define the following sets.

$$\mathrm{TExc}(\pi)=\bigl\{i\in[n]\,|\,i+1 \text{ is a top exceedance for } \pi\bigr\};$$
$$\mathrm{Fix}(\pi)=\bigl\{i\in[n]\,|\,\pi\bigl([i]\bigr)=[i]\bigr\};$$
$$\mathrm{NFix}(\pi)=\bigl\{i\in[n]\setminus\mathrm{Fix}(\pi)\,|\,\pi(i+1)=i+1\bigr\}.$$

Then by Theorem 4.1 and Lemma 5.1 we have

Corollary 5.2 *Let $\pi,\rho\in A_n^J$ be two boolean permutations of $[n+1]$ such that $\pi\le\rho$ in the Bruhat order. Then the Kazhdan–Lusztig polynomial $P^J_{\pi,\rho}$ is zero if and only if there exists an index $i\le n$ such that one of the following conditions is satisfied (we identify each $s_i\in J$ with i):*

– $i \in \mathrm{TExc}(\rho) \cap \mathrm{Fix}(\pi)$ *and* $i+1 \in J \cap \mathrm{NFix}(\rho)$;
– $i, i+1 \in \mathrm{TExc}(\rho) \cap \mathrm{Fix}(\pi)$ *and* $i+1 \in J$;
– $i \in \mathrm{TExc}(\rho^{-1}) \cap J$, $i, i+1 \in \mathrm{Fix}(\pi)$, *and* $i-1 \notin \mathrm{TExc}(\pi) \cap \mathrm{TExc}(\rho)$;
– $i, i+1 \in \mathrm{NFix}(\rho) \cap \mathrm{TExc}(\pi^{-1})$ *or* $i, i+1 \in \mathrm{NFix}(\rho) \cap \mathrm{TExc}(\pi)$ *and* $i+1 \in J$;
– $i, i+1 \in \mathrm{NFix}(\rho)$, $\#(\{i, i+1\} \cap \mathrm{TExc}(\pi^{-1})) = 1$, $\#(\{i, i+1\} \cap \mathrm{Fix}(\pi)) = 1$ *and* $i+1 \in J$.

In all other cases, let

$$A_{\pi,\rho} = \{ i \in [n] | i, i+1 \in \mathrm{NFix}(\rho), i+1 \in \mathrm{Fix}(\pi) \}. \tag{8}$$

Then

$$P^J_{\pi,\rho} = q^{\#(A_{\pi,\rho} \cap J)} (1+q)^{\#(A_{\pi,\rho} \cap (S \setminus J))}.$$

For example, let $\pi, \rho \in A_9$ defined by $\pi = [2,1,3,6,4,7,5,8,9,10]$ and $\rho = [4,2,3,10,5,6,7,8,1,9]$. By Lemma 5.1 we have that π, ρ are boolean elements. Since the descents of $\pi^{-1} = [2,1,3,5,7,4,6,8,9,10]$ are $1,5$ and the descents of $\rho^{-1} = [9,2,3,1,5,6,7,8,10,4]$ are $1,3,9$, then π, ρ are both in A^J_n for all J such that $J \cap \{s_1, s_3, s_5, s_9\} = \emptyset$. By [2, Theorem 2.1.5] we get $\pi \leq \rho$ and finally by Corollary 5.2 we have $P^J_{\pi,\rho} = 0$, if and only if $J \cap \{s_4, s_6, s_8\} \neq \emptyset$. In fact, $\mathrm{TExc}(\pi) = \{1,5\}$, $\mathrm{TExc}(\pi^{-1}) = \{1,4,6\}$, $\mathrm{TExc}(\rho) = \{3,9\}$, $\mathrm{TExc}(\rho^{-1}) = \{8,9\}$, $\mathrm{Fix}(\pi) = \{2,3,7,8,9\}$, $\mathrm{Fix}(\rho) = \emptyset$, $\mathrm{NFix}(\pi) = \emptyset$ and $\mathrm{NFix}(\rho) = \{2,3,5,6,7,8\}$. For $J = \{s_2, s_4\} \equiv \{2,4\}$ we have $P^J_{\pi,\rho} = q(q+1)$.

For the ordinary Kazhdan–Lusztig polynomials Corollary 5.2 becomes

Corollary 5.3 *Let $\pi, \rho \in A_n$ be two boolean permutations of $[n+1]$ such that $\pi \leq \rho$ in the Bruhat order. Then $P_{\pi,\rho} = (1+q)^{\#A_{\pi,\rho}}$, where $A_{\pi,\rho}$ is defined in* (8).

Now we consider the Coxeter group B_n. It is easy to check that there are two boolean reflections in B_n which are maximal in the Bruhat order: they are $s_0 s_1 \cdots s_{n-1} \cdots s_1 s_0$ and $s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$, where s_0 is the transposition $(1,-1)$ and s_i is the product $(i, i+1)(-i, -i+1)$ in disjoint cycle notation (equivalently $(s_0 s_1)^4 = \epsilon$ and $(s_i s_{i+1})^3 = \epsilon$ for $i > 0$). In fact, given any boolean word t, if there is a letter s_1 between two occurrences of s_0 then move both elements s_0 to the beginning and to the end of t (it is possible since s_0 commutes with all other elements) and then manipulate the remaining letters as a subword in A_{n-1}; if s_0 is between two occurrences of s_1 (and therefore there is exact one s_0) then necessarily there are no occurrences of s_{i+1} between the two letters s_i for all $i \geq 1$, otherwise t is not a reduced word.

Lemma 5.4 *Let $t_1, t_2 \in B_n$, $t_1 = s_0 \cdots s_{n-1} \cdots s_0$, $t_2 = s_{n-1} \cdots s_0 \cdots s_{n-1}$. Let $\pi \in B_n$. Then π is a boolean element $\pi \leq t_1$ if and only if $\#(|\pi([i])| \cap [i]) \geq i-1$ for all $i \leq n$ and the only negative elements in the window notation of π may be the first entry or the element -1.*

Moreover, in this case, if $\overline{\pi}$ is the reduced word of π, which is a subword of t_1, then $\overline{\pi}(s_i) = 1_l$ if $i+1$ is a top exceedance of π (if $i = 0$ then the window notation of π has only one negative entry which is -1); $\overline{\pi}(s_i) = 1_r$ if $i+1$ is a top exceedance *of π^{-1} (if $i=0$ then the window notation of π has only one negative entry in the first place); $\overline{\pi}(s_i)=2$ if and only if $\pi(i+1)=\pi(i+1)$ and $\pi([i+1,n])\neq[i+1,n]$ (if $i=0$ then there are exactly two negative entries in the window notation of π); $\overline{\pi}(s_i)=0$ if and only if $\pi([i+1,n])=[i+1,n]$ (if $i=0$ then there is no negative element in the window notation of π).*

The permutation π is a boolean element $\pi\leq t_2$ if and only if $\#(|\pi([i])|\cap[i])\geq i-1$ and the only negative entry in the window notation of π (if it exists) is the element $-m-1$ or the element in the $(m+1)$th entry, if $\pi(i)=i$ for all $i\leq m$ and $\pi(m+1)\neq m+1$.

Moreover, in this case, if $\overline{\pi}$ is a reduced word of π, which is a subword of t_2 then for all $i\geq 1$, $\overline{\pi}(s_i)=1_l$ if i is a bottom exceedance of π^{-1}; $\overline{\pi}(s_i)=1_r$ if $i+1$ is a bottom exceedance of π; $\overline{\pi}(s_i)=2$ if and only if $\pi(i)=\pi(i)$ and $\pi([i+1,n])\neq[i+1,n]$; $\overline{\pi}(s_i)=0$ if and only if $\pi([i+1,n])=[i+1,n]$.

The proof is essentially the same as that of Lemma 5.1. We give the Corollary of Theorem 4.1 only for ordinary Kazhdan–Lusztig polynomials. The parabolic case could be done as in Corollary 5.2.

Let $\pi\in B_n$. We set

$$\mathrm{Fix}(\pi)=\big\{i\in[0,n-1]|\pi\big([i+1,n]\big)=[i+1,n]\big\},$$
$$\mathrm{NFix}(\pi)=\big\{i\in[n-1]\setminus\mathrm{Fix}(\pi)|\pi(i)=i\big\}\cup\big\{0 \text{ if } \#\{\pi(i)<0\}=2\big\}.$$

Corollary 5.5 *Let $\pi,\rho\in B_n$ two boolean elements in B_n such that $\pi\leq\rho$ in the Bruhat order. Then the Kazhdan–Lusztig polynomial $P_{\pi,\rho}$ is given by*

$$P_{\pi,\rho}=\begin{cases}(1+q)^{B_{\pi,\rho}} & \text{if } \pi\leq\rho\leq t_1,\\ (1+q)^{B'_{\pi,\rho}} & \text{if } \pi\leq\rho\leq t_2,\end{cases}$$

where $B_{\pi,\rho}=\{i\in[0,n-1]|i,i+1\in\mathrm{NFix}(\rho),i+1\in\mathrm{Fix}(\pi)\}$, $B'_{\pi,\rho}=\{i\in[0,n-1]|i,i+1\in\mathrm{NFix}(\rho),i\in\mathrm{Fix}(\pi)\}$.

Note that intervals from the identity to t_1 and from the identity to t_2 are isomorphic to the interval from the identity to the maximal transposition in type A. The ordinary Kazhdan–Lusztig polynomial indexed by u and v depends only on the interval from the identity to v (see [5]). The Kazhdan–Lusztig polynomials of type B can be computed identifying the indexing boolean elements with boolean elements in the symmetric group (and we may apply Corollary 5.3).

Now we consider the Coxeter group D_n. It is easy to check that the only boolean reflection of length $2n-1$ is $s_0s_1s_2\cdots s_{n-1}s_{n-2}\cdots s_2s_1s_0$, where s_0 is the transposition $(1,-2)(-1,2)$ and s_i is the product $(i,i+1)(-i,-i+1)$ in disjoint cycle notation (equivalently $(s_0s_1)^2=\epsilon$, $(s_0s_2)^3=\epsilon$ and $(s_is_{i+1})^3=\epsilon$ for $i>0$): in fact, let t any boolean reflection with the same length. Then any reduced word of t contain both occurrences of s_0 or s_1 outside the occurrences (maybe only one) of s_2. Then move, by commutativity, these occurrences to the leftmost and rightmost place. The central part can be identified with an element of A_{n-1} and we can conclude easily.

Lemma 5.6 *Let $\pi \in D_n$. Then π is a boolean element if and only if $\#(|\pi([i])| \cap [i]) \geq i-1$ for all $i \leq n$ and the only negative elements in the window notation are in the first two columns and in the entries containing $-1, -2$ (if the first two entries are not $\pm 1, \pm 2$ then these have the same sign).*

Let $\overline{\pi}$ be a reduced word of π, subword of $s_0 s_1 \cdots s_{n-1} \cdots s_1 s_0$, and let $i \geq 2$. Then $\overline{\pi}(s_i) = 1_l$ if $i+1$ is a top exceedance of π; $\overline{\pi}(s_i) = 1_r$ if $i+1$ is a top exceedance of π^{-1}; $\overline{\pi}(s_i) = 2$ if $\pi(i+1) = i+1$ and $\pi([i+1,n]) \neq [i+1,n]$; $\overline{\pi}(s_i) = 0$ if $\pi([i+1,n]) = [i+1,n]$. If $i \leq 1$ then there is an occurrence of s_1 on the right if $\pi(1) \geq 3$ or $\pi(2) \leq -3$; there is an occurrence of s_1 on the left if $\pi^{-1}(1) \geq 3$ or $\pi^{-1}(2) \leq -3$ or $\pi(1,2) \in \{(2,1),(-1,-2)\}$; there is an occurrence of s_0 on the right if $\pi(1) \leq -3$ or $\pi(2) \leq -3$; there is an occurrence of s_0 on the left if $\pi^{-1}(1) \leq -3$ or $\pi^{-1}(2) \leq -3$ or $\pi(1,2) \in \{(-2,-1),(-1,-2)\}$.

Corollary 5.7 *Let $\pi, \rho \in D_n$ be two boolean elements such that $\pi \leq \rho$. Then the Kazhdan–Lusztig polynomial $P_{\pi,\rho}$ is given by*

$$P_{\pi,\rho}(q) = (1+q)^{D_{\pi,\rho}}(1+2q)^{D'_{\pi,\rho}},$$

where $D_{\pi,\rho}$ is the number of indices i such that $\rho(i) = i$, $\rho(i+1) = i+1$, $\rho([i+2,n]) \neq [i+2,n]$ and $\pi([i+2,n]) = [i+2,n]$ incremented by 1 if $\rho(1) = 1$, $\rho(2) < 2$, $\rho(3) \neq 3$ and $\pi((1,2)) \neq (-1,-2)$ or $\rho^{-1}(2) < -2$, $|\rho^{-1}(1)| = 2$ and $\pi([3,n]) = [3,n]$ or $|\rho^{-1}(1)| > 2$, $\rho(2) \in \{-n,\dots,-3,-1,1\}$ and $\pi([3,n]) = [3,n]$; $D'_{\pi,\rho}$ is 1 if $\rho(1) = 1$, $\rho(2) < 2$, $\rho(3) = 3$ and $\pi([3,n]) = [3,n]$ and $D'_{\pi,\rho} = 0$ in all other cases.

Note that interval from the identity to $s_0 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_0$ in D_n is isomorphic to the interval from the identity to the maximal transposition in type A. We can apply the same considerations immediately below Corollary 5.5.

6 Comparison with maximally-clustered permutations

In a recent paper [14] Jones applied the Deodhar rule to the case of a maximally-clustered hexagon-avoiding permutations in S_n. Here we recall the definitions and the results necessary to show that a boolean element in S_n is a maximally-clustered hexagon-avoiding permutations and have a comparison between Jones's results and Theorem 4.1.

Suppose that $\sigma = [\sigma_1, \dots, \sigma_n] \in S_n$ and $\rho = [\rho_1, \dots, \rho_k] \in S_k$ for $k \leq n$. We say that σ *contains the permutation pattern* ρ whenever there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$\sigma_{i_a} < \sigma_{i_b} \quad \text{if and only if } \rho_a < \rho_b$$

for all $1 \leq a < b \leq k$. For example, $[\underline{4}5\underline{3}1\underline{2}]$ contains the pattern [321] in several ways, including the underlined subsequence. If σ does not contain the pattern ρ, we say that σ avoids ρ.

Let w an element of a Coxeter group W and fix a reduced expression $w = w_1 w_2 \cdots w_k$. Define a *mask* α associated with the reduced expression w to be any binary vector $(\alpha_1, \dots, \alpha_k)$ of length $k = l(w)$. Every mask corresponds to a subexpression of w defined by $w^\alpha = w_1^{\alpha_1} \cdots w_k^{\alpha_k}$ where

$$w_j^{\alpha_j} = \begin{cases} w_j & \text{if } \alpha_j = 1, \\ \epsilon & \text{if } \alpha_j = 0. \end{cases}$$

We say that a position j (for $2 \leq j \leq k$) of the fixed reduced expression w is a *defect* with respect to the mask α if

$$l\big(w_1^{\alpha_1} \cdots w_{j-1}^{\alpha_{j-1}} w_j\big) < l\big(w_1^{\alpha_1} \cdots w_{j-1}^{\alpha_{j-1}}\big).$$

Note that the defect status of position j does not depend on the value of α_j. We denote the number of all defects of the mask α by $d(\alpha)$.

Definition 6.1 Let $s_1, \dots, s_{n-1}$ be the canonical generators of S_n. A *braid cluster* is an expression of the form

$$s_{i_1} s_{i_2} \cdots s_{i_k} s_{i_{k+1}} s_{i_k} \cdots s_{i_2} s_{i_1},$$

where each s_{i_p} for $1 \leq p \leq k$ has only one s_{i_q} with $p < q \leq k+1$ such that $|i_p - i_q| = 1$.

Let σ be a permutation and let $N(\sigma)$ denote the number of [321] pattern instances in σ. We say that σ is *maximally clustered* if there is a reduced expression for σ of the form

$$a_0 c_1 a_1 c_2 a_2 \cdots c_M a_M,$$

where each a_i is a reduced expression, each c_i is a braid cluster with length $2n_i + 1$ and $N(\sigma) = \sum_{i=1}^{M} n_i$. Such an expression is called *contracted*.

The maximally clustered permutations are characterized by avoiding the permutation patterns

$$[3421], \qquad [4321] \quad \text{and} \quad [4321].$$

By Lemma 5.1 it follows that all boolean elements are maximally clustered.

Let σ be a contracted expression for a maximally clustered permutation, where each braid cluster has the form $s_{m+1} s_{m+2} \cdots s_{m+k} s_{m+k+1} s_{m+k} \cdots s_{m+1}$ (in [14, Lemma 6] it is shown that it is always possible to find any such decomposition). We say that a mask is 10*-*avoiding* if it never has the values 1 and 0 (respectively) on the first two entries in any central braid. If $\alpha_{m+k} = 1$ and $\alpha_{m+k+1} = 0$ on some central braid instance $s_{m+k} s_{m+k+1} s_{m+k}$. Otherwise we say that σ is a 10*-*avoiding* mask for σ.

Let $\rho_i = s_{i+3} s_{i+2} s_{i+1} s_{i+5} s_{i+6} s_{i+7} s_{i+4} s_{i+3} s_{i+2} s_{i+5} s_{i+6} s_{i+4} s_{i+3} s_{i+5} \in S_k$, with $k \geq i+7$. We say that a permutation $\sigma \in S_n$ is *hexagon-avoiding* if $\rho_i \not\leq \sigma$ for any i (this is not the original definition, see e.g. [14] and references cited there for a general definition, including other Coxeter groups).

Note that $l(\rho_i) = 14$ for all i and that any subword with length 14 of a boolean element has necessarily at least 8 distinct elements. Therefore all boolean elements of type A are hexagon-avoiding.

The following result was proved in [14, Corollary 9].

Theorem 6.2 *Let σ be a contracted expression for a maximally-clustered hexagon-avoiding permutation in S_n, and let $\mathcal{E}_\sigma$ be the set of 10^*-avoiding mask on σ. Then for any $\rho \in S_n$,*

$$P_{\rho,\sigma}(q) = \sum_{\alpha \in \mathcal{E}_\sigma, \sigma^\alpha = \rho} q^{d(\alpha)}.$$

Since all boolean elements of type A are maximally-clustered and hexagon-avoiding, it is possible to apply Theorem 6.2 to them.

Theorem 4.1 asserts that in type A (non-parabolic case) the only non-constant factors in the Kazhdan–Lusztig polynomials of two elements are given by all subdiagrams of the form

$$\begin{array}{cc} \times & \times \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{0} \\ \hline \end{array} . \tag{9}$$

We assume that s_1 and s_2 are the elements associated with the two columns of the diagram. Let $s_1\sigma's_1$ be a reduced expression of a boolean permutation σ with $l(\sigma) = k$. For any mask $\alpha \in \mathcal{E}_\sigma$ we denote by α' the submask of α given by $\alpha'_i = \alpha_{i+1}$ for $i \le k-1$. Let π be a permutation, $\pi \le \sigma$ and denote by π' be the greatest element such that $\pi' \le \pi$ and $\pi' \le \sigma'$. Then it is clear that if $\alpha \in \mathcal{E}_\sigma$ is such that $\sigma^\alpha = \pi$ then $\sigma'^{\alpha'} = \pi'$.

Now, if $s_1 \not\le \pi$ then any such α is of the form $(0, \alpha', 0)$ (in this case $d(\alpha) = d(\alpha')$) and $(1, \alpha', 1)$ (in this case $d(\alpha) = 1 + d(\alpha')$): the second case is possible only if $s_2 \not\le \pi$. If $s_1 \le \pi$ and $s_2 \not\le \pi$ then a mask α with $\sigma^\alpha = \pi$ is of the form $(1, \alpha', 0)$ (in this case $d(\alpha) = d(\alpha') + 1$) and $(0, \alpha', 1)$ (in this case $d(\alpha) = d(\alpha')$).

Therefore, a diagram such as in (9) contributes in the computation of the Kazhdan–Lusztig polynomial with a factor $(1 + q)$. In all other cases the mask α is uniquely determined by its submask α' and $d(\alpha) = d(\alpha')$. This concludes the comparison between Theorems 4.1 and 6.2 for boolean elements in type A.

7 Poincaré polynomials

Given $v \in W$, let $F_v(q) = \sum_{u \le v} q^{l(u)} P_{u,v}$. It is well known that, if W is any Weyl or affine Weyl group, $F_v(q)$ is the intersection homology Poincaré polynomial of the Schubert variety indexed by v (see [17]). In this section we compute the Poincaré polynomial for any boolean element in a Coxeter group whose Coxeter graph is a tree with at most one vertex having more than two adjacent vertices (such groups include all classical finite Coxeter and affine Weyl groups except $\widetilde{A}_n$ and $\widetilde{D}_n$).

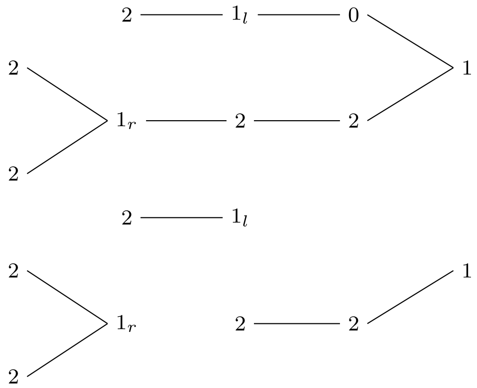


Fig. 4 An example of diagram and its three essential components. It is depicted with the same rules used for all elements in the first rows of a diagram like in Fig. 2

Let $v \in W$ be a boolean element and consider the diagram of $(\epsilon_W, \overline{v})$. For convenience we will not depict the second row of each column which is always 0 and we omit all symbols $\times$. We will call it the diagram of v.

Let v be a boolean element and let s be the element of S associated with one of the leftmost vertices in the diagram of v. We set $F^{\setminus}_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) \neq 0$ and $F^{0}_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) = 0$.

Now consider a diagram d. Delete all entries equal to 0 and delete all edges whose left vertex is not a cell containing 2. Let $d_1, \dots, d_k$ be the remaining connected components. We refer to them as the *essential components* of d. In Fig. 4 there is an example of essential components of a diagram.

Lemma 7.1 *Let $v \in W$ be a boolean element and let d be the diagram of $\overline{v}$. Let $d_1, \dots, d_k$ be the essential components of the diagram d and $v_1, \dots, v_k$ be the boolean reflections corresponding to $d_1, \dots, d_k$. Then*

$$F_v(q) = \prod_{i=1}^{k} F_{v_i}(q).$$

Proof We use induction on $l(v)$. If $l(v) = 1$, there is nothing to prove. Now let $l(v) > 1$ and let $d_1, \dots, d_k$ be the essential components of d associated with v. Let s be the element associated with one of the leftmost vertices of v and let d_1 be the essential component containing such vertex. In this proof we denote by $\overline{F}_v(q)$ the polynomial corresponding to the diagram $d \setminus d_1$ (known by induction) and by $\widehat{F}_v(q)$ the polynomial corresponding to d_1. If $\overline{v}(s) = 1$ then by Theorem 4.1 or by Lemmas 3.2 and 3.1 and recursion in Proposition 2.2 we have that

$$F_v(q) = (1+q)\overline{F}_v(q), \qquad F^{\setminus}_{v,s}(q) = q\overline{F}_v(q), \qquad F^{0}_{v,s}(q) = \overline{F}_v(q).$$

If $\overline{v}(s) = 2$ then we can assume that d_1 starts with

$$(2)^h \;\text{———}\; *,$$

for $h \geq 1$ (otherwise choose another element in S). The previous diagram is depicted according to the same conventions used for example in (3).

Denote by s' the only element on the right of s. By Theorem 4.1 and by induction we have

$$\begin{aligned} F_v(q) &= (1+q)^{2h} F'^{\setminus}_{v,s'}(q) + (1+q)^h f_{h-\delta} F'^{0}_{v,s'}(q) \\ &= (1+q)^{2h} \widehat{F'}^{\setminus}_{v,s'}(q)\overline{F}_v(q) + (1+q)^h f_{h-\delta} \widehat{F'}^{0}_{v,s'}(q)\overline{F}_v(q) \\ &= \widehat{F'}_v(q)\overline{F}_v(q), \end{aligned}$$

where F'_v is the polynomial associated with d after deleting all the vertices $(2)^h$, and δ is determined uniquely by v (and Theorem 4.1). The first factor $(1+q)^{2h}$ denotes the possibility to have all pairs $(2,0)$, $(2,1_l)$, $(2,1_r)$ and $(2,2)$ in the diagram of $(\overline{u},\overline{v})$ in all h leftmost columns; the second factor $(1+q)^h$ denotes the possibility to have only the pairs $(2,0)$ and $(2,1_l)$. Similar formulas can be computed for $F^{\setminus}_{v,s}(q)$ and $F^{0}_{v,s}$. Therefore we can apply the induction (it is possible that more superscripts $\setminus$ or 0 are necessary; the proof does not change). □

Lemma 7.1 shows that it is simple to compute $F_v(q)$ for any boolean elements v by knowing $F_t(q)$ for all boolean reflection t.

Lemma 7.2 *Let $v \in W$ be a boolean reflection and suppose that its diagram d has one leftmost vertex s such that if*

$$(2)^h \;\text{———}\; *$$

is a subdiagram of d containing s, then necessarily $h = 1$. Then

$$F_v(q) = (1+q)^2 F'_v(q),$$

where $F'_v(q)$ is the polynomial associated with diagram d after deleting the vertex s.

Proof By Proposition 2.2, it is easy to check that

$$F_v(q) = (1+q)^2 F'^{\setminus}_{v,s} + (1+q) F'^{0}_{v,s}(q)(1+q) = (1+q)^2 F'_v(q),$$

where the last factor $(1+q)$ is due to the contribution of $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline * & 0 \\ \hline\end{array}$ in the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ according to Theorem 4.1. □

As corollary of Lemmas 7.1 and 7.2 we have the following result due to Marietti [21, Theorem 8.1]

Corollary 7.3 *Let $v \in S_{n+1}$ be a boolean element. Let t be the boolean reflection $s_1 \cdots s_n \cdots s_1$ with s_i be the transposition $(i, i+1)$. Let $\overline{v}$ be the reduced word of v subword of t. Then*

$$F_v(q) = (1+q)^{l(v)-2a(v)}\left(1+q+q^2\right)^{a(v)},$$

where $a(v)$ is the number of patterns $(2, 1_*)$ in $\overline{v}$.

By Lemmas 7.1 and 7.2 its proof reduces to compute $F_{s_1s_2s_1}(q) = (q^2+q+1)\times(1+q)$ and $F_{s_1} = (1+q)$ in S_3.

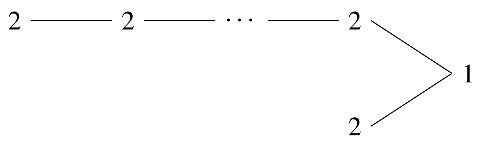
To prove the next result we have to compute the polynomials $F^{\setminus}_{v,s}(q)$ and $F^{0}_{v,s}(q)$ with v associated with the diagram 2 —— 2 —— ··· —— 1 with i vertices. Let $s \in S$ be the element corresponding to the first vertex and let $s' \in S$ be the element associated with the second vertex. If $i = 2$ then by direct computation we have

$$F^{\setminus}_{v,s}(q) = q(1+q)^2 \qquad F^{0}_{v,s}(q) = (1+q).$$

By induction it is easy to compute that

$$\begin{aligned} F^{\setminus}_{v,s} &= \left(2q+q^2\right)F'^{\setminus}_{v,s'}(q) + qF'^{0}_{v,s'}(q)(1+q) = q(1+q)^{2i-2}, \\ F^{0}_{v,s} &= F'^{\setminus}_{v,s'}(q) + F'^{0}_{v,s'}(q)(1+q) = (1+q)^{2i-3}, \end{aligned} \tag{10}$$

where $F'_v(q)$ denotes, as usual, the polynomial associated with the diagram without the first vertex. Similarly, let v be the boolean reflection corresponding to the diagram



with $i+1$ vertices. Then

$$F^{\setminus}_{v,s} = q(1+q)^{2i} \quad \text{and} \quad F^{0}_{v,s} = (1+q)^{2i-1}. \tag{11}$$

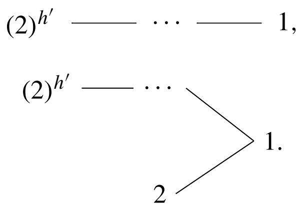
Proposition 7.4 *Let W be a Coxeter group such that its Coxeter graph is a tree and all vertices except at most one have degree less than 3. Denote by w such an exceptional vertex. Let $v \in W$ be a boolean element. Then*

$$F_v(q) = \left(1+q+q^2\right)^{k-1}\left(q(1+q)^{h+1} + f_h(q)\right)(1+q)^{l(v)-2k-h-2},$$

where k is the number of essential components of the diagram d of v with at least two vertices and h is the number of entries equal to 2 in the adjacent cells of w (also consider the cell on the right).

The formula is also true when there is no vertex of degree greater than 2: in this case let w be any vertex of degree 2.

Proof By Lemma 7.1 it suffices to compute the polynomial associated with the only non-trivial component. By Lemma 7.2 it suffices to consider only the following two cases.



In the first case we compute

$$\begin{aligned} F_v(q) &= (1+q)^{2h'} F'^{\setminus}_{v,s'}(q) + (1+q)^{h'} f_h F'^{0}_{v,s'}(q) \\ &= (1+q)^{h'+2i-3}\big(q(1+q)^{h+1} + f_h\big) \quad \text{by (10)}, \end{aligned}$$

where $F'_v(q)$ is the polynomial associated with the diagram without the h' leftmost cells and i is an integer. The second case is similar; use (11). □

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