Spectral properties of descent algebra elements

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Abstract In this article, we determine the eigenvalues and their corresponding multiplicities of the action on the group algebra of a finite Coxeter group of an element of its descent algebra. Meanwhile, we identify a slight error in the paper of Bergeron, Bergeron, Howlett, and Taylor in a formula for certain structure coefficients in the descent algebra. We provide the correct formula, and give an example which explicitly shows the difference between both formulas.

Keywords Descent algebra · Coxeter group · Group algebra · Regular representation · Eigenvalues · Multiplicities

1 Introduction

We keep the usual notations $\mathbb{K}$ for an algebraically closed field of characteristic 0, and (W, S) for a finite Coxeter system, that is to say, W is a finite group generated by the elements of S subject to the defining relations

$$(sr)^{m_{sr}} = e, \quad \text{for all } s, r \in S$$

where e is the identity element of W , the m_{sr} are positive integers, and $m_{ss} = 1$ for all $s \in S$.

Let $J \subseteq S$. We naturally use the notations W_J for the parabolic subgroup of W generated by the elements of J , and c_J for a Coxeter element of W_J which is a product of all elements of J taken in some fixed order. We write $\overline{c_J}$ for the conjugacy class of c_J .

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Let $J \subseteq S$. We write $\tilde{J}$ for the set of subsets $K \subseteq S$ such that the parabolic subgroups W_J and W_K are conjugate. Let $\tilde{J}_1, \dots, \tilde{J}_p$ be pairwise different such that

$$\{\tilde{J}_i\}_{i \in [p]} = \{\tilde{J}\}_{J \subseteq S}.$$

Let $J, K \subseteq S$. We write ${}^J W^K$ for the distinguished cross section for the double coset space $W_J \backslash W / W_K$. If $J = \{\emptyset\}$ resp. $K = \{\emptyset\}$, we just write W^K resp. ${}^J W$.

Let $J \subseteq S$. We write

$$x_J := \sum_{w \in W^J} w.$$

Let $\mathcal{E}_W := \{x_J \mid J \subseteq S\}$. We know that $\mathbb{K}[\mathcal{E}_W]$ is the descent algebra of W which is a subalgebra of the group algebra $\mathbb{K}[W]$ multiplicatively equipped with [9, Theorem 1]:

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L$$

for $J, K \subseteq S$, with

$$a_{JKL} := |\{x \in {}^J W^K \mid x^{-1} W_J x \cap W_K = W_L\}|.$$

Let $u = \sum_{w \in W} \lambda_w w \in \mathbb{K}[W]$. We write $R_W(u) = (\lambda_{ww'^{-1}})_{w, w' \in W}$ for the matrix of the regular representation of the left-multiplication action of u on $\mathbb{K}[W]$ relatively to the standard basis $\{w \mid w \in W\}$. Let $I_{|W|}$ be the identity matrix of $\mathbb{K}^{|W| \times |W|}$, and x a variable. Recall that the spectrum of $R_W(u)$ is the set of the roots of the polynomial $\det(R_W(u) - x I_{|W|})$, and the multiplicity of an element of the spectrum is the multiplicity of this polynomial root. The purpose of this article is to prove the following theorem:

Theorem 1.1 *Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\mathcal{E}_W]$. Then the spectrum of $R_W(d)$ is*

$$Sp(R_W(d)) = \left\{ \Delta_j = \sum_{i=1}^p a_{J_i J_j J_j} \left(\sum_{K_i \in \tilde{J}_i} \lambda_{K_i} \right) \right\}_{j \in [p]}$$

with corresponding multiplicities

$$\{m_{\Delta_j} = |\overline{c_{J_j}}|\}_{j \in [p]}.$$

Let $w \in W$. Recall that the set of left resp. right descent set of w is

$$\text{DES}_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\} \quad \text{resp.} \quad \text{DES}_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

ℓ being the statistic which gives the length of the minimal expression of the elements of W in terms of elements of S .

Let $J \subseteq S$. We write D_J for the set of elements of W with right descent set J . We have the disjoint union

$$W^{S \setminus K} = \bigcup_{J \subseteq K} D_J.$$

Thus setting

$$y_J := \sum_{w \in D_J} w,$$

we get

$$x_{S \setminus K} = \sum_{J \subseteq K} y_J,$$

and hence by Möbius inversion formula

$$y_J = \sum_{K \subseteq J} (-1)^{|J \setminus K|} x_{S \setminus K}.$$

The descent algebra is a nowadays well studied object on the intersection of combinatorics and algebra. As examples, Blessenohl, Hohlweg, and Schocker proved in [3, Main Theorem] a symmetry property of the descent algebra, when it is linked to the representation theory of its Coxeter group. Using hyperplane arrangements and random walk properties, Brown determined in [4, Theorem 5] the condition of diagonalizability of the action of element of the descent algebra of a finite Coxeter group, and gave a remarkable approach of the eigenvalues and corresponding multiplicities of its regular representation. However, the interest is particularly showed for the descent algebra relative to the Coxeter system of the symmetric group $(A_n, S_{A_n} = \{t_1, \dots, t_n\})$, where t_i is the transposition $(i \ i + 1)$. In a discussion with Krattenthaler, Thibon explained the proof of [6, Theorem 56] which states the eigenvalues and their multiplicities of the action on the group algebra of symmetric group of the descent algebra element

$$\sum_{J \subseteq S_{A_n}} q^{\text{Maj}(J)} y_J \in \mathbb{R}(q)[\mathcal{E}_{A_n}],$$

with

$$\text{Maj}(J) := \sum_{j \in \{i \in [n] \mid t_i \in J\}} j.$$

In [8, Theorem 4.1], Schocker worked on the descent algebra of the symmetric group. And even the author diagonalized in [7, Theorem 1.1] the regular representation of the element

$$\sum_{J \subseteq S_{A_n}} \text{Des}_X(J) y_J \in \mathbb{R}(X_1, \dots, X_n)[\mathcal{E}_{A_n}],$$

with

$$\text{Des}_X(J) := \sum_{j \in \{i \in [n] \mid t_i \in J\}} X_j.$$

This paper is organized as follows. We begin with the calculation of the formula for the coefficients $a_{J_i J_j J_j}$. Then we determine the eigenvalues and their corresponding multiplicities of the regular representation of an element of the descent algebra of

a finite Coxeter group. In an Appendix we treat the Coxeter group F_4 as complete example and we give a counterexample to the formula in [2, Theorem 6.5].

2 Special coefficients of the descent algebra

In this section, we determine a formula for the values of a_{JKK} . Through a slight modification, our formula corrects a mistake in the proposed formula [2, Theorem 6.5].

Let $J \subseteq S$. We write N_J for the subgroup [5, Corollary 3]

$$N_J := \{w \in W \mid w^{-1}W_J w = W_J\} \cap W^J.$$

Let $w \in W$ and U, V be subgroups of W . We write

$${}^U w^V := \{x \in U w V \mid 1(x) \leq 1(uxv), \forall u \in U, \forall v \in V\}$$

for the set of minimal double coset representatives of w relative to U and V . If $U = \{e\}$ resp. $V = \{e\}$, we just write w^V resp. ${}^U w$. For the case of parabolic subgroups, we just write ${}^{W_J} w^{W_K} = {}^J w^K$ with $J, K \subseteq S$.

Lemma 2.1 *Let $w \in W$ and $J, K \subseteq S$. Then the set ${}^J w^K$ contains a unique element. In this case, we consider ${}^J w^K$ no more as a subset of W but as an element of W .*

Proof It can be read off in [9, 1. Introduction]. □

Let $K \subseteq S$ and $K' \in \tilde{K}$. We write

$$C_{K'K} := \{w \in W \mid w^{-1}W_{K'} w = W_K\}.$$

Lemma 2.2 *Let $c \in C_{K'K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}} c^{N_K W_K}$. Then $c_{K'K} \in C_{K'K}$.*

Proof If $c = n'k'c_{K'K}nk$ with $n'k' \in N_{K'}W_{K'}$ and $nk \in N_K W_K$, then

$$(n'k'c_{K'K}nk)^{-1}W_{K'}n'k'c_{K'K}nk = W_K,$$

$$(c_{K'K}nk)^{-1}W_{K'}c_{K'K}nk = W_K,$$

$$c_{K'K}^{-1}W_{K'}c_{K'K} = nkW_K(nk)^{-1},$$

$$c_{K'K}^{-1}W_{K'}c_{K'K} = W_K. \quad \square$$

Let $E \subseteq W$ and U, V be subgroups of W . We write

$${}^U E^V := \bigcup_{w \in E} {}^U w^V.$$

For the case of parabolic subgroups, we just write ${}^{W_J} E^{W_K} = {}^J E^K$, with $J, K \subseteq S$.

Lemma 2.3 Let $K \subseteq S$, $K' \in \widetilde{K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_KW_K}$. Then

$${}^{K'}(c_{K'K}N_K)^K = c_{K'K}N_K \quad \text{and} \quad {}^{K'}(N_{K'}c_{K'K})^K = N_{K'}c_{K'K}.$$

Proof It is clear that $(c_{K'K}N_K)^K = c_{K'K}N_K$. Effectively, since $c_{K'K}$ is a left coset representative of the subgroup N_KW_K , and the elements of N_K are left coset representatives of W_K , then the elements of $c_{K'K}N_K$ are left coset representatives of W_K .

We just then have to prove that ${}^{K'}(c_{K'K}N_K) = c_{K'K}N_K$. Let $c_{K'K}n \in c_{K'K}N_K$ and let us suppose that $c_{K'K}n = k'b$, where $k' \in W_{K'}$ and $b \in {}^{K'}W$. Then $1(c_{K'K}n) = 1(k'b) = 1(k') + 1(b)$ i.e.

$$1(c_{K'K}n) \geq 1(b).$$

On the other hand, we have $(k')^{-1}c_{K'K}n = c_{K'K}k_1n = c_{K'K}nk_2 = b$ with $k_1, k_2 \in W_K$. Then $1(c_{K'K}nk_2) = 1(c_{K'K}n) + 1(k_2) = 1(b)$, i.e.

$$1(c_{K'K}n) \leq 1(b).$$

The only possibility is then $k_2 = k_1 = k' = e$, so we get the result.

The proof for ${}^{K'}(N_{K'}c_{K'K})^K = N_{K'}c_{K'K}$ is analogous. $\square$

Lemma 2.4 Let $K \subseteq S$, $K' \in \widetilde{K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_KW_K}$. Then

$$\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}w = W_K\} = c_{K'K}N_K = N_{K'}c_{K'K}.$$

Proof It is clear that $\{w \in {}^KW^K \mid w^{-1}W_Kw = W_K\} = N_K$ for all $K \subseteq S$.

- The map $\phi : \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}w = W_{K'}\} \rightarrow \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}w = W_K\}$

$$n \mapsto nc_{K'K}$$

is clearly injective.

- The map $\phi' : \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}w = W_K\} \rightarrow \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}w = W_{K'}\}$,

$$x \mapsto {}^{K'}(c_{K'K}x^{-1})^{K'}$$

is injective. Effectively, $\phi'(x) = \phi'(y)$ means $c_{K'K}x^{-1} = u_1c_{K'K}y^{-1}u_2$ with $u_1, u_2 \in W_{K'}$. Then $c_{K'K}x^{-1} = c_{K'K}vy^{-1}u_2$ and $x^{-1} = vy^{-1}u_2$ with $v \in W_K$. The only possibility is $v = u_2 = e$.

Then we deduce that ϕ is bijective and $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}w = W_K\} = N_{K'}c_{K'K}$.

The proof is analogous for $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}w = W_K\} = c_{K'K}N_K$. $\square$

Let $J, K \subseteq S$, and, for all $K' \in \widetilde{K}$, let us fix an element $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_KW_K}$. We write

$$H_{JK} := \{K' \in \widetilde{K} \mid c_{K'K} = {}^Jc_{K'K}\}.$$

We can now give the formula for the determination of a_{JKK} .

Theorem 2.5 *Let $J, K \subseteq S$. We have*

$$a_{JKK} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.$$

Proof Recall that

$$a_{JKK} := \left| \left\{ w \in {}^J W^K \mid W_J \cap w W_K w^{-1} = w W_K w^{-1} \right\} \right|,$$

with $w W_K w^{-1} = W_{K'}$ and $K' \in \tilde{K} \cap 2^J$.

We have $w \in c_{K'K} N_K = N_{K'} c_{K'K}$. But we must also have $w = {}^J w$. That means:

- On the one hand, we must have $c_{K'K} = {}^J c_{K'K}$. Otherwise ${}^J(c_{K'K} N_K) \cap C_{K'K} = \emptyset$.
- On the other hand, if $c_{K'K} = {}^J c_{K'K}$, then $w \in {}^J(N_{K'} c_{K'K}) = ({}^J N_{K'}) c_{K'K}$.

Since

$$|{}^J N_{K'}| = \frac{|N_{K'}|}{|W_J \cap N_{K'}|},$$

it follows that

$$\begin{aligned} a_{JKK} &= \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_{K'}|}{|W_J \cap N_{K'}|} \\ &= \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}. \end{aligned}$$

since W_K and $W_{K'}$ are conjugate. □

Let $J' \in \tilde{J}$ and $K' \in \tilde{K}$. From Theorem 2.5, we deduce that

$$a_{JKK} = a_{J'K'K'}.$$

This last equality can also be found in [2, Theorem 6.2].

3 Eigenvalues and multiplicities

We are now able to determine the eigenvalues and their corresponding multiplicities.

Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\mathcal{E}_W]$. We write $v_{\mathcal{E}_W}(d)$ for the column vector of d relative to the basis $\mathcal{E}_W$, and $M_{\mathcal{E}_W}(d)$ for the matrix of the left-multiplication action of d on $\mathbb{K}[\mathcal{E}_W]$ relative to the basis $\mathcal{E}_W$ i.e.

$$v_{\mathcal{E}_W}(d) = (\lambda_K)_{K \subseteq S} \quad \text{and} \quad M_{\mathcal{E}_W}(d) = \left(\sum_{J \subseteq S} \lambda_J a_{JKL} \right)_{L, K \subseteq S}.$$

Let $n \geq 2$ and $(d_i)_{i \in [n]} \in \mathbb{K}[\mathcal{E}_W]^n$. We have

$$v_{\mathcal{E}_W} \left(\prod_{i \in [n]}^{\rightarrow} d_i \right) = \left(\prod_{i \in [n-1]}^{\rightarrow} M_{\mathcal{E}_W}(d_i) \right) \cdot v_{\mathcal{E}_W}(d_n).$$

Recall that the noncommutative multiplication is defined in the following way:

$$\prod_{i \in [n]}^{\rightarrow} v_i = v_1 v_2 \dots v_n$$

for elements v_i of a noncommutative group algebra.

We write $M_{\mathcal{E}_W}(d)|_{\bullet, K}$ for the column of $M_{\mathcal{E}_W}(d)$ corresponding to the basis vector x_K , i.e.

$$M_{\mathcal{E}_W}(d)|_{\bullet, K} = \left(\sum_{J \subseteq S} \lambda_J a_{JKL} \right)_{L \subseteq S} = v_{\mathcal{E}_W}(d \cdot x_K).$$

Lemma 3.1 *Let $d \in \mathbb{K}[\mathcal{E}_W]$. Then $R_W(d)$ and $M_{\mathcal{E}_W}(d)$ have the same spectrum.*

Proof It is clear that $Sp(M_{\mathcal{E}_W}(d)) \subseteq Sp(R_W(d))$.

We write $0_{2|S|}$ for the matrix with entry 0 of $\mathbb{K}^{2|S| \times 2|S|}$, and $I_{2|S|}$ for the identity matrix of $\mathbb{K}^{2|S| \times 2|S|}$. Let $\sum_{i=0}^{2|S|} \mu_i t^i$ be the characteristic polynomial of $M_{\mathcal{E}_W}(d)$ in the variable t with $(\mu_i)_{i \in \{0\} \cup [2|S|]} \in \mathbb{K}^{2|S|+1}$. We have $\sum_{i=0}^{2|S|} \mu_i M_{\mathcal{E}_W}(d)^i = 0_{2|S|}$, especially

$$\mu_0 I_{2|S|}|_{\bullet, S} + \sum_{i=1}^{2|S|} \mu_i M_{\mathcal{E}_W}^{i-1}(d) \cdot M_{\mathcal{E}_W}(d)|_{\bullet, S} = 0_{2|S|}|_{\bullet, S}.$$

Since $I_{2|S|}|_{\bullet, S} = v_{\mathcal{E}_W}(e)$, and $M_{\mathcal{E}_W}(d)|_{\bullet, S} = v_{\mathcal{E}_W}(d)$, then

$$\mu_0 v_{\mathcal{E}_W}(e) + \sum_{i=1}^{2|S|} \mu_i M_{\mathcal{E}_W}^{i-1}(d) \cdot v_{\mathcal{E}_W}(d) = v_{\mathcal{E}_W}(0).$$

This means $\mu_0 e + \sum_{i=1}^{2|S|} \mu_i d^i = 0$ and $Sp(R_W(d)) \subseteq Sp(M_{\mathcal{E}_W}(d))$. □

For the rest of the section, we need a total order $\succ$ on the subsets of $S = \{s_i\}_{i \in [|S|]}$ which was introduced by F. and N. Bergeron at the end of the second section of [1]: We define $\min J := \min\{i \in [|S|] \mid s_i \in J\}$, and assume that $\min \emptyset = |S| + 1$. Let $J, K \subseteq S$ such that $J \neq K$.

- If $\min J > \min K$ then $J \succ K$.
- Otherwise $J \succ K$ if and only if $J \setminus \{s_{\min J}\} \succ K \setminus \{s_{\min K}\}$.

We have already seen the definition of the set $\{\tilde{J}_i\}_{i \in [p]}$ in the introduction.

Let K_i be the element of $\tilde{J}_i$ such that $L_i \succ K_i$ for all $L_i \in \tilde{J}_i \setminus \{K_i\}$. We order the sets $\{\tilde{J}_i\}_{i \in [p]}$ such that $K_i \succ K_j$ if $i < j$.

Proposition 3.2 Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\mathcal{E}_W]$. Then the spectrum of the matrix $R_W(d)$ is

$$Sp(R_W(d)) = \left\{ \sum_{i=1}^p a_{K_i K_j K_j} \left(\sum_{L_i \in \tilde{J}_i} \lambda_{L_i} \right) \right\}_{j \in [p]}.$$

Proof We order the basis $(x_J)_{J \subseteq S}$ of $\mathbb{K}[\mathcal{E}_W]$ according to the total order $\succ$ of F and N . Bergeron that means we get a new ordered basis $(x_{L_i})_{i \in [2^{|S|}]}$ such that $L_i \succ L_j$ if $i < j$. We assume that the rows and columns of the matrix $M_{\mathcal{E}_W}(d)$ are ordered in increasing order by the order $<$. We get $d \cdot x_{L_j} \in \langle \{x_{L_i}\}_{i \in [j]} \rangle$. Then the matrix of d on the basis $(x_{L_i})_{i \in [2^{|S|}]}$ is an upper triangular matrix. The characteristic polynomial of this matrix in the variable t is

$$\prod_{K \subseteq S} \left(\left(\sum_{J \subseteq S} \lambda_J a_{J K K} \right) - t \right),$$

and

$$Sp(M_{\mathcal{E}_W}(d)) = \left\{ \sum_{J \subseteq S} \lambda_J a_{J K K} \right\}_{K \subseteq S}.$$

Since $a_{J K K} = a_{J' K' K'}$ for all $J, J' \in \tilde{J}_i$, and all $K, K' \in \tilde{J}_j$, we get the result. $\square$

Proposition 3.3 Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\mathcal{E}_W]$, and $\Delta_j = \sum_{i=1}^p a_{K_i K_j K_j} (\sum_{L_i \in \tilde{J}_i} \lambda_{L_i})$. Then the multiplicity of the eigenvalue Δ_j of $R_W(d)$ is

$$m_{\Delta_j} = |\overline{c_{J_j}}|.$$

Proof Let $A = (a_{K_i K_j K_j})_{i, j \in [p]}$, $m = (m_{\Delta_j})_{j \in [p]}$, $u = (|W|)_{j \in [p]}$, and $c = (|\overline{c_{J_j}}|)_{j \in [p]}$. At the end of the sixth section of [2], it is proved that $A^{-1}u = c$.

Let tr be the trace map of square matrix. We have $tr(R_W(d)) = |W| \sum_{J \subseteq S} \lambda_J$. Then, $\sum_{i=1}^j a_{K_j K_i K_i} m_{\Delta_i} \lambda_{L_j} = |W| \lambda_{L_j}$ for $L_j \in \tilde{J}_j$ i.e.

$$\sum_{i=1}^j a_{K_j K_i K_i} m_{\Delta_i} = |W|.$$

In matrix form, we get $Am = u$. Thus $A^{-1}u = m$. $\square$

We note that with the matrix relation $Am = u$, we can also get the cardinalities of the conjugacy classes of W .

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Appendix A: Example of the symmetry group of the 24-cell

Recall that the Coxeter system of the symmetry group of the 24-cell with cardinality 1152 is $(F_4, S_{F_4} = \{s_i\}_{i \in [4]})$, and its Coxeter graph is

$$s_1 \longleftrightarrow s_2 \overset{4}{\longleftrightarrow} s_3 \longleftrightarrow s_4.$$

Using Theorem 2.5 and the values of $|N_K|$ in [5, p. 74], we obtain the values of a_{JKK} for the case of F_4 in Table 1.

We consider the element

$$d = \sum_{J \subseteq S_{F_4}} \lambda_J x_J \in \mathbb{K}[\mathcal{E}_{F_4}].$$

The eigenvalues of $R_{F_4}(u)$ are

$$\begin{aligned} \Delta_1 &= 1152\lambda_\emptyset + 576(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}} + \lambda_{\{s_4\}}) + 192(\lambda_{\{s_1, s_2\}} + \lambda_{\{s_3, s_4\}}) \\ &\quad + 144\lambda_{\{s_2, s_3\}} + 288(\lambda_{\{s_1, s_3\}} + \lambda_{\{s_1, s_4\}} + \lambda_{\{s_2, s_4\}}) + 24(\lambda_{\{s_1, s_2, s_3\}} + \lambda_{\{s_2, s_3, s_4\}}) \\ &\quad + 96(\lambda_{\{s_1, s_3, s_4\}} + \lambda_{\{s_1, s_2, s_4\}}) + \lambda_{S_{F_4}}, \\ \Delta_2 &= 48(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_1, s_2\}}) + 24(\lambda_{\{s_2, s_3\}} + \lambda_{\{s_1, s_3\}} + \lambda_{\{s_1, s_4\}} + \lambda_{\{s_2, s_4\}} \\ &\quad + \lambda_{\{s_1, s_2, s_3\}} + \lambda_{\{s_1, s_2, s_4\}}) + 6\lambda_{\{s_2, s_3, s_4\}} + 8\lambda_{\{s_1, s_3, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_3 &= 48(\lambda_{\{s_3\}} + \lambda_{\{s_4\}} + \lambda_{\{s_3, s_4\}}) + 24(\lambda_{\{s_2, s_3\}} + \lambda_{\{s_1, s_3\}} + \lambda_{\{s_1, s_4\}} + \lambda_{\{s_2, s_4\}} \\ &\quad + \lambda_{\{s_2, s_3, s_4\}} + \lambda_{\{s_1, s_3, s_4\}}) + 6\lambda_{\{s_1, s_2, s_3\}} + 8\lambda_{\{s_1, s_2, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_4 &= 12\lambda_{\{s_1, s_2\}} + 12\lambda_{\{s_1, s_2, s_3\}} + 6\lambda_{\{s_1, s_2, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_5 &= 8\lambda_{\{s_2, s_3\}} + 4(\lambda_{\{s_1, s_2, s_3\}} + \lambda_{\{s_2, s_3, s_4\}}) + \lambda_{S_{F_4}}, \\ \Delta_6 &= 12(\lambda_{\{s_3, s_4\}} + \lambda_{\{s_2, s_3, s_4\}}) + 6\lambda_{\{s_1, s_3, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_7 &= 4(\lambda_{\{s_1, s_3\}} + \lambda_{\{s_1, s_4\}} + \lambda_{\{s_2, s_4\}}) + \lambda_{\{s_1, s_3, s_4\}} + \lambda_{\{s_1, s_2, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_8 &= 2\lambda_{\{s_1, s_2, s_3\}} + \lambda_{S_{F_4}}, \\ \Delta_9 &= 2\lambda_{\{s_2, s_3, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_{10} &= 2\lambda_{\{s_1, s_3, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_{11} &= 2\lambda_{\{s_1, s_2, s_4\}} + \lambda_{S_{F_4}}, \\ \Delta_{12} &= \lambda_{S_{F_4}}, \end{aligned}$$

Table 1 The values of a_{JKK} for the symmetry group of the 24-cell

$\emptyset$	$\{s_1\}$	$\{s_4\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_2, s_3\}$	$\{s_2, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_2, s_4\}$	SF_4
$\emptyset$	1152	0	0	0	0	0	0	0	0	0	0
$\{s_1\}$	576	48	0	0	0	0	0	0	0	0	0
$\{s_4\}$	576	0	48	0	0	0	0	0	0	0	0
$\{s_1, s_2\}$	192	48	0	12	0	0	0	0	0	0	0
$\{s_2, s_3\}$	144	24	24	0	8	0	0	0	0	0	0
$\{s_3, s_4\}$	192	0	48	0	12	0	0	0	0	0	0
$\{s_1, s_4\}$	288	24	24	0	0	4	0	0	0	0	0
$\{s_1, s_2, s_3\}$	24	24	6	12	0	0	2	0	0	0	0
$\{s_2, s_3, s_4\}$	24	6	24	0	12	0	0	2	0	0	0
$\{s_1, s_3, s_4\}$	96	8	24	0	6	4	0	0	2	0	0
$\{s_1, s_2, s_4\}$	96	24	8	6	0	4	0	0	0	2	0
SF_4	1	1	1	1	1	1	1	1	1	1	1

with corresponding multiplicities

$$\begin{aligned}
 m_{\Delta_1} &= 1 = |\bar{e}| \\
 m_{\Delta_2} &= 12 = |\bar{s}_1| \\
 m_{\Delta_3} &= 12 = |\bar{s}_4| \\
 m_{\Delta_4} &= 32 = |\bar{s}_1 \bar{s}_2| \\
 m_{\Delta_5} &= 54 = |\bar{s}_2 \bar{s}_3| \\
 m_{\Delta_6} &= 32 = |\bar{s}_3 \bar{s}_4| \\
 m_{\Delta_7} &= 72 = |\bar{s}_1 \bar{s}_4| \\
 m_{\Delta_8} &= 84 = |\bar{s}_1 \bar{s}_2 \bar{s}_3| \\
 m_{\Delta_9} &= 84 = |\bar{s}_2 \bar{s}_3 \bar{s}_4| \\
 m_{\Delta_{10}} &= 96 = |\bar{s}_1 \bar{s}_3 \bar{s}_4| \\
 m_{\Delta_{11}} &= 96 = |\bar{s}_1 \bar{s}_2 \bar{s}_4| \\
 m_{\Delta_{12}} &= 577 = |\bar{s}_1 \bar{s}_2 \bar{s}_3 \bar{s}_4|.
 \end{aligned}$$

Appendix B: Counterexample for the special coefficients

Let $J, K \subseteq S$. The proposed formula in [2, Theorem 6.5] is

$$a_{JKK} = \sum_{K' \in \tilde{K} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|},$$

and recall that the formula in our Theorem 2.5 is

$$a_{JKK} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|},$$

with $H_{JK} = \{K' \in \tilde{K} \mid c_{K'K} = {}^J c_{K'K}\}$.

The Coxeter system of the symmetry group of the dodecahedron with cardinality 120 is $(H_3, S_{H_3} = \{s_i\}_{i \in [3]})$, and its Coxeter graph is

$$s_1 \overset{5}{\longleftrightarrow} s_2 \longleftrightarrow s_3.$$

We have the values of $|N_K|$ for H_3 in [5, p. 79]. If we use the formula in [2, Theorem 6.5], we get the following values of a_{JKK} for H_3 :

	$\emptyset$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\emptyset$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	8	2	0	0	0
$\{s_2, s_3\}$	20	8	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let

$$A = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 8 & 2 & 0 & 0 & 0 \\ 20 & 8 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We know from Proposition 3.3 that

$$A^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix}$$

gives the cardinalities of the conjugacy classes of H_3 . However, we get

$$A^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \\ -6 \\ -10 \\ 15 \\ 105 \end{pmatrix},$$

which is absurd.

But if we use Theorem 2.5, the values of $a_{J_K K}$ calculated for H_3 are

	$\emptyset$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\emptyset$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	4	2	0	0	0
$\{s_2, s_3\}$	20	4	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let

$$B = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 4 & 2 & 0 & 0 & 0 \\ 20 & 4 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then we get the following cardinalities of the conjugacy classes of H_3 :

$$B^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \\ 24 \\ 20 \\ 15 \\ 45 \end{pmatrix},$$

which are the correct values.

References

1. Bergeron, F., Bergeron, N.: Symbolic manipulation for the study of the descent algebra of finite Coxeter groups. *J. Symb. Comput.* **14**, 127–139 (1992)
2. Bergeron, F., Bergeron, N., Howlett, R.B., Taylor, D.E.: A decomposition of the descent algebra of a finite Coxeter group. *J. Algebr. Comb.* **1**, 23–44 (1992)
3. Blessenohl, D., Hohlweg, C., Schocker, M.: A symmetry of the descent algebra of a finite Coxeter group. *Adv. Math.* **193**, 416–437 (2005)
4. Brown, K.: Semigroups, rings, and Markov chains. *J. Theor. Probab.* **13**(4), 871–938 (2000)
5. Howlett, R.: Normalizers of parabolic subgroups of reflection groups. *J. Lond. Math. Soc.* (2) **21**, 62–80 (1980)
6. Krattenthaler, C.: Advanced determinant calculus. *Sémin. Lothar. Comb.* **42**, Article B42q (1999), 67 pp.
7. Randriamaro, H.: Diagonalization of matrices of statistics. *Ann. Comb.* (2011, to appear). <http://arxiv.org/abs/1104.4099>
8. Schocker, M.: The descent algebra of the symmetric group. *Fields Inst. Commun.* **00**. <http://www.mathematik.uni-bielefeld.de/~ringel/schocker-neu.ps>
9. Solomon, L.: A Mackey formula in the group ring of a Coxeter group. *J. Algebra* **41**, 255–268 (1976)