

Critical groups of graphs with reflective symmetry

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Abstract The critical group of a graph is a finite Abelian group whose order is the number of spanning forests of the graph. For a graph G with a certain reflective symmetry, we generalize a result of Ciucu–Yan–Zhang factorizing the spanning tree number of G by interpreting this as a result about the critical group of G . Our result takes the form of an exact sequence, and explicit connections to bicycle spaces are made.

Keywords Critical group · Graph Laplacian · Spanning trees · Graph involution · Bicycle space · Involution

1 Introduction and statement of results

A graph with *reflective symmetry* is a graph $G = (V, E)$ with a distinguished, non-degenerate drawing in \mathbf{R}^2 such that

- (1) reflection about a line ℓ takes the drawing into itself, and
- (2) every edge that is fixed by this reflection about ℓ is fixed point-wise.

For a graph with reflective symmetry, the reflection of the distinguished drawing gives rise to an involution. This involution will always be denoted ϕ , and is a map $V \rightarrow V$ that induces a map $E \rightarrow E$.

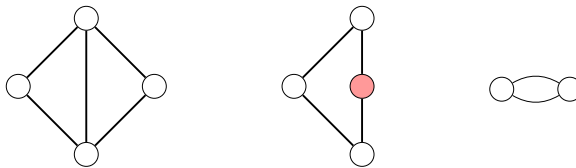
Condition (2) above means that no edge of G crosses the axis of symmetry. Assuming that the line of reflection is vertical, the drawing of G gives rise to a partition of the edges E of G into three blocks: $E = E_L \cup E^\phi \cup E_R$. The sets E_L and E_R denote the edges on the left and right sides of the reflection line ℓ , respectively. The

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set E^ϕ denotes the set of ϕ -fixed edges. Similarly, there is a partition of the vertices of G as $V = V_L \cup V^\phi \cup V_R$.

A subgraph of G is specified by the edges of G it contains, its vertices being the endpoints of the specified edges. A graph $G_+ = (V_+, E_+)$ is obtained from $E_L \cup E^\phi$ by subdividing each ϕ -fixed edge (i.e., the edges in E^ϕ). A graph $G_- = (V_-, E_-)$ is obtained from E_R by identifying its ϕ -fixed vertices to a single vertex.

Example 1.1 Below we have a graph G with reflective symmetry, along with G_+ and G_- (shown left-to-right). The shaded vertex is the one obtained by subdividing the ϕ -fixed edge of G .



This paper will be concerned with the critical group of a graph possessing reflective symmetry. The critical group of a graph G is a finite Abelian group, denoted $K(G)$, whose order is the number of spanning forests of G . The number of spanning forests of a graph G will be denoted $\kappa(G)$.

The critical groups is also known as the *Jacobian group* or *Picard group* of the graph, and is intimately connected with the *Abelian sandpile model* and a *chip firing game* played on the vertices of G . We will define $K(G)$ formally in Sect. 2, and discuss functorial properties in some depth in the appendix. We will not discuss any connections to chip firing games.

A theorem of Ciucu, Yan and Zhang [3] motivates our work.

Theorem 1.2 (Ciucu–Yan–Zhang) *Let G be a planar graph with reflective symmetry, with axis of symmetry ℓ . Then*

$$\kappa(G) = 2^{\omega(G)} \kappa(G_+) \kappa(G_-),$$

where $\omega(G)$ is the number of bounded regions intersected by ℓ .

We offer a generalization of their result at the level of critical groups.

Main Theorem *Let G be a graph with reflective symmetry, and G_+, G_- the two graphs this symmetry gives rise to. There is a group homomorphism*

$$f^* : K(G_+) \oplus K(G_-) \rightarrow K(G),$$

such that $\ker(f^*)$ and $\text{coker}(f^*)$ are all 2-torsion.

The kernel and cokernel of f can be explicitly determined as the space of bicycles in G and $G_+ \cup G_-$, respectively, possessing certain symmetry properties. If G_+ is

connected, then

$$\frac{|K(G)|}{|K(G_+) \oplus K(G_-)|} = \frac{|\text{coker}(f^*)|}{|\text{ker}(f^*)|} = 2^{|V^\phi| - |E^\phi| - 1}.$$

One obtains a version of Theorem 1.2 by taking the orders of the groups in the exact sequence

$$0 \rightarrow \text{ker}(f^*) \rightarrow K(G_+) \oplus K(G_-) \rightarrow K(G) \rightarrow \text{coker}(f^*) \rightarrow 0.$$

Corollary 1.3 *If G is a graph with reflective symmetry and G_+ is connected, then*

$$\kappa(G) = 2^{|V^\phi| - |E^\phi| - 1} \kappa(G_+) \kappa(G_-).$$

The reader will notice that the main theorem really pertains to graphs with an involutive automorphism whose fixed edges have fixed vertices. Indeed any such graph possess a drawing that makes it a graph with reflective symmetry, as we have defined it: Draw the quotient G/ϕ so that its fixed edges lay along a straight line ℓ . This drawing along with its reflection across ℓ yields the desired drawing of G .

This paper is organized as follows. In Sect. 2 we recall the definition of the critical group of a graph. In Sect. 3 we define the group homomorphism alluded to in the Main Theorem. We then identify some basic properties of this map. In Sect. 4 we explicitly identify the kernel and cokernel of the maps, and in Sect. 5 we relate the order of these two objects. Finally, we include an appendix that contains many technical facts on critical groups that are not available in the published literature.

2 Critical groups of graphs

For this section only, $G = (V, E)$ is an arbitrary graph. Orient the edges of G arbitrarily, and form the usual boundary map

$$\partial = \partial(G) : \mathbf{Z}E \rightarrow \mathbf{Z}V.$$

We follow the convention that the negative of an oriented edge $e \in \mathbf{Z}E$ corresponds to that edge with opposite orientation. Thus, if $e = uv$ is an oriented edge, then $uv = -vu$.

Both $\mathbf{Z}E$ and $\mathbf{Z}V$ come with distinguished bases, and hence orthonormal forms. There is, thus, an adjoint (or transpose) map,

$$\partial^t : \mathbf{Z}V \rightarrow \mathbf{Z}E,$$

given by the coboundary operator. The *bond (or cut) space* B of G is the image of ∂^t . The *cycle space* Z of G is the kernel of ∂ . These spaces are free modules and are orthogonal under the above form.

The bond space of G is generated, as a free \mathbf{Z} -module, by the fundamental bonds at the vertices of G , omitting one vertex from each connected component. Given a

vertex of G , this is the element

$$b_G(v) := \sum_{u \sim v} uv \in \mathbf{Z}E.$$

More generally, the fundamental bond of a subset $S \subset V$ is $b_G(S) = \sum_{v \in S} b_G(v)$.

The cycle space of G is generated, as free \mathbf{Z} -module, by oriented circuits of G . That is, if $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\ell \rightarrow v_{\ell+1} = v_1$ is an oriented circuit in G then $\sum_{i=1}^\ell v_i v_{i+1}$ is an element of the cycle space of G , and such elements generate Z .

Following Appendix A, we define the *critical group of G* to be the quotient

$$K(G) := (\mathbf{Z}E)/(Z + B).$$

This is a finite Abelian group whose order is the number of spanning forests of G , which we denote by $\kappa(G)$.

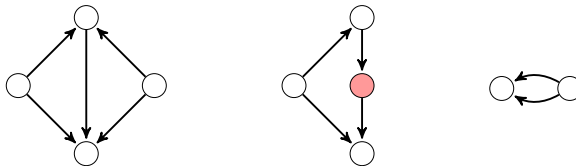
3 A map arising from a reflective symmetry

Let $G = (V, E)$ be a graph with reflective symmetry, and $G_+ = (V_+, E_+)$, $G_- = (V_-, E_-)$ the left and right graphs this gives rise to, as in the introduction. Let ϕ denote the involution determined by the reflective symmetry.

The edges of G_+ come in two flavors: Those that are simply edges from E_L , the left half G , and those that were obtained by subdividing ϕ -fixed edges of G . Edges of G_- correspond uniquely to edges in E_R , the right half of G . We will often abuse notation and identify edges in G_+ or G_- that come from edges in G with the corresponding edge in G . A similar identification will be made for vertices in G_+ and G_- arising from vertices in G .

Orient the edges of G in such a way that ϕ is an involution of directed graphs. In this way we obtain induced orientations of G_+ and G_- .

Example 3.1 Here we display an orientation of G that is ϕ -fixed, and the induced orientations on G_+ and G_- .



Our primary goal in this section is to define a map

$$f^* : K(G_+) \oplus K(G_-) \rightarrow K(G).$$

For this we define a \mathbf{Z} -linear map

$$f : \mathbf{Z}E_+ \oplus \mathbf{Z}E_- \rightarrow \mathbf{Z}E,$$

that will take cycles to cycles and bonds to bonds, and f^* will be the induced map on critical groups. If $e \in E_+$ is an edge obtained from a non-fixed edge of E , define

$$f(e, 0) := e + \phi(e).$$

If $e \in E_+$ is obtained by subdividing an edge $e' \in E$, we set $f(e, 0) := e'$. For an edge $e \in E_-$, which we think of as an edge in E , we set

$$f(0, e) := e - \phi(e).$$

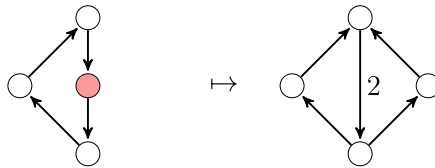
There is the usual adjoint map $f^t : \mathbf{Z}E \rightarrow \mathbf{Z}E_+ \oplus \mathbf{Z}E_-$, characterized by the property that $\langle f(e', e''), e \rangle = \langle (e', e''), f^t(e) \rangle$. Specifically, for $e \in E^\phi$, let e' and e'' be the edges of G_+ this gives rise to. Then $f^t(e) = (e' + e'', 0)$. If $e \in E_L$ then $f^t(e) = (e, -\phi(e))$, and for $e \in E_R$ we have $f^t(e) = (\phi(e), e)$.

Denote the cycle and bond spaces of G_+ by Z_+ and B_+ . Similarly denote the cycle and bond spaces of G_- by Z_- and B_- .

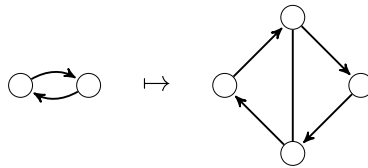
Proposition 3.2 *The map f takes $Z_+ \oplus Z_-$ into Z , and takes $B_+ \oplus B_-$ into B .*

Example 3.3 We illustrate the proposition in our running example. Edges with arrows have coefficient +1 oriented in the indicated direction. Edges with larger coefficients are indicated. Edges without arrows have coefficient zero.

Here we map a cycle z in G_+ to a cycle in G as $f(z, 0)$.



Here we map a cycle z in G_- to a cycle in G as $f(0, z)$.



Proof We leave the proof of the statement about cycles to the reader, confident that the example will guide their proof.

For the second part of the proposition it is sufficient to observe the following. First, if $v \in V_+$ is one of the vertices obtained by subdividing a ϕ -fixed edge of G , then $f(b_{G_+}(v), 0) = 0$. If $v \in V_+$ comes from a ϕ -fixed vertex of G then $f(b_{G_+}(v), 0) = b_G(v)$. If $v \in V_+$ is any other vertex then $f(b_{G_+}(v), 0) = b_G(v) + b_G(\phi(v))$.

We now consider the case of $v \in V_-$. If v is the vertex obtained by contracting $V^\phi \subset V$ to a point then $f(0, b_{G_-}(v)) = b_G(V_L) - b_G(V_R)$. For any other vertex, $f(0, b_{G_-}(v)) = b_G(v) - b_G(\phi(v))$. \square

It follows that f induces a natural map,

$$f^* : K(G_+) \oplus K(G_-) \rightarrow K(G),$$

on the quotient spaces. The adjoint f^t induces a map going the opposite direction

$$(f^t)^* : K(G) \rightarrow K(G_+) \oplus K(G_-).$$

Proposition 3.4 *The kernel and cokernel of f^* are 2-torsion.*

Proof By Proposition A.3 it is sufficient to prove that f^* and $(f^t)^*$ have cokernels that are 2-torsion.

Choose an edge $e \in E$. If e is ϕ -fixed then $e \in \text{im}(f)$. If e is not ϕ -fixed, suppose that e is on the left half of G . We may view e and $\phi(e)$ as edges in G_+ and G_- and compute,

$$f(e, -\phi(e)) = e + \phi(e) - \phi(e) + e = 2e.$$

We conclude from this that $\text{coker}(f)$ is 2-torsion, and hence $\text{coker}(f^*)$ is too.

Choose an edge $e' \in E_+$. If e' was obtained by subdividing $e \in E$, let $e'' \in E_+$ be the other edge obtained in this way. We compute,

$$f^t(e) = e' + e'' \equiv e' + e'' + (e' - e'') = 2e' \pmod{B_+},$$

since $e' - e''$ is a bond of G_+ .

If $e \in E_+$ did not arise from subdividing an edge of G , then we may identify e with an edge e of G . We have

$$f^t(e - \phi(e)) = f^t(e) - f^t(\phi(e)) = (e, \phi(e)) - (-e, \phi(e)) = 2(e, 0).$$

A similar computation shows that if $e \in E_-$ then $2(0, e) \in \text{im}(f^t)$. It follows that $\text{coker}((f^t)^*)$ is 2-torsion. □

We have thus proved the first and second part of the Main Theorem.

4 Identifying the kernel and cokernel

To ease the notation within this section and the next we make the following convention.

Convention *In this section and the next, Z and B will denote the reduction of the usual cycle and bond spaces of G by the prime 2. Thus $Z = \ker(\partial : (\mathbf{Z}/2)E \rightarrow (\mathbf{Z}/2)V)$ and $B = \text{im}(\partial^t : (\mathbf{Z}/2)V \rightarrow (\mathbf{Z}/2)E)$. The same notation is used for Z_{\pm} and B_{\pm} . We will also write f and f^t for the reduction of these \mathbf{Z} -linear maps by 2.*

Since the kernel and cokernel of f^* are 2-torsion their structure is intimately related to the reduction of their critical groups by 2. The 2-bicycle space (hereafter

the *bicycle space*) of G is $Z \cap B \subset (\mathbf{Z}/2)E$, which by Proposition B.2 is naturally isomorphic to $K(G)/2K(G)$.

An element h of $(\mathbf{Z}/2)E$ can be identified with a subgraph of $H \subset G$ via its support. Note that this subgraph does not come with an orientation. An element of $Z \cap B$ corresponds to a graph H satisfying the properties:

- (1) H is the set of edges connecting a bipartition of V .
- (2) Every vertex of G is incident to an even number of edges of H .

The following algebraic result is proved in a more general context as Proposition B.2, and it follows since the kernel and cokernel of f^* are known to be 2-torsion.

Proposition 4.1 *There are group isomorphisms,*

$$\begin{aligned} \text{coker}(f^*) &\approx \ker(f^t : Z \cap B \rightarrow (Z_+ \oplus Z_-) \cap (B_+ \oplus B_-)), \\ \ker(f^*) &\approx \ker(f : (Z_+ \oplus Z_-) \cap (B_+ \oplus B_-) \rightarrow Z \cap B). \end{aligned}$$

We are now in a position to identify $\text{coker}(f^*)$.

Proposition 4.2 *The kernel of $f^t : (\mathbf{Z}/2)E \rightarrow (\mathbf{Z}/2)E_+ \oplus (\mathbf{Z}/2)E_-$ has a basis given by the ϕ -fixed elements $e + \phi(e)$. The kernel of f^t restricted to $Z \cap B$ consists of the ϕ -fixed bicycles of G .*

Proof It is sufficient to prove the first claim. A basis for $(\mathbf{Z}/2)E$ is given by $\{e + \phi(e) : e \in E_L\} \cup E_L \cup E^\phi$. Likewise, a basis of $(\mathbf{Z}/2)E_+ \oplus (\mathbf{Z}/2)E_-$ is given by $\{(e, \phi(e)) : e \in E_L\} \cup E_+ \cup E_L$.

The matrix of f^t becomes diagonal in this basis, and it is clear that the kernel of f^t has the stated form. □

To identify the cokernel of $(f^t)^*$ we need another involution. Define

$$\psi : (\mathbf{Z}/2)(E_+ \cup E_-) \rightarrow (\mathbf{Z}/2)(E_+ \cup E_-)$$

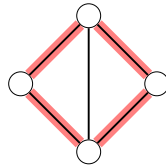
as follows. If $e \in E_+$ is obtained by subdividing an edge of E then set $\psi(e)$ equal to the other edge obtained in this way. If $e \in E_+$ arises from an edge that is not ϕ -fixed, then we define $\psi(e) := \phi(e) \in E_-$ and $\psi(\phi(e)) = e$. This map is *not* determined by a graph automorphism.

Proposition 4.3 *The kernel of $f : (\mathbf{Z}/2)E_+ \oplus (\mathbf{Z}/2)E_- \rightarrow (\mathbf{Z}/2)E$ consists of the elements ψ -fixed elements. The kernel of f restricted to $(Z_+ \cap Z_-) \cap (B_+ \cap B_-)$ consists of the ψ -fixed bicycles of $G_+ \cup G_-$.*

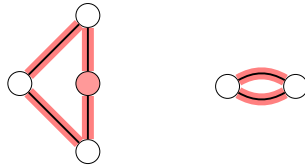
Proof Compute the matrix of f in terms of the basis used in the proof of Proposition 4.2. The first statement follows from inspection of the matrix representing f and the second follows from the first. □

Propositions 4.1, 4.2 and 4.3 give a complete combinatorial description of $\text{coker}(f^*)$ and $\ker(f^*)$. We illustrate them with an example.

Example 4.4 We continue with our running example, starting with $\text{coker}(f^*)$. A ϕ -fixed bicycle in G is indicated by the shaded edges below.



It follows that $\text{coker}(f^*) \approx \mathbf{Z}/2$. For the kernel of f^* , we investigate bicycles in $G_+ \cup G_-$. The graph $G_+ \cup G_-$ itself is a ψ -fixed bicycle.



Although both $G_+ \subset G_+ \cup G_-$ and $G_- \subset G_+ \cup G_-$ are bicycles, they are not ψ -fixed. It follows that $\text{ker}(f^*) \approx \mathbf{Z}/2$. The ker–coker exact sequence for f^* takes the form,

$$0 \rightarrow \mathbf{Z}/2 \rightarrow \mathbf{Z}/4 \oplus \mathbf{Z}/2 \xrightarrow{f^*} \mathbf{Z}/8 \rightarrow \mathbf{Z}/2 \rightarrow 0.$$

Example 4.5 Let G be a $(2n)$ -cycle with the obvious reflective symmetry. Then G_+ is a path on $n + 1$ vertices and G_- is an n -cycle. We see that G is a ϕ -fixed bicycle. Since G_+ is a path it has no (non-empty) bicycles, and hence there are no ψ -fixed bicycles.

The ker–coker exact sequence for the map f^* takes the form

$$0 \rightarrow \mathbf{Z}/n \rightarrow \mathbf{Z}/(2n) \rightarrow \mathbf{Z}/2 \rightarrow 0,$$

which is never split if n is even.

We close this section with alternate presentations of $\text{ker}(f^*)$ and $\text{coker}(f^*)$.

Proposition 4.6 *There are isomorphisms,*

$$\begin{aligned} \text{ker}(f^*) &\approx \frac{((\mathbf{Z}/2)(E_+ \cup E_-))^\psi}{(\mathbf{Z}_+ \oplus \mathbf{Z}_-)^\psi + (B_+ \oplus B_-)^\psi}, \\ \text{coker}(f^*) &\approx \frac{((\mathbf{Z}/2)E)^\phi}{\mathbf{Z}^\phi + B^\phi}. \end{aligned}$$

Proof The proofs amount to the fact that $\text{ker}(f^*)$ and $\text{coker}(f^*)$ are succinctly described as the ψ and ϕ fixed elements of $K(G_+)/2K(G_+) \oplus K(G_-)/2K(G_-)$ and

$K(G)/2K(G)$. This is true because the isomorphisms relating the various presentations of the critical groups in Appendix B are equivariant with respect to ψ and ϕ .

We then see that $((\mathbf{Z}/2)(E_+ \cup E_-))^\psi$ and $((\mathbf{Z}/2)E)^\phi$ surject onto these critical groups. The kernels of these maps are evident. □

5 Relating $\ker(f^*)$ and $\text{coker}(f^*)$

Our final goal is to relate the orders of $\ker(f^*)$ and $\text{coker}(f^*)$ in a concrete fashion. An easy and immediate result is that $|\text{coker}(f^*)|/|\ker(f^*)|$ is a positive integer power of 2.

Proposition 5.1 *There is an injective map*

$$\ker(f^*) \rightarrow \text{coker}(f^*).$$

Proof We use the above presentation of these groups as ψ and ϕ fixed bicycles. If $(x, x') \in (\mathbf{Z}/2)E_+ \oplus (\mathbf{Z}/2)E_-$ is ψ -fixed, set $g(x, x') := f(x, 0) = f(0, x')$. This restricts to a map on the ψ -fixed bicycles whose image is in the space of ϕ -fixed bicycles. The map is injective since $f|_{(Z_+ \oplus Z_-)^\psi}$ is injective. □

We will use the map g occurring in the proof of the proposition in what follows. Consider the commutative diagram below, whose horizontal arrows are those induced by g , and whose vertical arrows are the natural ones.

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 (Z_+ \oplus Z_-)^\psi \cap (B_+ \oplus B_-)^\psi & \longrightarrow & Z^\phi \cap B^\phi \\
 \downarrow & & \downarrow \\
 (Z_+ \oplus Z_-)^\psi \oplus (B_+ \oplus B_-)^\psi & \longrightarrow & Z^\phi \oplus B^\phi \\
 \downarrow & & \downarrow \\
 (Z_+ \oplus Z_-)^\psi + (B_+ \oplus B_-)^\psi & \longrightarrow & Z^\phi + B^\phi \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array} \tag{1}$$

The columns in this diagram are exact. We wish to identify the order of the cokernel in the top row. For this, we need to compute the kernel and cokernel in the middle row.

Proposition 5.2 *The dimension of $(B_+ \oplus B_-)^\psi$ is $|V_R| + |E^\phi|$. The dimension of B^ϕ is $|V_R| + |V^\phi| - 1$. It follows that*

$$|B^\phi| / |(B_+ \oplus B_-)^\psi| = 2^{|V^\phi| - |E^\phi| - 1}.$$

Proof A basis for the bond space of G_- is obtained by taking the fundamental bonds at all of its vertices except one. We exclude the vertex obtained by contracting all of V^ϕ to a point. If we take these bonds and symmetrize them by ψ we obtain $|V_R| = |V_L|$ many linearly independent bonds in $(B_+ \oplus B_-)^\psi$. Any ψ -fixed bond not contained in the span of these cannot be supported on B_- . It is clear that the bonds at the vertices obtained by subdividing ϕ -fixed edges complete our description of a basis of $(B_+ \oplus B_-)^\psi$.

A basis for the bond space of G is given by all but one of the fundamental bonds at vertices of G . We omit a ϕ -fixed vertex from our basis. The remaining ϕ -fixed vertices have ϕ -fixed bonds. Symmetrizing the bonds of vertices in V_R yields the rest of a basis for B^ϕ . □

Proposition 5.3 *Suppose that G_+ is connected. There is an equality,*

$$\frac{|(Z_+ \oplus Z_-)^\psi|}{|Z^\phi|} = 2^{|V^\phi| - |E^\phi| - 1}.$$

Proof The idea is to consider the injection $g : (Z_+ \oplus Z_-)^\psi \rightarrow Z^\phi$, and compute a basis for its cokernel. For this we note that $|V^\phi| - |E^\phi| - 1$ is the number of connected components of G^ϕ .

Choose one vertex from each connected component of G^ϕ , v_0, v_1, \dots, v_m . In G_+ , take a path p_{ij} connecting v_i to v_j . Viewing p_{ij} as a path in G , we form the cycle $z_{ij} = p_{ij} + \phi(p_{ij}) \in Z^\phi$. We claim that the cycles $\{z_{ij}\}$ are not in the image of g . If there was a cycle in G_- lifting z_{ij} then it would differ from p_{ij} by a sum of bonds of vertices obtained by subdividing ϕ -fixed edges. Since v_i and v_j are not connected by a path in G^ϕ we see that $z_{ij} \notin \text{im}(g)$.

Let v'_i and v'_j be two vertices in the same connected component of G^ϕ as v_i and v_j , respectively. If p'_{ij} is a path connecting v'_i to v'_j and $z'_{ij} = p'_{ij} + \phi(p'_{ij})$, then $z_{ij} + z'_{ij} \in \text{im}(g)$. This is because we have a cycle of G_+ , $p_{ij} + p'_{ij} +$ (a subdivided path in G^ϕ from v_i to v'_i) + (a subdivided path in G^ϕ from v_j to v'_j). Applying ψ to this cycle yields a cycle in G_- , since p_{ij} must touch the axis of symmetry an even number of times. We conclude from this $z_{ij} = z'_{ij} \in Z^\phi / \text{im}(g)$ and that $z_{i(i+1)} + \dots + z_{(j-1)j}$ is equivalent to $z_{i,j}$ in $Z^\phi / \text{im}(g)$.

We now claim that (the images of) $z_{01}, z_{12}, \dots, z_{(m-1)m}$ form a basis for $Z^\phi / \text{im}(g)$. They are linearly independent since the paths $\{p_{i(i+1)}\}$ are linearly independent in $(\mathbf{Z}/2)E_+$. If z represents a cycle in G that is not in $\text{im}(g)$, then z visits each of some even number of connected component of G^ϕ a twice odd number of times. Subtracting off elements of the form z_{ij} , where i and j label two components visited an odd number of times by z , we conclude that $\{z_{ij}\}$ spans $Z^\phi / \text{im}(g)$. From this we have $\{z_{i(i+1)} : i = 0, \dots, m - 1\}$ spans $Z^\phi / \text{im}(g)$ □

Proposition 5.4 *There is an equality,*

$$\frac{|(Z_+ \oplus Z_-)^\psi + (B_+ \oplus B_-)^\psi|}{|Z^\phi + B^\phi|} = \frac{|\ker(f^*)|}{|\operatorname{coker}(f^*)|}.$$

Proof This follows from Proposition 4.6 by multiplying and dividing the left side by $|((\mathbf{Z}/2)E)^\phi|$, which is equal to $|((\mathbf{Z}/2)(E_+ \cup E_-))^\psi|$. □

We are *finally* in a position to prove the remaining part of the Main Theorem.

Theorem 5.5 *Suppose that G_+ is connected. Then,*

$$\frac{|K(G_+) \oplus K(G_-)|}{|K(G)|} = \frac{|\ker(f^*)|}{|\operatorname{coker}(f^*)|} = 2^{|V^\phi| - |E^\phi| - 1}.$$

Proof Take the alternating product of the orders of the groups in the first column of the diagram (1) and divide by the alternating product for the second column. We obtain

$$1 = \frac{|(Z_+ \oplus Z_-)^\psi \cap (B_+ \oplus B_-)^\psi| \cdot |(Z_+ \oplus Z_-)^\psi + (B_+ \oplus B_-)^\psi| \cdot |Z^\phi \oplus B^\phi|}{|(Z_+ \oplus Z_-)^\psi \oplus (B_+ \oplus B_-)^\psi| \cdot |Z^\phi \cap B^\phi| \cdot |Z^\phi + B^\phi|}.$$

Applying Propositions 5.1, 5.2, 5.3, and 5.4 this yields

$$1 = 2^{2(|V^\phi| - |E^\phi| - 1)} \frac{|\ker(f^*)|^2}{|\operatorname{coker}(f^*)|^2}.$$

Manipulating this fraction and taking the square root proves the theorem. □

6 Open problems

It would be desirable to actually exhibit bicycles forming a basis of the cokernel of $g : \ker(f^*) \rightarrow \operatorname{coker}(f^*)$. This appears to be difficult and subtle, since it requires producing linearly independent bicycles in G . When G is planar the left-right tours of Shank [4] could possibly be used to furnish the needed bicycles.

There is a more general version of Theorem 1.2 given by Yan and Zhang [6]. It allows for an arbitrary involution on a weighted graph drawn in the plane, essentially meaning that we relax the condition that edges cannot cross the axis of symmetry.

The construction of an appropriate version of G_+ and G_- is more involved in this case, but a result of the form $\kappa(G) = 2^m \kappa(G_+) \kappa(G_-)$ is obtained. The integer m appearing in this formula might be negative in general, and positive integer weights for G might involve half integer weights for G_+ and G_- . It would be interesting to see a critical group generalization of this result.

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Appendix A: Critical groups of adjoint pairs

The point of these appendices is to gather and prove algebraic results about critical groups for the previous work. Some of these results can be found in Bacher, de la Harpe, Nagnebda [1] and Treumann’s bachelor’s thesis [5]

Following Treumann [5], we consider the category **Adj**, whose objects are adjoint pairs (∂, ∂^t) of linear maps,

$$\partial : C_1 \rightarrow C_0, \quad \partial^t : C_0 \rightarrow C_1$$

between two finitely generated free \mathbf{Z} -modules C_1 and C_0 . We assume that both C_1 and C_0 are both equipped with a positive definite inner product (both denoted $\langle -, - \rangle$) and have bases which are orthonormal with respect to these inner products. The adjointness of the maps ∂, ∂^t means that for all $v \in C_0$ and $e \in C_1$,

$$\langle \partial e, v \rangle = \langle e, \partial^t v \rangle.$$

Let (∂, ∂^t) be an adjoint pair as above. We define $Z := \ker(\partial)$ and $B := \text{im}(\partial^t)$. The *critical group* of (∂, ∂^t) is

$$K = K(\partial, \partial^t) := C_1 / (Z + B).$$

A morphism between two adjoint pairs $(\partial, \partial^t), (\partial', (\partial')^t)$ is a pair of linear maps

$$f = (f_1 : C_1 \rightarrow C'_1, f_0 : C_0 \rightarrow C'_0),$$

subject to the intertwining conditions

$$f_0 \partial = \partial' f_1, \quad f_1 \partial^t = (\partial')^t f_0 \pmod{B'}.$$

A morphism $f = (f_1, f_0)$ between two pairs (∂, ∂^t) and $(\partial', (\partial')^t)$ induces maps

$$f_1^* : K \rightarrow K',$$

as one checks that f_1 takes Z into Z' and B into B' .

Proof Suppose that $z \in Z$. Then $\partial^t f_1(z) = f_0(\partial z) = 0$. Suppose that $b = (\partial')^t v$. Then $f_1((\partial')^t v) = \partial^t f_0(v) \pmod{B'}$, hence $f_1((\partial')^t v)$ is an element of B' . \square

In this way, the critical group is a functor $\mathbf{Adj} \rightarrow \mathbf{Ab}$ from **Adj** to the category of finitely generated Abelian groups.

There is an alternate definition of the critical group in terms of the Laplacian operator $\partial \partial^t : C_0 \rightarrow C_0$.

Proposition A.1 (Treumann [5]) *The induced map $\partial : K \rightarrow \text{coker}(\partial \partial^t)$ is injective and there is a direct sum decomposition,*

$$K \oplus \text{coker}(\partial) = \text{coker}(\partial \partial^t).$$

The order of K is the absolute value of the maximal minors of $\partial \partial^t$.

When $\partial = \partial(G)$ for a graph G , the cokernel of ∂ is a free \mathbf{Z} -module whose rank is the number of connected components of G . In this case the absolute value of the maximal minors of $\partial\partial^t$ is the given by Kirchhoff’s Theorem as the number of spanning forests of G .

A morphism $f = (f_1, f_0)$ between two pairs (∂, ∂^t) and $(\partial', (\partial')^t)$ gives rise to a natural map on the quotient

$$f_0^* : \text{coker}(\partial\partial^t) \rightarrow \text{coker}(\partial'(\partial')^t).$$

This map restricts to the critical group summands, and we have the following result.

Theorem A.2 (Treumann [5]) *The induced maps*

$$f_1^* : K \rightarrow K', \quad f_0^* : K \rightarrow K'$$

are equal.

As such, we will drop the subscript on f when referring to the map it induces on critical groups.

Let f be a morphism in **Adj**, taking (∂, ∂^t) to $(\partial', (\partial')^t)$. The map f has an adjoint f^t taking $(\partial', (\partial')^t)$ to (∂, ∂^t) , given by taking the adjoint of the constituent functions. This means that

$$\langle f_1e, e' \rangle = \langle e, f_1^t e' \rangle, \quad \langle f_0v, v' \rangle = \langle v, f_0^t v' \rangle.$$

The pair $f^t = (f_1^t, f_0^t)$ is a morphism in **Adj**. The maps $f^* : K \rightarrow K'$ and $(f^t)^*$ are related by the following result.

Proposition A.3 *There is a commutative square*

$$\begin{array}{ccc} K & \xleftarrow{\quad} & K' \\ \downarrow \approx & \begin{array}{c} f^{t*} \\ \circlearrowleft f^* \end{array} & \approx \downarrow \\ \text{Hom}_{\mathbf{Z}}(K, \mathbf{Q}/\mathbf{Z}) & \xleftarrow{\quad} & \text{Hom}_{\mathbf{Z}}(K', \mathbf{Q}/\mathbf{Z}) \end{array}$$

There are natural isomorphisms $\ker(f^*) \approx \text{coker}(f^{t*})$, $\text{coker}(f^*) \approx \ker(f^{t*})$.

It is a fact that K is equipped with a non-degenerate \mathbf{Q}/\mathbf{Z} -valued bilinear form $(-, -)$, see [1, p.170], [5, p.3]. The identification of K with $\text{Hom}_{\mathbf{Z}}(K, \mathbf{Q}/\mathbf{Z})$ is via $x \mapsto (x, -)$. For this inner product we have the relation

$$(f^*(x), y) = (x, f^{t*}(y)).$$

See [2, Proposition 2.5] or [5, Proposition 9] for the proof of the proposition.

We now come to an important technical lemma.

Lemma A.4 *Let (∂, ∂^t) and $(\partial', (\partial')^t)$ be two objects in **Adj**. Suppose that $f_1 : C_1 \rightarrow C'_1$ is a \mathbf{Z} -linear map satisfying $f_1 Z \subset Z'$ and $f_1 B \subset B'$.*

*Define $f_0 : \text{im } \partial \rightarrow C'_0$ by $f_0(\partial x) := \partial' f_1(x)$. Then $f := (f_1, f_0)$ defines a morphism in **Adj**,*

$$(\partial : C_1 \rightarrow \text{im } \partial, \partial^t : \text{im } \partial \rightarrow C_1) \xrightarrow{f} (\partial', (\partial')^t).$$

Proof This is well defined since f_1 takes Z into Z' . We only need to check that the second intertwining condition is satisfied, since the first is satisfied by definition. For this we must find that f_1 commutes, up to B' , with the down-up Laplacian:

$$f_1((\partial^t \partial)x) = ((\partial')^t \partial') f_1(x) \pmod{B'}.$$

On the left, since $(\partial^t \partial)x$ is in B and f_1 takes B to B' we obtain an element of B' . The element of the right is patently in B' , thus both sides are equal modulo B' . \square

The point of this result is that if one has defined f_1 and it behaves sufficiently well then f_0 is essentially determined by f_1 , modulo data that the critical group cannot see. Indeed, if (∂, ∂^t) is in **Adj** then its critical group is equal to that of

$$(\partial : C_1 \rightarrow \text{im } \partial, \partial^t : \text{im } \partial \rightarrow C_1) \in \mathbf{Adj}.$$

This follows since the image of $\partial^t : \text{im } \partial \rightarrow C_1$ is equal to the image of $\partial^t : C_0 \rightarrow C_1$.

Appendix B: Bicycles

We maintain the notation of adjoint pairs from the previous section. Let p be a prime number. The p -bicycles of an adjoint pair (∂, ∂^t) are the elements of

$$\text{Hom}(K, \mathbf{Z}/p).$$

Since K is equipped with a non-degenerate \mathbf{Q}/\mathbf{Z} -valued bilinear form the p -bicycles are naturally identified with K/pK . Indeed, every map $K \rightarrow \mathbf{Z}/p$ can be thought of as a map $K \rightarrow \{0, 1/p, \dots, (p-1)/p\} \subset \mathbf{Q}/\mathbf{Z}$, and this map is of the form $(x, -)$ for some $x \in K$. Mapping $(x, -)$ to $x + pK$ gives the desired identification.

A morphism $f : (\partial, \partial^t) \rightarrow (\partial', (\partial')^t)$ in **Adj** gives rise to a natural map

$$f : K/pK \rightarrow K'/pK'$$

which is just reduction of f by p . Given an adjoint pair (∂, ∂^t) we will denote the reduction by p of the associated modules Z and B by $Z^{\mathbf{Z}/p}$ and $B^{\mathbf{Z}/p}$.

Proposition B.1 *There is a commutative square*

$$\begin{array}{ccc} K/pK & \longleftarrow & K'/pK' \\ \downarrow \approx & f^{t*} & \approx \downarrow \\ \text{Hom}(K, \mathbf{Z}/p) & \longleftarrow \circ f^* & \text{Hom}(K', \mathbf{Z}/p) \end{array}$$

If $\text{coker}(f^*)$ is all p -torsion, there is a natural isomorphism $\text{coker}(f^*) \approx \ker(f^{t*})$.

Proof Let $(-, -)$ denote the \mathbf{Q}/\mathbf{Z} -valued bilinear forms on K and K' . The commutativity of the diagram boils down to the equality $(f(x), y) = (x, f^t(y))$, which holds by Proposition A.3, reduced modulo $\frac{1}{p}(\mathbf{Q}/\mathbf{Z}) \subset \mathbf{Q}/\mathbf{Z}$.

Since $\text{Hom}_{\mathbf{Z}}(-, \mathbf{Z}/p)$ is right exact, the kernel of $(f^t)^* : K'/pK' \rightarrow K/pK$ is identified with $\text{Hom}_{\mathbf{Z}}(\text{coker}(f^*), \mathbf{Z}/p)$. Since $\text{coker}(f^*)$ is assumed to be p -torsion, $\text{Hom}_{\mathbf{Z}}(\text{coker}(f^*), \mathbf{Z}/p) = \text{coker}(f^*)$. □

Proposition B.2 *The p -bicycles of (∂, ∂^t) are naturally identified with $Z^{\mathbf{Z}/p} \cap B^{\mathbf{Z}/p}$. This association is functorial in the sense that if $f : (\partial, \partial^t) \rightarrow (\partial', (\partial')^t)$ is a morphism, then there is a commutative diagram,*

$$\begin{array}{ccc}
 K/pK & \xrightarrow{f^*} & K'/pK' \\
 \approx \downarrow & & \downarrow \approx \\
 \mathbf{Z}^{\mathbf{Z}/p} \cap B^{\mathbf{Z}/p} & \xrightarrow{f_1} & (Z')^{\mathbf{Z}/p} \cap (B')^{\mathbf{Z}/p}
 \end{array}$$

Proof There is a direct sum decomposition

$$C_0/pC_0 = \ker(\partial\partial^t) \oplus \text{im}(\partial\partial^t),$$

since $\partial\partial^t$ is self-adjoint. This yields an isomorphism of \mathbf{Z}/p -vector spaces

$$\ker(\partial\partial^t) \approx \text{coker}(\partial\partial^t) = K/pK \oplus \text{coker}(\partial).$$

Now, map an element x in $\ker \partial\partial^t$ to $\partial^t x$. The kernel of this map is precisely $\text{coker}(\partial)$, and its image is all of $Z^{\mathbf{Z}/p} \cap B^{\mathbf{Z}/p}$. This proves the first part of the result.

The commutativity of the square follows directly from the relations

$$f_0\partial = \partial' f_1, \quad f_1\partial^t = (\partial')^t f_0,$$

and the fact that f^* is represented either by f_0^* or f_1^* . □

By Lemma A.4, we do not need to have the full data of a morphism in **Adj** to reach the conclusion of the previous proposition.

Corollary B.3 *Let (∂, ∂^t) and $(\partial', (\partial')^t)$ be two objects in **Adj**. Suppose that $f_1 : C_1 \rightarrow C'_1$ is a \mathbf{Z} -linear map satisfying $f_1 Z \subset Z'$ and $f_1 B \subset B'$. Then, there is a*

commutative diagram

$$\begin{array}{ccc}
 K/pK & \xrightarrow{f^*} & K'/pK' \\
 \approx \downarrow & & \downarrow \approx \\
 \mathbb{Z}^{\mathbb{Z}/p} \cap B^{\mathbb{Z}/p} & \xrightarrow{f_1} & (\mathbb{Z}')^{\mathbb{Z}/p} \cap (B')^{\mathbb{Z}/p}
 \end{array}$$

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