Compactified Jacobians and q, t -Catalan numbers, II

Evgeny Gorsky · Mikhail Mazin

Received: 14 September 2012 / Accepted: 4 April 2013 / Published online: 20 April 2013
© Springer Science+Business Media New York 2013

Abstract We continue the study of the rational-slope generalized q, t -Catalan numbers $c_{m,n}(q, t)$. We describe generalizations of the bijective constructions of J. Haglund and N. Loehr and use them to prove a weak symmetry property $c_{m,n}(q, 1) = c_{m,n}(1, q)$ for $m = kn \pm 1$. We give a bijective proof of the full symmetry $c_{m,n}(q, t) = c_{m,n}(t, q)$ for $\min(m, n) \leq 3$. As a corollary of these combinatorial constructions, we give a simple formula for the Poincaré polynomials of compactified Jacobians of plane curve singularities $x^{kn \pm 1} = y^n$. We also give a geometric interpretation of a relation between rational-slope Catalan numbers and the theory of (m, n) -cores discovered by J. Anderson.

Keywords q, t -Catalan numbers · Compactified Jacobian · Semigroup

1 Introduction

Let m and n be two coprime positive integers.

Definition 1 Consider a rectangle with width m and height n . The rational-slope Catalan number

$$c_{m,n} = \frac{(m+n-1)!}{m!n!}$$

E. Gorsky
Mathematics Department, Stony Brook University, Stony Brook, NY 11794, USA
e-mail: egorsky@math.sunysb.edu

M. Mazin (✉)
Institute for Mathematical Sciences, Stony Brook University, Stony Brook, NY 11794, USA
e-mail: mmazin@math.sunysb.edu

Fig. 1 Definition of $a(c)$ and $l(c)$

is defined (e.g. [3]) as the number of lattice paths from the northwest corner to the southeast corner of this rectangle, staying below the diagonal connecting these corners. We will denote by $Y_{m,n}$ the set of all such paths, so that $|Y_{m,n}| = c_{m,n}$.

In the case $m = n + 1$ the number $c_{m,n}$ coincides with the usual Catalan number. In [7] M. Haiman and A. Garsia proposed a bivariate generalization of Catalan numbers whose combinatorial definition [8] can be naturally extended to the rational-slope case.

Definition 2 Let D be a Young diagram. Following [18], we define the statistics

$$h_+^{\frac{m}{n}}(D) := \sharp \left\{ c \in D : \frac{a(c)}{l(c) + 1} \leq \frac{m}{n} < \frac{a(c) + 1}{l(c)} \right\},$$

where $a(c)$ and $l(c)$ denote the arm- and leg-length (see Fig. 1) for a box $c \in D$.

Definition 3 [9] We define the rational-slope q, t -Catalan number by the formula

$$c_{m,n}(q, t) = \sum_{D \in Y_{m,n}} q^{\delta - |D|} t^{h_+^{\frac{m}{n}}(D)},$$

where $\delta = \frac{(m-1)(n-1)}{2}$.

Note that the polynomials $c_{m,n}(q, t)$ are special cases of more general polynomials considered in [18]. For the reader's convenience, from now on we will write $h_+^{\frac{m}{n}}$ instead of $h_+^{\frac{m}{n}}$.

It has been proved in [8] that $c_{m,n}(q, t)$ coincides with the q, t -Catalan numbers of A. Garsia and M. Haiman [7] for $m = n + 1$. It has been conjectured in [17] that $c_{n, kn+1}(q, t)$ coincides with the k -analogue of them. The following two conjectures are motivated by the above coincidences. Both conjectures appeared previously in the literature, but in different setups.

Conjecture 1 (Symmetry conjecture) *The function $c_{m,n}(q, t)$ satisfies the functional equation*

$$c_{m,n}(q, t) = c_{m,n}(t, q).$$

This conjecture is a special case of Conjecture 22 of Loehr and Warrington [18]. The second conjecture is a weaker version of the first one:

Conjecture 2 (Weak symmetry conjecture) *The function $c_{m,n}(q, t)$ satisfies the functional equation*

$$c_{m,n}(q, 1) = c_{m,n}(1, q).$$

Conjecture 1 was proved in [8] for $m = n + 1$. It was deduced by Garsia and Haglund from some nontrivial identities for q, t -Catalan numbers. For $m = kn + 1$ a similar statement was conjectured in [17]. No bijective proof in any of these cases is known yet.

Theorem 1 *The symmetry conjecture holds for $\min(m, n) \leq 3$.*

In the proof we construct an explicit bijection exchanging the area and h_+ statistics (see Sect. 4). After the first version of this paper was submitted, K. Lee, L. Li and N. Loehr obtained an independent proof of this result and extended it to the case $m = 4$ (see [16]).

The weak symmetry property was proved by an explicit bijective construction for $m = n + 1$ by J. Haglund [13] and for $m = kn + 1$ by N. Loehr [17]. Following constructions from [9], we introduce a map $G_{m,n} : Y_{m,n} \rightarrow Y_{m,n}$. It was conjectured in [9] that $G_{m,n}$ is a bijection. Let $T : Y_{m,n} \rightarrow Y_{n,m}$ be the transposition map.

Theorem 2 *The maps $G_{m,n}$ and $G_{n,m}$ satisfy the following properties:*

1. $|G_{m,n}(D)| = |G_{n,m}(D^T)| = \delta - h_+(D)$.
2. *There exists an involution $D \mapsto \widehat{D}$ such that $G_{m,n}(D) = (G_{n,m}(\widehat{D}^T))^T$.*
3. *If $m = n + 1$, then $Y_{n,n+1} = Y_{n+1,n}$ and, moreover, $G_{n,n+1} = G_{n+1,n}$.*
4. *If $m = kn \pm 1$, then $G_{m,n}$ is bijective.*
5. *If $m = kn + 1$, then $G_{m,n}$ coincides with the Haglund–Loehr bijection.*

Our description of the map $G_{m,n}$ is independent of the Haglund–Loehr construction. We define these maps in terms of semimodules over the semigroup generated by m and n . Constructing the inverse of $G_{m,n}$ translates into the reconstruction of a semimodule from a certain collection of data. This approach allowed us to prove bijectivity in the case $m = kn - 1$, which is a new result. Using the property (1), one can show that the bijectivity of $G_{m,n}$ (or, equivalently, $G_{n,m}$) is sufficient to prove the weak symmetry conjecture.

Corollary 1 *The weak symmetry conjecture holds for $m = kn \pm 1$.*

We apply the above combinatorial results to study the geometry of the *Jacobi factor*, certain moduli spaces associated with plane curve singularities (see [1, 4, 23] and Sect. 2 for the definitions). Of particular interest is the Jacobi factor of the plane curve singularity $\{x^m = y^n\}$, which can be identified with a certain affine Springer fiber [10, 14, 19]. It was conjectured in [21] (see also [22]) that the homology of this Jacobi factor is related to the invariants of the (m, n) torus knot. The following theorem is the main result of [9]:

Theorem 3 [9] *Consider a plane curve singularity $\{x^m = y^n\}$. Then its Jacobi factor admits an affine cell decomposition. The cells are parametrized by Young diagrams $D \in Y_{m,n}$, and the dimension for the cell C_D in the Jacobi factor can be computed in terms of D as follows:*

$$\dim C_D = \frac{(m-1)(n-1)}{2} - h_+(D).$$

It was pointed out in [9] that the weak symmetry conjecture implies a remarkably simple formula for the Poincaré polynomial of the Jacobi factor:

$$\begin{aligned} P_{\overline{JC}_x}(t) &= \sum_D t^{2\dim C_D} = \sum_D t^{2(\delta - h_+(D))} \\ &= t^{2\delta} c_{m,n}(1, t^{-2}) \stackrel{\text{WS}}{=} t^{2\delta} c_{m,n}(t^{-2}, 1) = \sum_D t^{2|D|}. \end{aligned}$$

A similar formula for the Poincaré polynomial was conjectured by Lusztig and Smelt at the last page of [19]. In the case $m = kn \pm 1$ Corollary 1 implies the following result.

Corollary 2 *If $m = kn \pm 1$ then the Poincaré polynomial of the Jacobi factor of the plane curve singularity $\{x^m = y^n\}$ has the form*

$$P_{\overline{JC}_x}(t) = \sum_D t^{2|D|},$$

where the summation is done over all Young diagrams $D \in Y_{m,n}$.

Following the ideas of J. Anderson [2], we prove the following result. Recall that $\delta = \frac{(m-1)(n-1)}{2}$. There is a canonical embedding of the Jacobi factor into the Grassmannian $\text{Gr}(\delta, V)$, where V is the space of polynomials in one variable of degree less or equal to $2\delta - 1$. The Grassmannian $\text{Gr}(\delta, V)$ has a cell decomposition by Schubert cells enumerated by Young diagrams contained in a $\delta \times \delta$ square. The cell decomposition of $\overline{JC}_x$ is given by intersections with the Schubert cells [11, 23].

Theorem 4 *A cell in $\text{Gr}(\delta, V)$ has non-empty intersection with $\overline{JC}_x$ if and only if the corresponding Young diagram is a simultaneous (m, n) -core, i.e. it does not have hooks of length m or n .*

2 Homology of the compactified Jacobian

2.1 Cell decomposition

In our previous paper [9] we studied the combinatorics of cell decompositions of the Jacobi factors of some plane curve singularities.

Let $x \in C \subset \mathbb{C}^2$ be a unibranch plane curve singularity, let t be a normalizing parameter on C at x , and $R \subset \mathbb{C}[[t]]$ be the complete local ring at x . Let $\delta = \dim(\mathbb{C}[[t]]/R)$. Since $x \in C$ is a plane curve singularity, it follows that $t^{2\delta}\mathbb{C}[[t]] \subset R$. Let $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$.

Definition 4 The *Jacobi factor* $\overline{JC}_x$ is the space of R -submodules $M \subset \mathbb{C}[[t]]$, such that $M \supset t^{2\delta}\mathbb{C}[[t]]$ and $\dim(\mathbb{C}[[t]]/M) = \delta$.

In other words, $\overline{JC}_x$ is isomorphic to the subvariety of the Grassmannian $\text{Gr}(\delta, V)$, consisting of subspaces invariant under the R -action.

Beauville proved in [4] that the compactified Jacobian of a complete rational curve is homeomorphic to the direct product of Jacobi factors of its singularities. To study the topology of the compactified Jacobian of a singular rational curve it is sufficient to study the topology of a single Jacobi factor.

In [23] Piontowski showed that for some singularities the Jacobi factor admits an algebraic cell decomposition. Following his work, we gave a combinatorial description of the cell decomposition in the case of a plane curve singularity with one Puiseux pair. In particular, it applies to a curve singularity $x^m = y^n$ for m and n co-prime. In this case the Jacobi factor coincides with a certain subvariety of the affine Grassmannian considered first by Lusztig and Smelt in [19] (see also [14]).

Let

$$\Gamma = \Gamma_{m,n} = \{am + bn | a, b \geq 0\} \subset \mathbb{Z}_{\geq 0}$$

be the semigroup generated by m and n . A subset $\Delta \subset \mathbb{Z}_{\geq 0}$ is called a semimodule over Γ , if $\Delta + \Gamma \subset \Delta$. It is zero-normalized, if $\min(\Delta) = 0$. If not stated otherwise, all semimodules are assumed zero-normalized throughout the paper.

A number a is called an n -generator of Δ , if $a \in \Delta$ and $a - n \notin \Delta$. Every Γ -semimodule has n distinct n -generators.

Definition 5 Let Δ be a Γ -semimodule, and $x \in \mathbb{Z}$. We define

$$g_{n,m,\Delta}(x) := \sharp([x, x+m) \setminus \Delta).$$

If a semimodule Δ is fixed, we write $g_{n,m}(x)$ instead of $g_{n,m,\Delta}(x)$ for the reader's convenience.

Theorem 5 [23] *The Jacobi factor for a plane curve singularity with one Puiseux pair (m, n) admits an affine cell decomposition. The cells C_Δ are enumerated by the 0-normalized Γ -semimodules Δ , and their dimensions can be computed by the*

formula

$$\dim C_{\Delta} = \sum_{j=1}^n g_{n,m,\Delta}(a_j) \quad (1)$$

where a_j are the n -generators of Δ .

It has been remarked in [9] that this definition has a natural combinatorial interpretation. Let us label the box (x, y) of the positive quadrant by the number $mn - m(1+x) - n(1+y)$. The corner box $(0, 0)$ is labeled by $2\delta - 1 = mn - m - n$ and the numbers decrease by m in east direction and by n in north direction. One can check that every number from $\mathbb{N} \setminus \Gamma$ appears exactly once in the $m \times n$ rectangle below the diagonal (it follows e.g. from Lemma 1 below).

Given a Γ -semimodule Δ , let us mark the boxes labeled by numbers from $\Delta \setminus \Gamma$ and denote the resulting set of boxes by $D(\Delta)$.

Theorem 6 [9] *For a Γ -semimodule Δ the set $D(\Delta)$ is a Young diagram. The correspondence D between semimodules and $Y_{m,n}$ is bijective. The dimension of a cell C_{Δ} in the Jacobi factor can be computed in terms of $D(\Delta)$ as follows:*

$$\dim C_{\Delta} = \frac{(m-1)(n-1)}{2} - h_+(D(\Delta)). \quad (2)$$

Example 1 Consider the semigroup Γ generated by the numbers 5 and 7, and a Γ -semimodule $\Delta = \mathbb{Z}_{\geq 0} \setminus \{1, 2, 3, 4, 6, 9\} = \{0, 5, 7, 8, 10, 11, \dots\}$. The diagram $D(\Delta)$ is shown in Fig. 2. The 5-generators of Δ are $\{a_0, a_1, a_2, a_3, a_4\} = \{0, 7, 8, 11, 14\}$, and

$$\begin{aligned} g_{5,7}(0) &= \sharp([0, 7) \setminus \Delta) = 5, & g_{5,7}(7) &= \sharp([7, 14) \setminus \Delta) = 1, \\ g_{5,7}(8) &= \sharp([8, 15) \setminus \Delta) = 1, & g_{5,7}(11) &= g_{5,7}(14) = 0. \end{aligned}$$

Therefore

$$\dim C_{\Delta} = g_{5,7}(0) + g_{5,7}(7) + g_{5,7}(8) + g_{5,7}(11) + g_{5,7}(14) = 7.$$

On the other hand, let us compute $h_+(D(\Delta))$. For the corner box (labeled by 23) the arm-length equals 1 and the leg-length equals 3, so the inequality $\frac{5}{7} < \frac{a(c)+1}{l(c)} = \frac{2}{3}$ does not hold. All other boxes of $D(\Delta)$ contribute to h_+ and hence $h_+(D(\Delta)) = 5$, as prescribed by (2).

2.2 Duality

Let us recall a useful symmetry property for the plane curve semigroup Γ .

Lemma 1 (e.g. [5, 15]) *Let Γ be the semigroup of a unibranched plane curve singularity and let δ be its δ -invariant. Then*

$$a \in \Gamma \iff 2\delta - 1 - a \notin \Gamma.$$

Fig. 2 The diagram $D(\Delta)$ for $m = 5$, $n = 7$, and $\Delta = \mathbb{Z}_{\geq 0} \setminus \{1, 2, 3, 4, 6, 9\} = \{0, 5, 7, 8, 10, 11, \dots\}$

For every Γ -semimodule Δ we can consider the *dual semimodule*

$$\Delta^* := \{\phi : \phi + \Delta \subset \Gamma\}.$$

Remark that Δ^* is not zero-normalized.

Lemma 2 *The dual semimodule Δ^* can be characterised by the equation*

$$\Delta^* = (2\delta - 1) - (\mathbb{Z} \setminus \Delta).$$

Proof Note that $(2\delta - 1) - (\mathbb{Z} \setminus \Delta)$ is a Γ -semimodule. Indeed, if $x \in \mathbb{Z} \setminus \Delta$ then $x - m, x - n \in \mathbb{Z} \setminus \Delta$.

It follows from Lemma 1 that

$$\phi + \Delta \subset \Gamma \iff (2\delta - 1) \notin \phi + \Delta.$$

Indeed, if $(2\delta - 1) \in \phi + \Delta$ then $\phi + \Delta$ is not a subset of Γ . Conversely, if $\exists x \in (\phi + \Delta) \setminus \Gamma$, then by Lemma 1 $(2\delta - 1 - x) \in \Gamma$, so $(2\delta - 1) \in x + \Gamma \subset \phi + \Delta$. Therefore

$$\begin{aligned} \phi + \Delta \subset \Gamma &\iff (2\delta - 1) \notin \phi + \Delta \\ &\iff (2\delta - 1 - \phi) \notin \Delta \iff \phi \in (2\delta - 1) - (\mathbb{Z} \setminus \Delta). \quad \square \end{aligned}$$

Remark 1 Lemmas 2 and 1 are combinatorial analogues of the Gorenstein property for the plane curve singularities (e.g. [5, 15]).

We normalize Δ^* so that it starts from 0 and denote the normalized semimodule by $\hat{\Delta}$. Following Lemma 2, we have

$$\hat{\Delta} = \max(\mathbb{Z} \setminus \Delta) - (\mathbb{Z} \setminus \Delta). \quad (3)$$

Table 1 Semimodules Δ^* and $\widehat{\Delta}$ for $(m, n) = (3, 4)$

Δ	Δ^*	$\widehat{\Delta}$
0, 3, 4, 6, ...	0, 3, 4, 6, ...	0, 3, 4, 6, ...
0, 3, 4, 5, 6, ...	3, 4, 6, ...	0, 1, 3, 4, 5, 6, ...
0, 2, 3, 4, 5, 6, ...	4, 6, ...	0, 2, 3, 4, 5, 6, ...
0, 1, 3, 4, 5, 6, ...	3, 6, ...	0, 3, 4, 5, 6, ...
0, 1, 2, 3, 4, 5, 6, ...	6, ...	0, 1, 2, 3, 4, 6, ...

Lemma 3 One has $(\widehat{\widehat{\Delta}}) = \Delta$.

Proof It follows from (3) that

$$\mathbb{Z} \setminus \widehat{\Delta} = \max(\mathbb{Z} \setminus \Delta) - \Delta,$$

and therefore

$$\max(\mathbb{Z} \setminus \widehat{\Delta}) = \max(\mathbb{Z} \setminus \Delta).$$

□

Example 2 We list the combinatorial types of Γ -semimodules Δ , Δ^* and $\widehat{\Delta}$ for $(m, n) = (3, 4)$ in Table 1.

2.3 Maps $G_{m,n}$ and $G_{n,m}$

Let us return to the formula (1). Given a Γ -semimodule Δ , we can consider its m -generators $a_1 < \dots < a_m$ and compute

$$g_{m,n}(a_i) = \sharp([a_i, a_i + n) \setminus \Delta).$$

Theorem 7 [9] *The numbers $g_{m,n}(a_i)$ are decreasing. The Young diagram with columns $g_{m,n}(a_i)$ belongs to the set $Y_{m,n}$.*

This result allows us to consider the map $G_{m,n} : Y_{m,n} \rightarrow Y_{m,n}$, sending $D(\Delta)$ to the diagram with columns $g_{m,n}(a_i)$. We will also use the notation $G_{m,n}(\Delta) = G_{m,n}(D(\Delta))$ for a Γ -semimodule Δ . In a similar way, one can define the map $G_{n,m}$ by choosing n -generators instead of m -generators. Note that the diagram $G_{n,m}(\Delta)$ has n columns, and belongs to $Y_{n,m}$. In this case it is natural to transpose $D(\Delta)$ and consider it as an element of $Y_{n,m}$.

Theorem 8 *One has*

$$G_{n,m}(\Delta) = (G_{m,n}(\widehat{\Delta}))^T.$$

Proof Let us recall the notion of an m -cogenerator for the Γ -semimodule Δ introduced in [9]: we call b an m -cogenerator for Δ if $b \notin \Delta$, but $b + m \in \Delta$. Remark

that by Lemma 2 the m -cogenerators of Δ are in 1-to-1 correspondence with the m -generators of $\widehat{\Delta}$. More precisely, if b is an m -generator of $\widehat{\Delta}$, then $(\max(\mathbb{Z} \setminus \Delta) - b)$ is an m -cogenerator for Δ .

Let $a_1, \dots, a_n$ be the n -generators of Δ and let $b_1, \dots, b_m$ be the m -generators of $\widehat{\Delta}$. One can check that $g_{n,m}(a_i)$ is equal to the number of m -cogenerators of Δ greater than a_i :

$$\begin{aligned} g_{n,m}(a_i) &= \sharp([a_i, a_i + m) \setminus \Delta) = \sharp\{j \mid \max(\mathbb{Z} \setminus \Delta) - b_j > a_i\} \\ &= \sharp\{j \mid a_i + b_j < \max(\mathbb{Z} \setminus \Delta)\}. \end{aligned}$$

Analogously by Lemma 3

$$g_{m,n}(b_j) = \sharp\{i \mid a_i + b_j < \max(\mathbb{Z} \setminus \Delta)\}.$$

It is clear that the corresponding Young diagrams are transposed to each other. $\square$

Corollary 3 *The map $G_{n,m}$ is bijective if and only if the map $G_{m,n}$ is bijective.*

Corollary 4 *The dimensions of cells in $\overline{JC}_x$ labeled by Δ and $\widehat{\Delta}$ are the same:*

$$\dim C_\Delta = |G_{n,m}(\Delta)| = |G_{m,n}(\widehat{\Delta})| = \dim C_{\widehat{\Delta}}.$$

In the case $m = n + 1$ the sets $Y_{n,n+1}$ and $Y_{n+1,n}$ coincide. Moreover, we have the following result.

Theorem 9 *If $m = n + 1$ then $G_{n+1,n} = G_{n,n+1}$.*

Proof Let Δ be a Γ -semimodule and $x \in \Delta$. Let $a_+(x)$ be the minimal n -generator of Δ greater than or equal to x . We set $a_+(x) = \infty$ if there are no n -generators greater than x . Similarly, let $a_-(x)$ be the maximal $(n + 1)$ -generator of Δ less than or equal to x . Note that since 0 is an $(n + 1)$ -generator and $x \in \Delta \subset \mathbb{Z}_{\geq 0}$, $a_-(x)$ is always well defined.

If x is not an n -generator then $x - n \in \Delta$, so $(x - n) + (n + 1) = x + 1 \in \Delta$. If $x + 1$ is not an n -generator then $x + 2 \in \Delta$ etc. By continuing this process we conclude that $[x, a_+(x)] \subset \Delta$. Since $a_+(x) - n \notin \Delta$, either $a_+(x) + 1 \notin \Delta$ or $a_+(x) + 1$ is a $(n + 1)$ -generator of Δ .

Analogously, $[a_-(x), x] \subset \Delta$ and either $a_-(x) - 1 \notin \Delta$ or $a_-(x) - 1$ is an n -generator of Δ . Therefore n - and $(n + 1)$ -generators of Δ are split into pairs (a_-, a_+) such that $[a_-, a_+] \subset \Delta$, and this is a 1-to-1 correspondence except for the largest $(n + 1)$ -generator. Since $[a_-, a_+] \subset \Delta$, we have $[a_- + n, a_+ + n + 1] \subset \Delta$ and

$$g_{n+1,n}(a_-) = \sharp([a_-, a_- + n) \setminus \Delta) = \sharp([a_+, a_+ + n + 1) \setminus \Delta) = g_{n,n+1}(a_+). \quad \square$$

Corollary 5 *If $m = n + 1$, then*

$$G_{n,n+1}(\widehat{\Delta}) = (G_{n,n+1}(\Delta))^T.$$

Fig. 3 Maps $G_{m,n}$ and $G_{n,m}$

Proof By Theorems 8 and 9

$$G_{n,n+1}(\hat{\Delta}) = (G_{n+1,n}(\Delta))^T = (G_{n,n+1}(\Delta))^T. \quad \square$$

Example 3 Let us present an example where $G_{n,m}$ and $G_{m,n}$ are essentially different. Let $(m, n) = (3, 7)$ and $\Delta = \{0, 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, \dots\}$. Then the 3-generators are 0, 1, 8, and

$$g_{3,7}(0) = 2, \quad g_{3,7}(1) = 2, \quad g_{3,7}(8) = 0.$$

The 7-generators are 0, 1, 3, 4, 6, 9, 12, and

$$g_{7,3}(0) = g_{7,3}(1) = g_{7,3}(3) = g_{7,3}(4) = 1, \quad g_{7,3}(6) = g_{7,3}(9) = g_{7,3}(12) = 0.$$

The dual semimodule is $\hat{\Delta} = \{0, 3, 6, 7, 8, \dots\}$. Its 3-generators are 0, 7, 8, and

$$g_{3,7}(0) = 4, \quad g_{3,7}(7) = g_{3,7}(8) = 0.$$

The 7-generators are 0, 3, 6, 8, 9, 11, 12, and

$$g_{7,3}(0) = g_{7,3}(3) = 2, \quad g_{7,3}(6) = g_{7,3}(8) = g_{7,3}(9) = g_{7,3}(11) = g_{7,3}(12) = 0.$$

We illustrate this in Fig. 3.

2.4 (m, n) -Cores

Definition 6 A Young diagram is called a p -core if it does not have boxes with hook length equal to p .

The p -core partitions play an important role in the study of representations of symmetric groups over finite fields (see [2, 6, 24] and references therein). J. Anderson observed that the number of partitions that are simultaneously m - and n -cores is finite:

Fig. 4 (3, 4) core

Theorem 10 [2] *The number of partitions that are simultaneously m - and n -cores equals*

$$\frac{1}{m+n} \binom{m+n}{n}.$$

The proof in [2] uses explicit bijection between the set of (m, n) -cores and the set $Y_{m,n}$. Following the analogy between such paths and $\Gamma_{m,n}$ -semimodules, we would like to present Anderson's bijection in slightly different form.

Proof Given a set $0 \in \Delta \subset \mathbb{Z}_{\geq 0}$, we construct the partition $P(\Delta)$ by the following rule. We start from $0 \in \Delta$ and read all consecutive integers. If $x \in \Delta$, we move north by 1, if $x \notin \Delta$, we move west by 1. The resulting lattice path bounds from above the Young diagram of the partition $P(\Delta)$.

The partition $P(\Delta)$ has a hook of length m if and only if there are integers $x \in \Delta$ and $y \notin \Delta$ such that $y = x + m$. Therefore $P(\Delta)$ is a simultaneous (m, n) -core if and only if Δ is a $\Gamma_{m,n}$ -semimodule. $\square$

Example 4 The (3, 4)-core corresponding to $\Delta = \{0, 3, 4, 6, \dots\}$ is shown in Fig. 4.

Lemma 4 *The (m, n) -core corresponding to $\widehat{\Delta}$ is conjugate to the (m, n) -core for Δ .*

Proof By (3), $\widehat{\Delta} = \max(\mathbb{Z} \setminus \Delta) - (\mathbb{Z} \setminus \Delta)$. Therefore to $x \in \Delta$ we associate an N step in the boundary of the (m, n) -core corresponding to Δ and a W step in the boundary of the (m, n) -core corresponding to $\widehat{\Delta}$. Analogously, to $x \notin \Delta$ we associate a W step in the boundary of the (m, n) -core corresponding to Δ and an N step in the boundary of the (m, n) -core corresponding to $\widehat{\Delta}$. $\square$

Theorem 11 *The number of self-dual Γ -semimodules equals*

$$\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}.$$

Proof By Lemma 4 the Γ -semimodules such that $\Delta = \widehat{\Delta}$ correspond to the self-conjugate (m, n) -cores. The number of such cores was computed by B. Ford, H. Mai and L. Sze in [6], and it is equal to $\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}$. $\square$

Remark 2 [3] The number of self-dual modules equals

$$\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} = \frac{[(m+n-1)!]_q}{[m!]_q [n!]_q} \Big|_{q=-1}.$$

Let us give a geometric interpretation of the construction from Theorem 10. As it was discussed in the Sect. 1, the vector space $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$ comes with a natural filtration

$$V = V_{2\delta} \supset V_{2\delta-1} \supset \cdots \supset V_0 = 0, \quad V_i := t^{2\delta-i} V.$$

Therefore $\text{Gr}(\delta, V)$ has a natural Schubert cell decomposition (see e.g. [20, pp. 74–75]), which can be described as follows.

Let P be a Young diagram contained in a $\delta \times \delta$ square. Let $p_1 \leq \cdots \leq p_\delta$ be its rows. The Schubert cell $C_P \subset \text{Gr}(\delta, V)$ consists of subspaces $W \subset V$ such that

$$\dim(W \cap V_i) = \#\{j : p_j + j \leq i\}.$$

Equivalently, one can assign the diagram $P(W)$ to a subspace $W \subset V$ in the following way. Define the subset $\Delta(W) \subset \mathbb{Z}_{\geq 0}$ as follows:

$$\Delta(W) = \{d \in \mathbb{Z} : \exists p \in W, p \in V_{2\delta-d} \setminus V_{2\delta-d-1}\} \cup [t^{2\delta}, \infty).$$

Now let us construct the diagram $P(W)$ from the subset $\Delta(W)$ as in Theorem 10. One can check that W belongs to the Schubert cell $C_{P(W)} \subset \text{Gr}(\delta, V)$.

Note that according to the construction, $P(W)$ is a simultaneous (m, n) -core iff $\Delta(W)$ is a $\Gamma_{m,n}$ -semimodule. It follows immediately that if $W \in \overline{JC}_x$ then $\Delta(W)$ is a $\Gamma_{m,n}$ -semimodule. On the other hand, J. Piontkowski showed in [23] that for a fixed $\Gamma_{m,n}$ -semimodule Δ there always exist $W \in \overline{JC}_x$ such that $\Delta(W) = \Delta$. More precisely, modules $W \in \overline{JC}_x$ with a fixed semimodule $\Delta(W)$ form a cell in the Piontkowski's cell decomposition of $\overline{JC}_x$.

Therefore, one gets the following

Theorem 12 *Let P be a Young diagram contained in a $\delta \times \delta$ square. Let $C_P \subset \text{Gr}(\delta, V)$ be the corresponding Schubert cell. Then the intersection $C_P \cap \overline{JC}_x$ is non-empty iff P is a simultaneous (m, n) -core, in which case it is the corresponding Piontkowski's affine cell.*

3 Bounce statistics and Poincaré polynomials

Let Δ be a $\Gamma_{m,n}$ -semimodule. Let us recall the following definition:

Definition 7 A number $a \in \Delta$ is called an m -generator if $a - m \notin \Delta$. A number $a \notin \Delta$ is called an m -cogenerator if $a + m \in \Delta$.

In this section we construct the inverse maps $G_{m,n}^{-1}$ in the cases $m = kn \pm 1$. In other words, we reconstruct a $\Gamma_{m,n}$ -semimodule Δ from the collection of integers $g_{m,n}(a_0), g_{m,n}(a_1), \dots, g_{m,n}(a_{m-1})$, where $\{0 = a_0 < \dots < a_{m-1}\}$ are the m -generators of Δ . Recall that the function $g_{m,n}(x)$ is defined by the formula

$$g_{m,n}(x) = \sharp([x, x+n) \setminus \Delta).$$

For the reader's convenience, we will use $g(x)$ instead of $g_{m,n}(x)$ in this section.

We start with some general facts on generators and cogenerators.

Lemma 5 *Suppose that $x \in \Delta$ is not an m -generator. Then $x + n$ is not an m -generator as well.*

Proof Indeed, if $x \in \Delta$ is not an m -generator, then $x - m \in \Delta$. But then $x - m + n = x + n - m \in \Delta$. Therefore, $x + n$ is not an m -generator. $\square$

Lemma 6 *Fix an integer $x \in \mathbb{Z}$. We have*

- (a) *The number of m -generators in $[x, x+n)$ equals $g(x-m) - g(x)$.*
- (b) *The number of m -cogenerators in $[x, x+n)$ equals $g(x) - g(x+m)$.*
- (c) *The number of m -generators in the interval $[x, x+n)$ is equal to the number of n -cogenerators in the interval $[x-m, x)$.*

Proof

- (a) For any integer $y \in [x-m, x-m+n) \setminus \Delta$ the number $y+m$ is either an m -generator in $[x, x+n)$ or it is in $[x, x+n) \setminus \Delta$. Also, for any $z \in [x, x+n) \setminus \Delta$ one has $z-m \in [x-m, x-m+n) \setminus \Delta$. Therefore, the number of m -generators in $[x, x+m)$ is equal to the number of elements in $[x-m, x-m+n) \setminus \Delta$ minus number of elements in $[x, x+n) \setminus \Delta$, which is $g(x-m) - g(x)$.
- (b) Similar to (a): an m -cogenerator in $[x, x+n)$ is an integer $y \in [x, x+n) \setminus \Delta$, such that $y+m \notin [x+m, x+m+n) \setminus \Delta$.
- (c) Follows immediately from (a) and (b). Indeed, by (a) one gets that the number of m -generators in $[x, x+n)$ equals $g(x-m) - g(x)$, and by (b) the number of n -cogenerators in $[x-m, x)$ also equals $g(x-m) - g(x)$ (here we switched m and n before applying (b)). $\square$

In Sects. 3.1 and 3.3 we discuss the cases $m = kn + 1$ and $m = kn - 1$, respectively. The logic in both cases is quite similar: first we reconstruct the bounce tree (see Definitions 10 and 13), and then use it to reconstruct the semimodule Δ . In Sect. 3.5 we compare this procedure and the bounce tree with the constructions of Haglund and Loehr.

3.1 The case $m = kn + 1$

Let $m = kn + 1$ from now until the end of the Sect. 3.1. Our goal is to reconstruct the semimodule Δ from the numbers $g(a_0), \dots, g(a_{kn})$, where $a_0, \dots, a_{kn}$ are the $(kn+1)$ -generators of Δ listed in the increasing order.

Fig. 6 The 7-generators of the $\Gamma_{3,7}$ -semimodule $\Delta := \{0, 2, 3, 5, 6, 7, \dots\}$ are equal to $0, 2, 3, 5, 6, 8, 11$. The tree T_Δ is presented in the diagram

Definition 10 The oriented graph T_Δ is defined as follows. The vertices are the $(kn+1)$ -generators of $\Delta : V_\Delta = \{a_0, \dots, a_{kn}\}$. Let $a_i, a_j \in V_\Delta$ be $(kn+1)$ -generators. We draw an edge $a_i \rightarrow a_j$, if $i \neq kn$ and $a_j = a_i + n$. See Fig. 6 for an example of a tree T_Δ .

Lemma 9 The graph T_Δ satisfies the following properties:

1. If $a_i \rightarrow a_j$ is an edge, then $i < j$.
2. The graph T_Δ is a tree with the root a_{kn} , and all edges are oriented towards a_{kn} .
3. The leaves of T_Δ are the $(kn+1, n)$ -generators of Δ , i.e. they are simultaneous n - and $(kn+1)$ -generators.

Proof

1. By construction, $j \geq i$ and $i < kn$. If $i = j$, then there are no $(kn+1)$ -generators in the interval $[a_i + 1, a_i + n]$, hence by Corollary 6 $[a_i, a_i + n] \subset \Delta$ and therefore $[a_i, +\infty) \subset \Delta$. By Lemma 5 there are no $(kn+1)$ -generators greater than a_i . Therefore $i = kn$. Contradiction.
2. Follows from the fact that there is only one edge from each vertex except the root a_{kn} , and the previous part.
3. If $a_i - n \in \Delta$ then by Lemma 5, $a_i - n$ is a $(kn+1)$ -generator. But then the tree T_Δ contains the edge $(a_i - n) \rightarrow a_i$ and a_i is not a leaf. Therefore, every leaf is a $(kn+1, n)$ -generator.

Conversely, suppose that a_i is a $(kn+1, n)$ -generator, but not a leaf of T_Δ . Then there exists a $(kn+1)$ -generator a_j such that $a_j \rightarrow a_i$ is an edge of T_Δ , so $a_j + n > a_i$ and the interval $[a_i + 1, a_j + n]$ does not contain any $(kn+1)$ -generators. It then follows from Corollary 6 that the whole interval $[a_i + 1, a_j + n]$ is contained in Δ . Therefore, by Lemma 7 $a_i + 1 - (n+1) = a_i - n \in \Delta$. Contradiction. $\square$

We will need the following observation about paths in T_Δ :

Lemma 10 Suppose that $a_{i_0} \rightarrow a_{i_1} \rightarrow \dots \rightarrow a_{i_l}$ is a path in T_Δ .

1. The interval $[a_{i_l} + 1, a_{i_0} + ln]$ is a subset of Δ and it does not contain $(kn+1)$ -generators.

2. The number of $(kn + 1)$ -generators in the interval $[a_{i_0} + 1, a_{i_0} + ln]$ is equal to $i_l - i_0$.
3. If a_j is a $(kn + 1)$ -generator and $a_{i_{l-1}} < a_j \leq a_{i_l}$, then $N_{i_0j} = l$.

Proof The proof of the first part is by induction on l . The $l = 1$ case is by definition and Corollary 6. Suppose that we proved lemma for $l - 1$. Then one has

$$[a_{i_l} + 1, a_{i_0} + ln] = [a_{i_l} + 1, a_{i_{l-1}} + n] \cup ([a_{i_{l-1}} + 1, a_{i_0} + (l - 1)n] + n).$$

The interval $[a_{i_l} + 1, a_{i_{l-1}} + n]$ does not contain $(kn + 1)$ -generators by the definition of the edge $a_{l-1} \rightarrow a_l$. In turn, the interval $([a_{i_{l-1}} + 1, a_{i_0} + (l - 1)n] + n)$ does not contain $(kn + 1)$ -generators by the induction assumption and Lemma 5. Finally, the inclusion $[a_{i_l} + 1, a_{i_0} + ln] \subset \Delta$ follows from Corollary 6.

The second part follows immediately from the first. For the third part, observe that since there are no $(kn + 1)$ -generators in the interval $[a_{i_{l-1}} + 1, a_{i_0} + (l - 1)n]$, we get

$$a_{i_0} + (l - 1)n < a_j \leq a_{i_l} \leq a_{i_0} + ln.$$

Therefore, $N_{i_0j} = \lceil \frac{a_j - a_{i_0}}{n} \rceil = l$. $\square$

Theorem 13 One can reconstruct the tree T_Δ from the collection of numbers $g(a_0), \dots, g(a_{kn})$. More precisely, given these numbers one can list all pairs (i, j) such that $a_i \rightarrow a_j$ is an edge in T_Δ .

Proof We will list the edges of T_Δ in the following order. First, we reconstruct the path from a_0 to a_{kn} . Then we take the smallest $(kn + 1)$ -generator a_l , which is not covered yet, and reconstruct the path

$$a_l \rightarrow a_{l+b_0^l} \rightarrow a_{l+b_0^l+b_1^l} \rightarrow \dots \rightarrow a_{kn}.$$

We repeat this procedure until we run out of generators.

On each step we need to find numbers $b_0^l, b_1^l, b_2^l, \dots$. By Lemma 10, b_i^l is equal to the number of $(kn + 1)$ -generators in the interval

$$I_i^l := (a_l + in, a_l + (i + 1)n].$$

We use this to define b_i^l for all $i \in \mathbb{Z}$.

We will also need numbers c_i^l , counting n -cogenerators in the same intervals I_i^l . According to Lemma 6, b_i^l is equal to the number of n -cogenerators in the interval $J_i^l := (a_l + in - (kn + 1), a_l + in]$. Note that

$$J_i^l = \{a_l + in - kn\} \sqcup I_{i-k}^l \sqcup \dots \sqcup I_{i-1}^l.$$

By construction, a_l is a leaf of the tree T_Δ . By Lemma 9, a_l is a $(kn + 1, n)$ -generator of Δ . Therefore, $a_l + in - kn = a_l + n(i - k)$ is an n -cogenerator if and only if

$i = k - 1$. We conclude that b_i^l can be expressed via $c_{i-k}^l, \dots, c_{i-1}^l$ as follows:

$$b_i^l = \begin{cases} \sum_{j=i-k}^{i-1} c_j^l & \text{if } i \neq k - 1, \\ 1 + \sum_{j=i-k}^{i-1} c_j^l & \text{if } i = k - 1. \end{cases} \quad (4)$$

On the other hand, for $i \geq 0$ one can use Lemmas 6 and 10 to express c_i^l through $b_0^l, b_1^l, \dots, b_i^l$ using numbers $g(a_j)$:

$$c_i^l = g(a_{l+b_0^l+b_1^l+\dots+b_{i-1}^l}) - g(a_{l+b_0^l+b_1^l+\dots+b_i^l}), \quad c_0^l = g(a_l) - g(a_{l+b_0^l}), \quad (5)$$

Indeed, $g(a_{l+b_0^l+b_1^l+\dots+b_i^l})$ is equal to the number of integers not in Δ in the interval $[a_{l+b_0^l+b_1^l+\dots+b_i^l}, a_{l+b_0^l+b_1^l+\dots+b_i^l} + n)$, which is the same as in the interval $[a_l + in, a_l + (i + 1)n)$ by the first part of Lemma 10.

Equations (4) and (5) together provide recurrence relations on numbers b_i^l and c_i^l . To start the recursive algorithm, one needs to compute the initial values $c_{-k}^l, \dots, c_{-1}^l$.

Recall the numbers $N_{ij} = \lceil \frac{a_j - a_i}{n} \rceil$ and $K_{ij} = \lceil \frac{a_j - a_i}{kn+1} \rceil$. Since we know all edges of the tree T_Δ with initial points less than a_l , we can use Lemma 10 to find numbers N_{ij} for any $i, j \leq l$. Then, using Corollary 7, we can find numbers K_{ij} for any $i, j \leq l$. We can also compute

$$f(a_i) := \sharp(\Delta \cap (-\infty, a_i)) = \sum_{j < i} K_{ji},$$

for all $i \leq l$. Indeed, fix a remainder $0 \leq r < kn + 1$, such that the corresponding $(kn + 1)$ -generator $a_j \equiv r \pmod{kn + 1}$ is less than a_i . Then there are exactly K_{ji} integers in $\Delta \cap (-\infty, a_i)$ with remainder r modulo $kn + 1$.

Note that

$$c_{-i}^l = \sharp(I_{-i+1}^l \cap \Delta) - \sharp(I_{-i}^l \cap \Delta) \quad \text{for all } i > 0. \quad (6)$$

Since a_l is a $(kn + 1, n)$ -generator of Δ , we have $a_l - in \notin \Delta$ for $i > 0$. Consider the smallest element of Δ bigger than $a_l - in$. By Corollary 6, it is a $(kn + 1)$ -generator a_{α_i} , where $\alpha_i := \min\{j : N_{jl} \leq i\}$.

Finally, we compute

$$\sharp(I_{-i}^l \cap \Delta) = f(a_{\alpha_{i-1}}) - f(a_{\alpha_i}) \quad \text{for } i > 1,$$

$$\sharp(I_{-1}^l \cap \Delta) = f(a_l) - f(a_{\alpha_1}) + 1,$$

and the number $\sharp(I_0^l \cap \Delta) = n - g(a_l)$ is given. Therefore, we can use (6) to find numbers $c_{-k}^l, \dots, c_{-1}^l$. $\square$

Theorem 14 *The tree T_Δ completely determines the semimodule Δ .*

Proof Now that we know the whole tree T_Δ , we can use Lemma 10 and Corollary 7 to find numbers K_{ij} and N_{ij} for all $i < j \leq kn$. Note that by the definition of num-

bers K_{ij} , we have

$$a_i \in ((K_{0i} - 1)(nk + 1), K_{0i}(nk + 1)] \quad \text{for all } i.$$

Therefore, it suffices to recover the remainders $r_0, \dots, r_{kn}$ of the generators $a_0, \dots, a_{kn}$ modulo $kn + 1$.

We can use numbers K_{ij} recover the order of $r_0, \dots, r_{kn}$. Indeed, if $i < j$ then

$$r_i < r_j \iff K_{ij} > K_{0j} - K_{0i}.$$

Since the remainders $r_0, \dots, r_{kn}$ run through all numbers $0, 1, \dots, kn$ once, knowing the order of $r_0, \dots, r_{kn}$ is equivalent to knowing the remainders themselves. $\square$

Let us illustrate Theorems 13 and 14 in the following example.

3.2 Example: reconstruction of a $\Gamma_{4,9}$ -semimodule

Let $n = 4$ and $m = 9$ ($k = 2$). Suppose that $g(a_0) = 2$, $g(a_1) = g(a_2) = 1$, and $g(a_3) = \dots = g(a_8) = 0$. At the first step we reconstruct the path from $a_0 = 0$ to a_8 in the tree T_Δ . Following the algorithm, we first need to find numbers c_{-1}^0 and c_{-2}^0 , counting 4-cogenerators in intervals $(-4, 0]$ and $(-8, -4]$, respectively. Since there are no elements of Δ less than $a_0 = 0$, we immediately conclude that

$$c_{-2}^0 = \sharp\{\Delta \cap (-4, 0]\} - \sharp\{\Delta \cap (-8, -4]\} = 1 - 0 = 1,$$

and

$$c_{-1}^0 = \sharp\{\Delta \cap (0, 4]\} - \sharp\{\Delta \cap (-4, 0]\} = (4 - g(a_0)) - 1 = 4 - 2 - 1 = 1.$$

Using the recurrence relations (4) and (5) we immediately compute:

$$\begin{aligned} b_0^0 &= c_{-2}^0 + c_{-1}^0 = 2, & c_0^0 &= g(a_0) - g(a_2) = 2 - 1 = 1, \\ b_1^0 &= 1 + c_{-1}^0 + c_0^0 = 3, & c_1^0 &= g(a_2) - g(a_5) = 1, \\ b_2^0 &= c_0^0 + c_1^0 = 2, & c_2^0 &= g(a_5) - g(a_7) = 0, \\ b_3^0 &= c_1^0 + c_2^0 = 1, & c_3^0 &= g(a_7) - g(a_8) = 0. \end{aligned}$$

Therefore, the path from a_0 to a_8 is $a_0 \rightarrow a_2 \rightarrow a_5 \rightarrow a_7 \rightarrow a_8$. The smallest 9-generator not covered yet is a_1 . Therefore, our next step is to recover the path from a_1 to a_8 .

Again, we start by reconstructing numbers c_{-1}^1 and c_{-2}^1 . We have

$$c_{-2}^1 = \sharp\{\Delta \cap (a_1 - 4, a_1]\} - \sharp\{\Delta \cap (a_1 - 8, a_1 - 4]\},$$

and

$$c_{-1}^1 = \sharp\{\Delta \cap (a_1, a_1 + 4]\} - \sharp\{\Delta \cap (a_1 - 4, a_1]\}.$$

The only element of Δ less than a_1 is a_0 and, moreover, $a_0 > a_1 - 4$. Indeed, a_0 is the only 9-generator less than a_1 and $a_1 < a_2 \leq a_0 + 4$, because we have the arrow $a_0 \rightarrow a_2$ in the tree T_Δ . We conclude that

$$\begin{aligned}\sharp\{\Delta \cap (a_1 - 8, a_1 - 4]\} &= 0, \\ \sharp\{\Delta \cap (a_1 - 4, a_1]\} &= 2,\end{aligned}$$

and

$$\sharp\{\Delta \cap (a_1, a_1 + 4]\} = 4 - g(a_1) = 3.$$

Therefore,

$$c_{-2}^1 = 2 \quad \text{and} \quad c_{-1}^1 = 1.$$

We again use the recurrence relations (4) and (5):

$$\begin{aligned}b_0^1 &= c_{-2}^1 + c_{-1}^1 = 3, & c_0^1 &= g(a_1) - g(a_4) = 1, \\ b_1^1 &= 1 + c_{-1}^1 + c_0^1 = 3, & c_1^1 &= g(a_4) - g(a_7) = 0, \\ b_2^1 &= c_0^1 + c_1^1 = 1, & c_2^1 &= g(a_7) - g(a_8) = 0.\end{aligned}$$

Therefore, the path from a_1 to a_8 is $a_1 \rightarrow a_4 \rightarrow a_7 \rightarrow a_8$. The smallest 9-generator not covered yet is a_3 . Therefore, our next step is to recover the path from a_3 to a_8 .

Similarly to above, we need to reconstruct numbers c_{-2}^3 and c_{-1}^3 first:

$$c_{-2}^3 = \sharp\{\Delta \cap (a_3 - 4, a_3]\} - \sharp\{\Delta \cap (a_3 - 8, a_3 - 4]\},$$

and

$$c_{-1}^3 = \sharp\{\Delta \cap (a_3, a_3 + 4]\} - \sharp\{\Delta \cap (a_3 - 4, a_3]\}.$$

There are three 9-generators less than a_3 : a_0 , a_1 , and a_2 . Since we have the path $a_0 \rightarrow a_2 \rightarrow a_5$ in the tree T_Δ , we conclude that $a_0 + 8 \geq a_5 > a_3$. Therefore, the only 3 elements of Δ less than a_3 are the 9-generators. Furthermore, from the reconstructed part of the tree we see that $a_3 - 8 < a_0 < a_3 - 4$, $a_3 - 4 < a_1 < a_3$, and $a_3 - 4 < a_2 < a_3$. Therefore,

$$\begin{aligned}\sharp\{\Delta \cap (a_3 - 8, a_3 - 4]\} &= 1, \\ \sharp\{\Delta \cap (a_3 - 4, a_3]\} &= 3,\end{aligned}$$

and

$$\sharp\{\Delta \cap (a_3, a_3 + 4]\} = 4 - g(a_3) = 4.$$

We conclude that

$$c_{-2}^3 = 2 \quad \text{and} \quad c_{-1}^3 = 1.$$

Once again, we use the recurrence relations (4) and (5):

$$b_0^3 = c_{-2}^3 + c_{-1}^3 = 3, \quad c_0^3 = g(a_3) - g(a_6) = 0,$$

Fig. 7 The tree T_Δ for the case $n = 4$, $m = 9$, $g(a_0) = 2$, $g(a_1) = g(a_2) = 1$, and $g(a_3) = \dots = g(a_8) = 0$

$$b_1^3 = 1 + c_{-1}^3 + c_0^3 = 2, \quad c_1^3 = g(a_6) - g(a_8) = 0.$$

Therefore, the path from a_3 to a_8 is $a_3 \rightarrow a_6 \rightarrow a_8$. See Fig. 7 for the full tree T_Δ .

Finally, we reconstruct the semimodule Δ from the tree T_Δ following the algorithm from Theorem 14. First, we use Lemma 10 to find numbers $N_{0,i}$ for $i = 1, 2, \dots, 8$ and Corollary 7 to find $K_{0,i}$. We get

$$N_{0,1} = N_{0,2} = 1, \quad N_{0,3} = N_{0,4} = N_{0,5} = 2, \quad N_{0,6} = N_{0,7} = 3, \quad N_{0,8} = 4,$$

and

$$K_{0,1} = K_{0,2} = K_{0,3} = K_{0,4} = K_{0,5} = 1, \quad K_{0,6} = K_{0,7} = K_{0,8} = 2.$$

Therefore, we see that a_1, a_2, a_3, a_4 , and a_5 are in the interval $(0, 9)$, while a_6, a_7, a_8 are in the interval $(9, 18)$.

Now we need to compare the remainders $r_1, \dots, r_8$ of the 9-generators $a_1, \dots, a_8$. We already know that $r_1 < r_2 < r_3 < r_4 < r_5$ and $r_6 < r_7 < r_8$. Let us compare r_1 and r_6 . By Lemma 10 and the tree T_Δ , we get $N_{1,6} = 2$. Therefore, by Corollary 7, $K_{1,6} = 1$, which means that $a_6 - a_1 < 9$. Therefore, $r_6 = a_6 - 9 < a_1 = r_1$.

Similarly, one computes that $r_7 < r_1$ and $r_2 < r_8 < r_3$. Therefore,

$$r_6 < r_7 < r_1 < r_2 < r_8 < r_3 < r_4 < r_5,$$

or

$$\begin{array}{llll} r_6 = 1, & r_7 = 2, & r_1 = 3, & r_2 = 4, \\ r_8 = 5, & r_3 = 6, & r_4 = 7, & r_5 = 8. \end{array}$$

Finally,

$$\begin{array}{lll} a_0 = 0, & a_1 = r_1 = 3, & a_2 = r_2 = 4, \\ a_3 = r_3 = 6, & a_4 = r_4 = 7, & a_5 = r_5 = 8, \end{array}$$

and

$$a_6 = r_6 + 9 = 10, \quad a_7 = r_7 + 9 = 11, \quad a_8 = r_8 + 9 = 14.$$

Fig. 8 In this picture $n = 3$, $k = 3$, a_i is an 8-generator, and “o” indicates an integer not in Δ . Note that the interval $[a_i - 2kn + 1, a_i - 2(kn - 1)] = [a_i - 17, a_i - 16]$ has empty intersection with Δ

3.3 The case $m = kn - 1$.

Let now $m = kn - 1$. In this case the semimodule Δ can be reconstructed from numbers $g(a_0), \dots, g(a_{kn-2})$ in a way similar to the case $m = kn + 1$. However, some adjustments are required. We will omit some of the proofs, in the cases when they are identical to the $m = kn + 1$ case.

Lemma 11 Suppose that $x \in \Delta$ and $x - \alpha n + 1 \notin \Delta$ for some $\alpha \in \{0, \dots, k\}$. Then x is a $(kn - 1)$ -generator.

Proof Indeed, if x is not a $(kn - 1)$ -generator, then $x - kn + 1 \in \Delta$. But then $(x - kn + 1) + (k - \alpha)n = x - \alpha n + 1 \in \Delta$. Contradiction. $\square$

Definition 11 For $x \in \Delta$ we define $a_+(x)$ to be the minimal $(kn - 1)$ -generator greater than or equal to x . If there are no $(kn - 1)$ -generators greater or equal to x , we set $a_+(x) = \infty$.

Corollary 8 For any $x \in \Delta$ one has $[x, a_+(x)] \subset \Delta$.

Proof Follows from Lemma 11 with $\alpha = 0$. $\square$

Lemma 12 Consider a $(kn - 1)$ -generator a_i . Then for any $l > 0$, the interval $J_l := [a_i - lkn + 1, a_i - l(kn - 1)]$ has empty intersection with Δ .

Proof The proof is by induction on l . The case $l = 1$ is clear. Then observe that

$$J_{l+1} = \{a_i - (l + 1)(kn - 1)\} \cup (J_l - kn). \quad \square$$

We illustrate Lemma 12 with Fig. 8.

Definition 12 For $j > i$ we introduce the following notation:

$$N_{ij} = \left\lfloor \frac{a_j - a_i}{n} \right\rfloor + 1 \quad \text{and} \quad K_{ij} = \left\lfloor \frac{a_j - a_i}{kn - 1} \right\rfloor + 1.$$

Corollary 9 We have the following formula: $K_{ij} = \lceil \frac{N_{ij}}{k} \rceil$.

Proof Note that

$$\left\lceil \frac{N_{ij}}{k} \right\rceil = \left\lceil \frac{\left\lfloor \frac{a_j - a_i}{n} \right\rfloor + 1}{k} \right\rceil = \left\lfloor \frac{a_j - a_i}{kn} \right\rfloor + 1.$$

Fig. 9 The 8-generators of the $\Gamma_{3,8}$ -semimodule $\Delta := \{0, 3, 4, 6, 7, 8, \dots\}$ are equal to $0, 3, 4, 6, 7, 9, 10, 13$. The tree T_Δ is presented in the diagram

Suppose that $K_{ij} \neq \lceil \frac{N_{ij}}{k} \rceil$. Then there exists an integer l such that $a_j - a_i \geq l(kn - 1)$ and $a_j - a_i < lkn$, which is equivalent to $a_i \in (a_j - lkn, a_j - l(kn - 1)]$. Therefore by Lemma 12 $a_i \notin \Delta$. Contradiction. $\square$

Definition 13 The oriented graph T_Δ is defined as follows. The vertices of this graph are the $(kn - 1)$ -generators of Δ plus one extra vertex $a_\infty : V_\Delta = \{a_0, \dots, a_{kn-2}, a_\infty\}$. Let $a_i, a_j \in V_\Delta$ be $(kn - 1)$ -generators. We draw an edge $a_i \rightarrow a_j$, if the equation $a_j = a_+(a_i + n)$ holds. In addition, we draw edges $a_i \rightarrow a_\infty$ for every a_i , such that there are no $(kn - 1)$ -generators greater than or equal to $a_i + n$. See Fig. 9 for an example of a tree T_Δ .

Lemma 13 The graph T_Δ satisfies the following properties:

1. If $a_i \rightarrow a_j$ is an edge, then $i < j$.
2. The graph T_Δ is a tree with the root a_∞ , and all edges are oriented towards a_∞ .
3. The leaves of T_Δ are the $(kn - 1, n)$ -generators of Δ .

Proof The first two parts are immediate from the construction. For the third part, consider a $(kn - 1)$ -generator a_i . If $a_i - n \in \Delta$ then by Lemma 5 $a_i - n$ is a $(kn - 1)$ -generator. But then the tree T_Δ contains the edge $(a_i - n) \rightarrow a_i$ and a_i is not a leaf. Therefore, every leaf is a $(kn - 1, n)$ -generator.

Conversely, suppose that a_i is a $(kn - 1, n)$ -generator, but not a leaf of T_Δ . Then there exists a $(kn - 1)$ -generator a_j such that $a_j \rightarrow a_i$ is an edge of T_Δ , so $a_j + n < a_i$ and the interval $[a_j + n, a_i - 1]$ does not contain any $(kn - 1)$ -generators. It then follows from Corollary 8 that the whole interval $[a_j + n, a_i - 1]$ is contained in Δ . Therefore by Lemma 11 $a_i - 1 - (n - 1) = a_i - n \in \Delta$. Contradiction. $\square$

We will also need the following observation about paths in T_Δ :

Lemma 14 Suppose $a_{i_0} \rightarrow a_{i_1} \rightarrow \dots \rightarrow a_{i_l}$ is a path in T_Δ .

1. The interval $[a_{i_0} + ln, a_{i_l} - 1]$ is a subset of Δ and it does not contain $(kn - 1)$ -generators.

2. The number of $(kn - 1)$ -generators in the interval $[a_{i_0}, a_{i_0} + ln - 1]$ is equal to $i_l - i_0$.
3. If a_j is a $(kn - 1)$ -generator and $a_{i_{l-1}} \leq a_j < a_{i_l}$, then $N_{i_0j} = l$.

Proof The proof of the first part is by induction on l . The $l = 1$ case is by definition and Corollary 8. Suppose that we proved lemma for $l - 1$. Then one has

$$[a_{i_0} + ln, a_{i_l} - 1] = ([a_{i_0} + (l - 1)n, a_{i_{l-1}} - 1] + n) \cup [a_{i_{l-1}} + n, a_{i_l} - 1].$$

The interval $[a_{i_{l-1}} + n, a_{i_l} - 1]$ does not contain $(kn - 1)$ -generators by the definition of the edge $a_{l-1} \rightarrow a_l$. In turn, the interval $([a_{i_0} + (l - 1)n, a_{i_{l-1}} - 1] + n)$ does not contain $(kn - 1)$ -generators by the induction assumption and Lemma 5. Finally, the inclusion $[a_{i_0} + ln, a_{i_l} - 1] \subset \Delta$ follows from Corollary 8.

The second part follows immediately from the first. For the third part, observe that since there are no $(kn - 1)$ -generators in the interval $[a_{i_0} + ln, a_{i_l} - 1]$, we get

$$a_{i_0} + (l - 1)n \leq a_{i_{l-1}} \leq a_j < a_{i_0} + ln.$$

Therefore, $N_{i_0j} = \lfloor \frac{a_j - a_{i_0}}{n} \rfloor + 1 = l - 1 + 1 = l$. □

Theorem 15 *One can reconstruct the tree T_Δ from the collection of numbers $g(a_0), \dots, g(a_{kn-2})$. More precisely, given these numbers one can list all pairs (i, j) such that $a_i \rightarrow a_j$ is an edge in T_Δ .*

Proof We will reconstruct T_Δ in the following order. First, we reconstruct the path from a_0 to a_∞ . Then we take the smallest $(kn - 1)$ -generator a_l , which is not covered yet, and reconstruct the path

$$a_l \rightarrow a_{l+b_0^l} \rightarrow a_{l+b_0^l+b_1^l} \rightarrow \dots \rightarrow a_\infty.$$

We repeat this procedure until we run out of generators.

On each step all we need to do is to find numbers $b_0^l, b_1^l, b_2^l, \dots$. According to Lemma 14, b_i^l is equal to the number of $(kn - 1)$ -generators in the interval $I_i^l := [a_l + in, a_l + (i + 1)n]$. We use this to define b_i^l for all $i \in \mathbb{Z}$.

We will also need numbers c_i^l , counting n -cogenerators in the same intervals I_i^l . According to Lemma 6, b_i^l is equal to the number of n -cogenerators in the interval $J_i^l := [a_l + in - (kn - 1), a_l + in]$. Note that

$$J_i^l = (I_{i-k}^l \sqcup \dots \sqcup I_{i-1}^l) \setminus \{a_l + (i - k)n\}.$$

By construction, a_l is a leaf of the tree T_Δ . By Lemma 13, a_l is a $(kn - 1, n)$ -generator of Δ . Therefore $a_l + in - kn = a_l + n(i - k)$ is an n -cogenerator if and only if $i = k - 1$. We conclude that b_i^l can be expressed via $c_{i-k}^l, \dots, c_{i-1}^l$ as follows:

$$b_i^l = \begin{cases} \sum_{j=i-k}^{i-1} c_j^l & \text{if } i \neq k - 1, \\ -1 + \sum_{j=i-k}^{i-1} c_j^l & \text{if } i = k - 1. \end{cases} \quad (7)$$

On the other hand, for $i \geq 0$ one can use Lemmas 6 and 14 to express c_i^l through $b_0^l, b_1^l, \dots, b_i^l$ using numbers $g(a_j)$:

$$c_i^l = g(a_{l+b_0^l+b_1^l+\dots+b_{i-1}^l}) - g(a_{l+b_0^l+b_1^l+\dots+b_i^l}), \quad c_0^l = g(a_l) - g(a_{l+b_0^l}). \quad (8)$$

Indeed, $g(a_{l+b_0^l+b_1^l+\dots+b_i^l})$ is equal to the number of integers not in Δ in the interval $[a_{l+b_0^l+b_1^l+\dots+b_i^l}, a_{l+b_0^l+b_1^l+\dots+b_i^l} + n)$, which is the same as in the interval $[a_l + in, a_l + (i+1)n)$ by the first part of Lemma 14.

Equations (7) and (8) together provide recurrence relations on numbers b_i^l and c_i^l . To start the recursive algorithm, one needs to find initial values $c_{-k}^l, \dots, c_{-1}^l$.

Recall the numbers $N_{ij} = \lfloor \frac{a_j - a_i}{n} \rfloor + 1$ and $K_{ij} = \lfloor \frac{a_j - a_i}{kn+1} \rfloor + 1$. Since we know all edges of the tree T_Δ with initial points less than a_l , we can use Lemma 14 to find numbers N_{ij} for any $i, j \leq l$. Then, by Corollary 9 we can find numbers K_{ji} for any $i, j \leq l$. We can also compute

$$f(a_i) := \sharp(\Delta \cap (-\infty, a_i]) = \sum_{j \leq i} K_{ji}.$$

Indeed, fix a remainder $0 \leq r < kn - 1$, such that the corresponding $(kn - 1)$ -generator $a_j \equiv r \pmod{kn - 1}$ is less than or equal to a_i . Then there are exactly K_{ji} integers in $\Delta \cap (-\infty, a_i]$ with remainder r modulo $kn - 1$.

Note that

$$c_{-i}^l = \sharp(I_{-i+1}^l \cap \Delta) - \sharp(I_{-i}^l \cap \Delta) \quad \text{for all } i > 0. \quad (9)$$

Since a_l is a $(kn - 1, n)$ -generator of Δ , we have $a_l - in \notin \Delta$ for $i > 0$. Consider the biggest element of Δ less than $a_l - in$. By Corollary 8, it is a $(kn - 1)$ -generator a_{α_i} , where $\alpha_i := \max\{j | N_{jl} > i\}$. If the set $\{j | N_{jl} > i\}$ is empty, then $a_l - in \leq 0$ and there are no elements of Δ less than $a_l - in$. For consistency, we set $\alpha_i = -1$, $a_{-1} = -\infty$, and $f(-\infty) = 0$ in this case.

Finally, we compute

$$\begin{aligned} \sharp(I_{-i}^l \cap \Delta) &= f(a_{\alpha_{i-1}}) - f(a_{\alpha_i}) \quad \text{for } i > 1, \\ \sharp(I_{-1}^l \cap \Delta) &= f(a_l) - f(a_{\alpha_1}) - 1, \end{aligned}$$

and the number $\sharp(I_0^l \cap \Delta) = n - g(a_l)$ is given. Therefore, we can use (9) to find numbers $c_{-k}^l, \dots, c_{-1}^l$. $\square$

Theorem 16 *The tree T_Δ completely determines the semimodule Δ .*

Proof The same as in the $m = kn + 1$ case. $\square$

3.4 Example: reconstruction of a $\Gamma_{5,9}$ -semimodule

Let $n = 5$, $k = 2$, and $m = 2 \times 5 - 1 = 9$. Suppose that $g(a_0) = 3$, $g(a_1) = g(a_2) = g(a_3) = 2$, $g(a_4) = 1$, and $g(a_5) = \dots = g(a_8) = 0$. At the first step we reconstruct the path from $a_0 = 0$ to a_∞ in the tree T_Δ . Following the algorithm, we first

need to find numbers c_{-1}^0 and c_{-2}^0 , counting 5-cogenerators in intervals $[-5, 0)$ and $[-10, -5)$, respectively. Since there are no elements of Δ less than $a_0 = 0$, we immediately conclude that

$$c_{-2}^0 = \sharp\{\Delta \cap [-5, 0)\} - \sharp\{\Delta \cap [-10, -5)\} = 0 - 0 = 0,$$

and

$$c_{-1}^0 = \sharp\{\Delta \cap [0, 5)\} - \sharp\{\Delta \cap [-5, 0)\} = (5 - g(a_0)) - 0 = 2.$$

Using the recurrence relations (7) and (8) we immediately compute:

$$\begin{aligned} b_0^0 &= c_{-2}^0 + c_{-1}^0 = 2, & c_0^0 &= g(a_0) - g(a_2) = 3 - 2 = 1, \\ b_1^0 &= -1 + c_{-1}^0 + c_0^0 = 2, & c_1^0 &= g(a_2) - g(a_4) = 2 - 1 = 1, \\ b_2^0 &= c_0^0 + c_1^0 = 2, & c_2^0 &= g(a_4) - g(a_6) = 1, \\ b_3^0 &= c_1^0 + c_2^0 = 2. \end{aligned}$$

Therefore, the path from a_0 to a_∞ is $a_0 \rightarrow a_2 \rightarrow a_4 \rightarrow a_6 \rightarrow a_8 \rightarrow a_\infty$. The smallest 9-generator not covered yet is a_1 . Therefore, our next step is to recover the path from a_1 to a_∞ .

Again, we start by reconstructing numbers c_{-1}^1 and c_{-2}^1 . We have

$$c_{-2}^1 = \sharp\{\Delta \cap [a_1 - 5, a_1)\} - \sharp\{\Delta \cap [a_1 - 10, a_1 - 5)\},$$

and

$$c_{-1}^1 = \sharp\{\Delta \cap [a_1, a_1 + 5)\} - \sharp\{\Delta \cap [a_1 - 5, a_1)\}.$$

The only 9-generator of Δ less than a_1 is a_0 and, moreover, $a_0 > a_1 - 5$. In fact, $a_0 + 5 = a_2 > a_1$. Indeed, $a_0 + 5 = 5$ is a 9-generator, and we have the arrow $a_0 \rightarrow a_2$ in the tree T_Δ . We conclude that

$$\begin{aligned} \sharp\{\Delta \cap [a_1 - 10, a_1 - 5)\} &= 0, \\ \sharp\{\Delta \cap [a_1 - 5, a_1)\} &= 1, \end{aligned}$$

and

$$\sharp\{\Delta \cap [a_1, a_1 + 5)\} = 5 - g(a_1) = 3.$$

Therefore,

$$c_{-2}^1 = 1 \quad \text{and} \quad c_{-1}^1 = 2.$$

We again use the recurrence relations (7) and (8):

$$\begin{aligned} b_0^1 &= c_{-2}^1 + c_{-1}^1 = 3, & c_0^1 &= g(a_1) - g(a_4) = 2 - 1 = 1, \\ b_1^1 &= -1 + c_{-1}^1 + c_0^1 = 2, & c_1^1 &= g(a_4) - g(a_6) = 1, \\ b_2^1 &= c_0^1 + c_1^1 = 2, & c_2^1 &= g(a_6) - g(a_8) = 0. \end{aligned}$$

Therefore, the path from a_1 to a_∞ is $a_1 \rightarrow a_4 \rightarrow a_6 \rightarrow a_8 \rightarrow a_\infty$. The smallest 9-generator not covered yet is a_3 . Therefore, our next step is to recover the path from a_3 to a_∞ .

Similarly to the above, we need to reconstruct numbers c_{-2}^3 and c_{-1}^3 first:

$$c_{-2}^3 = \sharp\{\Delta \cap [a_3 - 5, a_3)\} - \sharp\{\Delta \cap [a_3 - 10, a_3 - 5)\},$$

and

$$c_{-1}^3 = \sharp\{\Delta \cap [a_3, a_3 + 5)\} - \sharp\{\Delta \cap [a_3 - 5, a_3)\}.$$

There are three 9-generators less than a_3 : a_0, a_1 , and a_2 . Since we already know the path $a_0 \rightarrow a_2 \rightarrow a_4$, we immediately conclude that $a_3 < a_0 + 10$. Since a_3 and a_0 have different remainders modulo 9, we get $a_3 < a_0 + 9$. Therefore, there are no elements of Δ less than or equal to $a_3 - 9$. We conclude that a_0, a_1 , and a_2 are the only elements of Δ less than a_3 . Moreover, $a_2 + 5 > a_3$ and $a_1 + 5 > a_3$, while $a_0 + 5 = a_2 < a_3 < a_0 + 10$. Therefore,

$$\begin{aligned}\sharp\{\Delta \cap [a_3 - 10, a_3 - 5)\} &= 1, \\ \sharp\{\Delta \cap [a_3 - 5, a_3)\} &= 2,\end{aligned}$$

and

$$\sharp\{\Delta \cap [a_3, a_3 + 5)\} = 5 - g(a_3) = 3.$$

We conclude that

$$c_{-2}^3 = 1 \quad \text{and} \quad c_{-1}^3 = 1.$$

Once again, we use the recurrence relations (7) and (8):

$$\begin{aligned}b_0^3 &= c_{-2}^3 + c_{-1}^3 = 2, & c_0^3 &= g(a_3) - g(a_5) = 2, \\ b_1^3 &= -1 + c_{-1}^3 + c_0^3 = 2, & c_1^3 &= g(a_5) - g(a_7) = 0, \\ b_2^3 &= c_0^3 + c_1^3 = 2.\end{aligned}$$

Therefore, the path from a_3 to a_∞ is $a_3 \rightarrow a_5 \rightarrow a_7 \rightarrow a_\infty$. See Fig. 10 for the full tree T_Δ .

Finally, we reconstruct the semimodule Δ from the tree T_Δ following the algorithm from Theorem 14. First, we use Lemma 14 to find numbers $N_{0,i}$ for $i = 1, 2, \dots, 8$ and Corollary 9 to find $K_{0,i}$. We get

$$\begin{aligned}N_{0,1} &= 1, & N_{0,2} &= N_{0,3} = 2, & N_{0,4} &= N_{0,5} = 3, \\ N_{0,6} &= N_{0,7} = 4, & N_{0,8} &= 5,\end{aligned}$$

and

$$K_{0,1} = K_{0,2} = K_{0,3} = 1, \quad K_{0,4} = K_{0,5} = K_{0,6} = K_{0,7} = 2, \quad K_{0,8} = 3.$$

Fig. 10 The tree T_Δ for the case $n = 5$, $m = 9$, $g(a_0) = 3$, $g(a_1) = g(a_2) = g(a_3) = 2$, $g(a_4) = 1$, and $g(a_5) = \dots = g(a_8) = 0$

Therefore, we find that a_1, a_2 , and a_3 are in the interval $(0, 9)$, a_4, a_5, a_6, a_7 are in the interval $(9, 18)$, and a_8 is in the interval $(18, 27)$.

Now we need to compare the remainders $r_1, \dots, r_8$ of the 9-generators $a_1, \dots, a_8$. We already know that $r_1 < r_2 < r_3$, and $r_4 < r_5 < r_6 < r_7$. Let us compare r_1 and r_4 . By Lemma 14 and the tree T_Δ , we get $N_{1,4} = 2$. Therefore, by Corollary 7, $K_{1,4} = 1$, which means that $a_4 - a_1 < 9$. Therefore, $r_4 = a_4 - 9 < a_1 = r_1$.

Similarly, one computes that $r_5 < r_1$, $r_2 < r_6 < r_3$, and $r_7 > r_3$. Therefore, we get

$$r_4 < r_5 < r_1 < r_2 < r_6 < r_3 < r_7.$$

To compare r_8 with the rest of the remainders, we compute

$$N_{1,8} = N_{2,8} = 4, \quad N_{3,8} = N_{4,8} = 3, \quad N_{5,8} = N_{6,8} = 2, \quad N_{7,8} = 1,$$

and

$$K_{1,8} = K_{2,8} = K_{3,8} = K_{4,8} = 2, \quad K_{5,8} = K_{6,8} = K_{7,8} = 1.$$

So, $r_8 = a_8 - 18 < a_i - 9 = r_i$ for $i = 5, 6, 7$, $r_8 = a_8 - 18 < a_i = r_i$ for $i = 1, 2, 3$, and $r_8 = a_8 - 18 > a_4 - 9 = r_4$. Therefore,

$$r_4 < r_8 < r_5 < r_1 < r_2 < r_6 < r_3 < r_7,$$

or

$$\begin{aligned} r_4 &= 1, & r_8 &= 2, & r_5 &= 3, & r_1 &= 4, & r_2 &= 5, \\ r_6 &= 6, & r_3 &= 7, & r_7 &= 8. \end{aligned}$$

Finally,

$$\begin{aligned} a_0 &= 0, & a_1 &= r_1 = 4, & a_2 &= r_2 = 5, & a_3 &= r_3 = 7, & a_4 &= r_4 + 9 = 10, \\ a_5 &= r_5 + 9 = 12, & a_6 &= r_6 + 9 = 15, & a_7 &= r_7 + 9 = 17, \end{aligned}$$

and

$$a_8 = r_8 + 18 = 20.$$

3.5 Bounce path and statistic

In the case $m = kn + 1$ the path $a_0 \rightarrow a_{b_0^0} \rightarrow a_{b_0^0+b_1^0} \rightarrow \cdots \rightarrow a_{b_0^0+\cdots+b_s^0}$ in the tree T_Δ can be compared with the *bounce path*, constructed by J. Haglund [12] in the case $m = n + 1$ and generalized by N. Loehr [17] for the case $m = kn + 1$. The numbers $b_0^0, \dots, b_s^0$ are equal to the horizontal steps in the bounce path. Here we recall Loehr's definition and check that it matches with the path from a_0 to a_{kn} in the tree T_Δ . We also generalize the bounce path and statistic to the case $m = kn - 1$ by considering the path from a_0 to a_∞ .

Definition 14 [17] Let $D \in Y_{m,n}$, where $m = kn + 1$. The bounce path for D is defined as follows. We start from the northwest corner. We alternate southward and eastward steps with the first step going southward. We always stay outside the diagram D .

On each southward step we go south until we hit a horizontal piece of the boundary of D . Each eastward step is equal to the sum of the last k southward steps (if there were less than k southward steps yet, then it is equal to the sum of all preceding southward steps).

Let us introduce the coordinates so that the southwest corner is $(0, 0)$. Then the bounce path starts from $(0, n)$ and finishes at $(kn, 0)$. The bounce statistic is defined as follows:

Definition 15 [17] Let $(v_0, \dots, v_a)$ and $(h_0, \dots, h_a)$ be the vertical and horizontal steps of the bounce path, respectively. Then $\text{bounce}(D)$ is given by the formula:

$$\text{bounce}(D) := (n - v_0) + (n - v_0 - v_1) + \cdots + \left(n - \sum_{0 \leq i \leq a} v_i\right).$$

In other words, $\text{bounce}(D)$ is equal to the sum of vertical coordinates of the southwest corners of the bounce path.

Let now Δ be a semimodule over $\Gamma_{m,n}$, $m = kn + 1$. Let D be the corresponding Young diagram, and, as in the Sect. 2.3, $G_{kn+1,n}(D)$ be the diagram with columns $g(a_0), \dots, g(a_{kn})$.

Let T_Δ be the bounce tree, and $a_0 \rightarrow a_{b_0^0} \rightarrow \cdots \rightarrow a_{b_0^0+\cdots+b_s^0} = a_{kn}$ be the path from a_0 to a_{kn} in it.

Theorem 17 The horizontal steps $(h_0, \dots, h_a)$ of the bounce path for the diagram $G_{kn+1,n}(D)$ are equal to $(b_0^0, \dots, b_s^0)$. In particular, $a = s$.

Proof Recall the formulas (4) and (5) (we plug $l = 0$):

$$b_i^0 = \begin{cases} \sum_{j=i-k}^{i-1} c_j^0 & \text{if } i \neq k-1, \\ 1 + \sum_{j=i-k}^{i-1} c_j^0 & \text{if } i = k-1, \end{cases}$$

and

$$c_i^0 = g(a_{b_0^0+b_1^0+\dots+b_{i-1}^0}) - g(a_{b_0^0+b_1^0+\dots+b_i^0}), \quad c_0^0 = g(a_0) - g(a_{b_0^0}).$$

where c_i^0 is the number of n -cogenerators in the interval $(in, (i+1)n]$.

We immediately see that

$$c_i^0 = \begin{cases} 0 & \text{if } i < -2, \\ 1 & \text{if } i = -2, \\ n - g(a_0) - 1 & \text{if } i = -1. \end{cases}$$

Indeed, $-n$ is the smallest n -cogenerator, and there are exactly $n - g(a_0)$ n -cogenerators less than zero.

For $i \geq 0$ the recurrence relations on the numbers c_i^0 and b_i^0 are almost the same as the definitions of the vertical and horizontal steps of the bounce path, respectively. There are two differences:

1. The first vertical step equals $v_0 = n - g(a_0) = c_{-1}^0 + 1 = c_{-1}^0 + c_{-2}^0$.
2. For $i = k - 1$ one has $b_{k-1}^0 = 1 + \sum_{j=-1}^{k-2} c_j^0 = \sum_{j=-2}^{k-2} c_j^0$.

One immediately sees that those differences cancel each other. Therefore one gets

$$v_i = \begin{cases} c_{i-1}^0 & \text{if } i > 0, \\ c_{-1}^0 + c_{-2}^0 & \text{if } i = 0, \end{cases}$$

and

$$h_i = b_i^0 \quad \text{for all } i \geq 0. \quad \square$$

Theorem 18 *We get the following relation:*

$$\text{bounce}(G_{kn+1,n}(D)) = \delta - |D|.$$

Proof Indeed, we have

$$\delta - |D| = \sharp(\mathbb{Z}_{>0} \setminus \Delta) = \sum_{i=0}^{\infty} \sharp((in, (i+1)n] \setminus \Delta).$$

By Lemma 10, $[a_{b_0^0+\dots+b_i^0}, ni] \subset \Delta$. We get

$$\sharp((in, (i+1)n] \setminus \Delta) = g(a_{b_0^0+\dots+b_{i-1}^0}) \quad \text{and} \quad \sharp((0, n] \setminus \Delta) = g(a_0).$$

Therefore,

$$\delta - |D| = g(a_0) + g(a_{b_0^0}) + g(a_{b_0^0+b_1^0}) + \dots + g(a_{kn}),$$

Note that this matches the definition of the bounce statistic. □

Fig. 12 The diagram $G_{8,3}(\Delta)$ with the bounce path for the $\Gamma_{3,8}$ -semimodule $\Delta = \{0, 3, 4, 6, 7, 8, 9, \dots\}$. Here the 8-generators are $0, 3, 4, 6, 7, 9, 10, 13$, $g(0) = 2$, $g(3) = g(4) = 1$, and $g(6) = g(7) = g(9) = g(10) = g(13) = 0$

Recall the formulas (7) and (8) (we plug $l = 0$):

$$b_i^0 = \begin{cases} \sum_{j=i-k}^{i-1} c_j^0 & \text{if } i \neq k-1, \\ -1 + \sum_{j=i-k}^{i-1} c_j^0 & \text{if } i = k-1, \end{cases}$$

and

$$c_i^0 = g(a_{b_0^0+b_1^0+\dots+b_{i-1}^0}) - g(a_{b_0^0+b_1^0+\dots+b_i^0}), \quad c_0^0 = g(a_0) - g(a_{b_0^0}),$$

where c_i^0 is the number of n -cogenerators in the interval $[in, (i+1)n)$.

We immediately see that

$$c_i^0 = \begin{cases} 0 & \text{if } i < -1, \\ n - g(a_0) & \text{if } i = -1. \end{cases}$$

Indeed, $-n$ is the smallest n -cogenerator, and there are exactly $n - g(a_0)$ n -cogenerators less than zero.

Comparing the recurrence relations one immediately gets

$$v_i = c_{i-1}^0 \quad \text{and} \quad h_i = b_i^0 \quad \text{for all } i \geq 0.$$

Therefore, the bounce path in the $m = kn - 1$ case is constructed in the same way as in the $m = kn + 1$ case, except that the $(k-1)$ st horizontal step is shorter by 1 (we count steps starting from 0):

$$h_i = \begin{cases} \sum_{j=i-k}^{i-1} v_j & \text{if } i \neq k-1, \\ -1 + \sum_{j=i-k}^{i-1} v_j & \text{if } i = k-1. \end{cases}$$

We illustrate the bounce path in Fig. 12.

Let us return to the $(3, 8)$ -semimodule $\Delta = \{0, 3, 4, 6, 7, 8, 9, \dots\}$ from example in Fig. 9. The 8-generators are $0, 3, 4, 6, 7, 9, 10, 13$, so the diagram $G_{8,3}(\Delta)$ with the bounce path is shown in Fig. 12.

4 Bijective proof of symmetry for $m \leq 3$

In this section we give a bijective proof of the identity $c_{n,m}(q, t) = c_{n,m}(t, q)$ for $m = 2$ and $m = 3$.

The $(2k + 1, 2)$ case is very simple. We should consider Young diagrams in the set $Y_{2k+1,2}$. Such a diagram D_i can have only one row with $i \leq k$ boxes in it, and $|D_i| = h_+(D_i) = i$. Therefore, the polynomial $c_{2k+1,2}(q, t)$ is given by the following formula:

$$c_{2k+1,2}(q, t) = q^k + q^{k-1}t + \cdots + qt^{k-1} + t^k,$$

which is obviously symmetric in q and t . The corresponding involution on diagrams sends the diagram D_i to D_{k-i} .

The case $(n, 3)$ turns out to be more subtle. We consider Young diagrams in the set $Y_{n,3}$. Such a diagram $D_{\alpha,\beta}$ has two rows of length α and β , $\alpha \leq \beta$. Moreover,

$$\alpha \leq k := \left\lfloor \frac{n}{3} \right\rfloor, \quad \beta \leq \left\lfloor \frac{2n}{3} \right\rfloor.$$

Consider the map $\phi : Y_{n,3} \rightarrow \mathbb{Z}^2$, $\phi(D) := (\delta - |D|, h_+(D))$.

Theorem 20 *The map ϕ is injective. The image of ϕ is the set of integer points inside the triangle:*

$$\mathbf{T}_\delta := \{(a, b) : a + b \leq \delta, a + 2b \geq \delta, 2a + b \geq \delta\}.$$

Proof The area of $D_{\alpha,\beta}$ equals $\alpha + \beta$. To compute $h_+(D_{\alpha,\beta})$, we consider three cases:

1. $0 \leq \alpha \leq \beta \leq k$. In this case one can check that $h_+(D_{\alpha,\beta}) = \beta$. Therefore, $\phi(D_{\alpha,\beta}) = (\delta - \alpha - \beta, \beta)$, and the image of ϕ is given by the formula

$$\{(a, b) \in \mathbf{T}_\delta \cap \mathbb{Z}^2 : b \leq k\}.$$

2. $\beta > k, \beta - \alpha \leq k$. In this case $h_+(D_{\alpha,\beta}) = 2\beta - k$, $\phi(D_{\alpha,\beta}) = (\delta - \alpha - \beta, 2\beta - k)$, and the image of ϕ is given by the formula

$$\{(a, b) \in \mathbf{T}_\delta \cap \mathbb{Z}^2 : b > k, b + k \text{ even}\}.$$

Indeed, the boxes contributing to $h_+(D_{\alpha,\beta})$ form three groups of sizes α , $\beta - \alpha$, and $\beta - k$, hence

$$h_+(D_{\alpha,\beta}) = (\beta - \alpha) + \alpha + (\beta - k) = 2\beta - k.$$

We show these boxes in Fig. 13.

3. $\beta - \alpha > k$. In this case $h_+(D_{\alpha,\beta}) = 2\alpha + k + 1$, $\phi(D_{\alpha,\beta}) = (\delta - \alpha - \beta, 2\alpha + k + 1)$, and the image of ϕ is given by the formula

$$\{(a, b) \in \mathbf{T}_\delta \cap \mathbb{Z}^2 : b > k, b + k \text{ odd}\}.$$

Note that these three image sets cover all integer points in $\mathbf{T}_\delta$ with no overlaps. $\square$

Note that the triangle $\mathbf{T}_\delta$ is symmetric with respect to the diagonal $a = b$. Therefore, one gets the following

Fig. 13 Boxes contributing to $h_+(D_{\alpha,\beta})$ for $\beta > k$, $\beta - \alpha \leq k$

Corollary 10 *There exists an involution $\mathbf{i}$ on $Y_{n,3}$ such that*

$$\delta - |D| = h_+(\mathbf{i}(D)).$$

In particular, the polynomial $c_{n,3}(q, t)$ is symmetric in q and t .

Proof The involution $\mathbf{i}$ is given by the formula $\mathbf{i} = \phi^{-1} \circ \mathbf{s} \circ \phi$, where $\mathbf{s} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the symmetry with respect to the diagonal $a = b$. $\square$

Acknowledgements The authors are grateful to D. Armstrong, L. Götsche, J. Haglund, A. Iarrobino, L. Migliorini, A. Oblomkov, J. Rasmussen and V. Shende for useful discussions. Special thanks to K. Lee, L. Li and N. Loehr for communicating us their proof of Conjecture 1 for $m \leq 4$. The research of E.G. was partially supported by the grants RFBR-10-01-00678, NSH-8462.2010.1 and the Simons foundation.

References

- Altman, A., Iarrobino, A., Kleiman, S.: Irreducibility of the compactified Jacobian. In: Nordic Summer School NAVF, Oslo, 1976. Noordhoff, Groningen (1977)
- Anderson, J.: Partitions which are simultaneously t_1 - and t_2 -core. *Discrete Math.* **248**(1–3), 237–243 (2002)
- Armstrong, D.: Rational Catalan Combinatorics, slides from a talk at JMM 2012, Boston
- Beauville, A.: Counting rational curves on K3-surfaces. *Duke Math. J.* **97**, 99–108 (1999)
- Delgado de la Mata, F.: Gorenstein curves and symmetry of the semigroup of values. *Manuscr. Math.* **61**(3), 285–296 (1988)
- Ford, B., Mai, H., Sze, L.: Self-conjugate p - and q -core partitions and blocks of A_n . *J. Number Theory* **129**, 858–865 (2009)
- Garsia, A., Haiman, M.: A remarkable q, t -Catalan sequence and q -Lagrange inversion. *J. Algebr. Comb.* **5**(3), 191–244 (1996)
- Garsia, A., Haglund, J.: A proof of the q, t -Catalan positivity conjecture. *Discrete Math.* **256**(3), 677–717 (2002)
- Gorsky, E., Mazin, M.: Compactified Jacobians and q, t -Catalan numbers. *J. Comb. Theory, Ser. A* **120**, 49–63 (2013)
- Gorsky, E., Oblomkov, A., Rasmussen, J., Shende, V.: Torus knots and the rational DAHA. [arXiv: 1207.4523](https://arxiv.org/abs/1207.4523)
- Grueel, G.-M., Pfister, G.: Moduli spaces for torsion free modules on curve singularities I. *J. Algebr. Geom.* **2**, 81–135 (1993)
- Haglund, J.: Conjectured statistics for the q, t -Catalan numbers. *Adv. Math.* **175**(2), 319–334 (2003)
- Haglund, J.: The q, t -Catalan Numbers and the Space of Diagonal Harmonics. With an Appendix on the Combinatorics of Macdonald Polynomials. University Lecture Series, vol. 41. American Mathematical Society, Providence (2008)
- Hikita, T.: Affine Springer fibers of type A and combinatorics of diagonal coinvariants. [arXiv: 1203.5878](https://arxiv.org/abs/1203.5878)
- Kunz, E.: The value-semigroup of a one-dimensional Gorenstein ring. *Proc. Am. Math. Soc.* **25**, 748–751 (1970)

16. Lee, K., Li, L., Loehr, N.: Combinatorics of certain higher q, t -Catalan polynomials: chains, joint symmetry, and the Garsia–Haiman formula. [arXiv:1211.2191](#)
17. Loehr, N.: Conjectured statistics for the higher q, t -Catalan sequences. *Electron. J. Comb.* **12**, 9 (2005)
18. Loehr, N., Warrington, G.: A continuous family of partition statistics equidistributed with length. *J. Comb. Theory, Ser. A* **116**, 379–403 (2009)
19. Lusztig, G., Smelt, J.M.: Fixed point varieties on the space of lattices. *Bull. Lond. Math. Soc.* **23**(3), 213–218 (1991)
20. Milnor, J.W., Stasheff, J.D.: *Characteristic Classes*. Annals of Mathematics Studies, vol. 76. Princeton University Press, Princeton (1974)
21. Oblomkov, A., Shende, V.: The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. *Duke Math. J.* **161**(7), 1277–1303 (2012)
22. Oblomkov, A., Rasmussen, J., Shende, V.: The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link. [arXiv:1201.2115](#)
23. Piontkowski, J.: Topology of the compactified Jacobians of singular curves. *Math. Z.* **255**(1), 195–226 (2007)
24. Puchta, J.C.: Partitions which are p - and q -core. *Integers* **1**, A6 (2001). 3 pp.