On stretching the interval simplex-permutohedron

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Abstract A family of polytopes introduced by E.M. Feichtner, A. Postnikov, and B. Sturmfels, which were named nestohedra, consists in each dimension of an interval of polytopes starting with a simplex and ending with a permutohedron. This paper investigates a problem of changing and extending the boundaries of these intervals. An iterative application of Feichtner–Kozlov procedure of forming complexes of nested sets is a solution of this problem. By using a simple algebraic presentation of members of nested sets it is possible to avoid the problem of increasing the complexity of the structure of nested curly braces in elements of the produced simplicial complexes.

Keywords Building set · Nested set · Hypergraph · Simple polytope · Truncation · Stellar subdivision · Combinatorial blowup · Simplex · Associahedron · Cyclohedron · Permutohedron

1 Introduction

A history of the family of polytopes that appear in the title of this paper under the name “the interval simplex-permutohedron,” may be traced to Appendix B of [26], where Stasheff and Shnider gave a procedure of truncation of an $(n - 2)$ -dimensional simplex, which realizes the associahedron K_n as a convex polytope. This procedure implicitly uses the fact that K_n , as an abstract polytope, is induced by (constructions of) the graph whose edges connect the neighbor symbols of binary operations in a term with n variables. It is also noted in [26] that this procedure could be modified in order to obtain cyclohedra out of simplices; the edges of graphs that induce cyclohedra would connect the neighbor symbols of binary operations in cyclic terms.

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Carr and Devadoss used graphs in [2] more explicitly to define a family of polytopes that includes simplices, associahedra, cyclohedra, permutohedra, and many others. This approach, which is based on the concept of tubings, is further developed in [5, 6] and [15].

The first definition of the notion of nested sets, which plays an important role in this paper, is given by Fulton and MacPherson [16]. De Concini and Procesi [4] used this notion and introduced the notion of building sets. Nonlinear nestohedra are obtained already in [17] and [18] by Gaiffi in the theory of real De Concini-Procesi models by gluing several manifolds with corners.

Feichtner and Kozlov [11] defined the notions of building sets and nested sets in a pure combinatorial manner in order to give an abstract framework for the incidence combinatorics occurring in some constructions in algebraic geometry (see, for example, [4]). Feichtner and Sturmfels [13] and Postnikov [24] used building sets independently, in a more narrow context, to describe the face lattices of a family of simple polytopes named nestohedra in [25].

An alternative, inductive, approach that leads to the same family of polytopes is given in [10]. This was an independent discovery motivated by some related work done in [7, 8], and [9].

For a fixed dimension n , the family of polytopes in question may be considered as an interval starting with an n -dimensional simplex and ending with an n -dimensional permutohedron, while the partial ordering of the interval is induced by the relation $\subseteq$ on the corresponding building sets. Such intervals contain some polytopes, like, for example, associahedra and cyclohedra, which are widely used in topology and category theory. This is the reason why we give the advantage to the families of simple polytopes (though the corresponding families of simplicial polytopes are easier to handle). However, if one prefers to simplify everything a bit, the mathematical content of the results would not change if we switch to the intervals starting with an n -dimensional simplex and ending with the polytope polar to an n -dimensional permutohedron.

This paper gives a solution to the problem of changing and extending these intervals. We find the Feichtner–Kozlov concept of building sets of arbitrary finite-meet semilattices, applied to the case of simplicial complexes, appropriate for this purpose.

In particular, we are interested in iterating the procedure of generating complexes of nested sets. The main obstacle is that every iteration of this procedure increases the complexity of the structure of nested curly braces in elements of the produced simplicial complex. This makes it almost impossible to write down the result of the procedure after just a couple of iterations. We show here how to avoid this problem by using a simple algebraic presentation of the members of nested sets as elements of a freely generated commutative semigroup.

Such an iterated procedure is applied to the boundary complex of a simplicial polytope. This provides an opportunity to replace the simplex at the beginning of the interval simplex-permutohedron by an arbitrary simple polytope that automatically replaces the permutohedron at the end of the interval by some other simple polytope. By iterating the procedure, one obtains interesting families of simplicial complexes.

Truncation of a polytope in its proper face is the operation that guarantees that all these simplicial complexes may be realized as the face lattices (with $\emptyset$ removed) of

some simple polytopes. In dimension 3, locally at each vertex, one may find one of the following six types of truncation (cf. the examples of Sect. 10).

2 Nested complexes and abstract polytopes of hypergraphs

In this section we compare the three approaches given in [13, 24], and [10], which all lead to the same family of polytopes. We start with some preliminary notions.

Let $\alpha_1, \dots, \alpha_n$ for $n \geq 1$ be finite and incomparable (with respect to $\subseteq$) sets. Then $C = P(\alpha_1) \cup \dots \cup P(\alpha_n)$, where $P(\alpha)$ is the power set of α , is a *finite abstract simplicial complex* based on the set $\{\alpha_1, \dots, \alpha_n\}$ of its *bases*.¹ Since we deal here only with finite abstract simplicial complexes, we call them just *simplicial complexes*. For C a simplicial complex, we have that $\langle C, \cap \rangle$ is a meet-semilattice that induces the poset $\langle C, \subseteq \rangle$. Depending on a context, by a simplicial complex we usually mean such a meet-semilattice or a poset.

For a subset B of the domain of a function f , we denote by $f[B]$ the set $\{f(b) \mid b \in B\}$. It is easy to see that two simplicial complexes C and D are isomorphic as meet-semilattices (or as posets) if and only if there is a bijection $\varphi : \bigcup C \rightarrow \bigcup D$ such that $\alpha \in C$ iff $\varphi[\alpha] \in D$ (or equivalently, such that α is a basis of C iff $\varphi[\alpha]$ is a basis of D). If ψ is a function such that the above φ is its restriction to $\bigcup C$, then we say that ψ *underlies* an isomorphism between C and D . So, we have the following remarks.

Remark 2.1 If C is a simplicial complex and ψ is a function such that its restriction to $\bigcup C$ is one-to-one, then ψ underlies an isomorphism between C and the simplicial complex $\{\psi[\alpha] \mid \alpha \in C\}$.

Remark 2.2 If ψ underlies isomorphisms between C and D_1 and between C and D_2 , then $D_1 = D_2$.

Let α be an arbitrary finite set. According to Definition 7.1 of [24], a set B of nonempty subsets of α is a *building set*² of $P(\alpha)$ when the following conditions hold:

¹The term “facet” is reserved here for a face of a polytope whose codimension is 1.

²Originally “a building set on α ” in [24], but I prefer to keep to the terminology of [11], where the building sets of semilattices are introduced.

- (B1) If $\beta, \gamma \in B$ and $\beta \cap \gamma \neq \emptyset$, then $\beta \cup \gamma \in B$;
 (B2) B contains all singletons $\{a\}$ for $a \in \alpha$.

In the terminology of [10] (Sects. 3–4), this notion corresponds to the notion of atomic saturated hypergraph, and it is easy to see that this is a restriction to the case of power sets of the more general notion of building sets of arbitrary finite-meet semi-lattices given in [11] (Definition 2.2).

Let N be a family of sets. As in [10], we say that $\{\beta_1, \dots, \beta_t\} \subseteq N$ is an N -antichain when $t \geq 2$ and $\beta_1, \dots, \beta_t$ are incomparable with respect to $\subseteq$. Also, for a family of sets B , we say that an N -antichain $\{\beta_1, \dots, \beta_t\}$ *misses* B when the union $\beta_1 \cup \dots \cup \beta_t$ does not belong to B .

2.1 Nested set complexes of Feichtner and Sturmfels

Let B be a building set of $P(\alpha)$, containing α . A subset N of B is *nested* when every N -antichain *misses* B . It is easy to see that for $M \subseteq N \subseteq B$, if N is nested, then M is nested too. So, the nested subsets of B form a simplicial complex whose bases are the maximal nested subsets of B .

In [13], for B as above, this simplicial complex is denoted by $\tilde{\mathcal{N}}(P(\alpha), B)$, and the link of α in $\tilde{\mathcal{N}}(P(\alpha), B)$ is denoted by $\mathcal{N}(P(\alpha), B)$ and called the *nested set complex* of $P(\alpha)$ with respect to B . (Nested set complexes of [13] are defined not only for power sets but for arbitrary finite lattices.)

Unfortunately, the name nested set complex and the symbol $\mathcal{N}$ were previously used in [12] (see also [3]) for the result and the name of the operation that generalizes $\tilde{\mathcal{N}}$ from above.

2.2 Nested complexes of Postnikov

Let B be a building set of $P(\alpha)$ not necessarily containing α . According to Definition 7.3 of [24], a subset N of B is a *nested set* when it satisfies the following:

- (N1) If $\beta, \gamma \in N$, then $(\beta \subseteq \gamma \text{ or } \gamma \subseteq \beta \text{ or } \beta \cap \gamma = \emptyset)$;
 (N2) If $\beta_1, \dots, \beta_t$ for $t \geq 2$ are mutually disjoint elements of N , then $\beta_1 \cup \dots \cup \beta_t$ does not belong to B ;
 (N3) N contains all maximal elements of B .

The *nested complex* $\mathcal{N}(B)$ is a poset of all nested sets ordered by inclusion. Let $\mathcal{N}^*(B)$ be obtained from $\mathcal{N}(B)$ by removing every maximal element of B from nested sets. Then $\mathcal{N}^*(B)$ is a simplicial complex whose bases are maximal nested sets with maximal elements of B removed.

2.3 Abstract polytopes of hypergraphs

This is an alternative, inductive approach to the same matters, which is given in [10]. For α a finite set, let $H \subseteq P(\alpha)$ be such that $\emptyset \notin H$ and $\alpha = \bigcup H$. Then H is a *hypergraph* on α (see [1], Sect. 1.1). A hypergraph H is *atomic* when for every $x \in \bigcup H$, we have that $\{x\} \in H$ (see [10], Sect. 3).

A *hypergraph partition* of a hypergraph H is a partition $\{H_1, \dots, H_n\}$, with $n \geq 1$, of H such that $\{\bigcup H_1, \dots, \bigcup H_n\}$ is a partition of $\bigcup H$. A hypergraph H is *connected* when it has only one hypergraph partition. (If H is nonempty, then this hypergraph partition is $\{H\}$.) For example, $H = \{\{x\}, \{y\}, \{z\}, \{x, y, z\}\}$ is connected.

A hypergraph partition $\{H_1, \dots, H_n\}$ of H is *finest* when every H_i is a connected hypergraph on $\bigcup H_i$. We say that $\bigcup H_i$ is a *connected component* of $\bigcup H$. For $H \subseteq P(\alpha)$ and $\beta \subseteq \alpha$, let

$$H_\beta =_{df} \{\gamma \in H \mid \gamma \subseteq \beta\} = H \cap P(\beta).$$

Let H be an atomic hypergraph. By induction on the cardinality $|\bigcup H|$ of $\bigcup H$ the *constructions* of H are defined as follows:

- (0) if $|\bigcup H| = 0$, then H is the empty hypergraph $\emptyset$, and $\emptyset$ is the only construction of $\emptyset$;
- (1) if $|\bigcup H| \geq 1$, H is connected, and K is a construction of $H_{\bigcup H - \{x\}}$ for $x \in \bigcup H$, then $K \cup \{\bigcup H\}$ is a construction of H ;
- (2) if $|\bigcup H| \geq 2$, H is not connected, and $\{H_1, \dots, H_n\}$, where $n \geq 2$, is the finest hypergraph partition of H , and for every $i \in \{1, \dots, n\}$, we have that K_i is a construction of H_i , then $K_1 \cup \dots \cup K_n$ is a construction of H .

A subset of a construction is called a *construct* when it contains every connected component of $\bigcup H$. The abstract polytope of H , denoted by $\mathcal{A}(H)$, is a poset of all the constructs of H ordered by $\supseteq$, plus the bottom element.

A hypergraph H is *saturated* when it satisfies condition (B1) of the definition of a building set with B replaced by H . So, the notions of an atomic saturated hypergraph on α and of a building set of $P(\alpha)$ coincide. It follows from Proposition 4.7 of [10] that for $\bar{H}$ being the intersection of all the atomic saturated hypergraphs, i.e., building sets, containing an atomic hypergraph H , we have $\mathcal{A}(H) = \mathcal{A}(\bar{H})$.

2.4 The interval simplex-permutohedron

Our task now is to compare the posets $\mathcal{N}(P(\alpha), B)$, $\mathcal{N}(B)$, and $\mathcal{A}(B)$ for B a building set of $P(\alpha)$. We start with some abbreviations. For a finite lattice L , let L_- be the poset obtained from L by removing the bottom element, and, analogously, let L^- be obtained from L by removing the top element. For a poset P , let P^{op} be the poset obtained by reversing the order of P .

Let B be a building set of $P(\alpha)$, and let $N \subseteq B$. Then it is easy to verify that

- (*) (N1) and (N2) of Sect. 2.2 hold iff every N -antichain misses B .

From (*) we infer that for B containing α , we have the isomorphism between $\mathcal{N}(B)$ and $\mathcal{N}(P(\alpha), B)$ obtained by removing α from the elements of $\mathcal{N}(B)$.

Although Proposition 6.12 of [10] is formulated for atomic, saturated, and connected hypergraphs, it is easy to verify that the connectedness is not essential, i.e., that the following proposition holds.

Proposition 2.3 *For every building set H and every $N \subseteq H$, we have: every N -antichain misses H iff for some construction K of H , we have that $N \subseteq K$.*

From Proposition 2.3 and (*) we infer that $(\mathcal{N}(B))^{op}$ is equal to $(\mathcal{A}(B))_-$.

Since $\mathcal{N}(B)$ and $\mathcal{A}(B)$ for B not containing α are easily defined in terms of $\mathcal{N}(H_1), \dots, \mathcal{N}(H_n)$, for $\{H_1, \dots, H_n\}$ being the finest hypergraph partition of B (see the definition of $\otimes$ given in [10], Sect. 5), we may consider only the building sets of $P(\alpha)$ that contain α (if α is nonempty). Such a building set is called in [10] an atomic saturated connected hypergraph or, shortly, an ASC-hypergraph. For H being an ASC-hypergraph, all the three posets $(\mathcal{N}(P(\alpha), H))^{op}$, $(\mathcal{N}(H))^{op}$ and $(\mathcal{A}(H))_-$ are isomorphic.

Let α be finite nonempty set. The ASC-hypergraphs on α , i.e., the building sets of $P(\alpha)$ containing α , make a meet-semilattice $\mathcal{H}$ with $\cap$ being the meet operation. We have that $H_\perp = \{\{a\} \mid a \in \alpha\} \cup \{\alpha\}$ and $H_\top = P(\alpha) - \{\emptyset\}$ are respectively the bottom and top elements of $\mathcal{H}$. (Since every pair of elements of $\mathcal{H}$ has an upper bound, it is easy to define the lattice structure on $\mathcal{H}$.)

Let $n = |\alpha| - 1$. We have that $\mathcal{A}(H_\perp)$ is isomorphic to the face lattice of an n -dimensional simplex. On the other hand, $\mathcal{A}(H_\top)$ is isomorphic to the face lattice of an n -dimensional permutohedron. For every $H \in \mathcal{H}$, we have that $\mathcal{A}(H)$ is isomorphic to the face lattice of some n -dimensional polytope $\mathcal{G}(H)$ (see [10], Sect. 9 and Appendix B). For a fixed dimension $n \geq 0$, we define the interval *simplex-permutohedron* to be the set $\{\mathcal{G}(H) \mid H \in \mathcal{H}\}$.

Since the function $\mathcal{G}$ defined in [10], Sect. 9, is one-to-one (although some members of the family are combinatorially equivalent), the interval simplex-permutohedron may be enriched with a poset or a lattice structure induced by the structure of $\mathcal{H}$. However, if we consider, as usual, polytopes only up to combinatorial equivalence, then the interval simplex-permutohedron is just a family of polytopes. We refer to [10], Appendix B, where one may find the complete intervals simplex-permutohedron in dimension 3 and lower, and a chart of a fragment of $\mathcal{H}$ tied to dimension 3.

3 Complexes of nested sets for simplicial complexes

In this section we define the notion of a building set of a simplicial complex and the notion of a corresponding complex of nested sets, which are just restrictions of the notions defined in [11] for arbitrary finite-meet semilattices.

Let C be a simplicial complex based on the set $\{\alpha_1, \dots, \alpha_n\}$ (cf. Sect. 2). A set $B \subseteq C$ is a *building set* of C when for every $i \in \{1, \dots, n\}$, we have that $B_{\alpha_i} = B \cap P(\alpha_i)$ is a building set of $P(\alpha_i)$ in the sense of the definition given at the beginning of Sect. 2. It is easy to verify that this is the notion to which the notion of a building set of a finite-meet semilattice introduced in [11] (Definition 2.2) is reduced when the semilattice is a simplicial complex.

For a family of sets X , let $\min X$ be the set of minimal (with respect to $\subseteq$) members of X , and let $\max X$ be defined analogously. Then we can prove the following proposition in a straightforward manner.

Proposition 3.1 *For a finite nonempty set α and a simplicial complex C , we have that:*

- (1) B is a building set of C iff B_γ is a building set of $P(\gamma)$ for every $\gamma \in C$;
- (2) if B is a building set of $P(\alpha)$, then $B - \{\alpha\}$ is a building set of $P(\alpha) - \{\alpha\}$;
- (3) $B_\perp = \{\{a\} \mid a \in \bigcup C\}$ is a building set of C ;
- (4) if $\alpha \in C$, then $B_\perp \cup \{\alpha\}$ is a building set of C ;
- (5) if B is a building set of C and $\beta \in \min(B - B_\perp)$, then $B - \{\beta\}$ is a building set of C ;
- (6) if B is a building set of C and $\beta \in \max(C - B)$, then $B \cup \{\beta\}$ is a building set of C .

(Note that if α is finite and nonempty, then $P(\alpha) - \{\alpha\}$ is a simplicial complex based on the set $\{\alpha - \{a\} \mid a \in \alpha\}$.)

Let B be a building set of a simplicial complex C . A subset N of B is *nested* when for every N -antichain $\{\beta_1, \dots, \beta_t\}$, the union $\beta_1 \cup \dots \cup \beta_t$ belongs to $C - B$. This is again in accordance with Definition 2.7 of [11].

It is easy to see that for $M \subseteq N \subseteq B$, if N is nested, then M is nested too. So, the nested subsets of B form again a simplicial complex whose bases are the maximal nested subsets of B . We denote this simplicial complex by $\tilde{\mathcal{N}}(C, B)$. (According to [11], it would be just $\mathcal{N}(B)$, but since we want to make a comparison with the notions given in Sect. 2.1, we find $\tilde{\mathcal{N}}(C, B)$ more appropriate since it is reduced to $\tilde{\mathcal{N}}(P(\alpha), B)$ when C is $P(\alpha)$ and B contains α .) We have that $\bigcup \tilde{\mathcal{N}}(C, B) = B$, since for every $\beta \in B$, we have that $\{\beta\}$ is nested. Then we can prove the following proposition in a straightforward manner.

Proposition 3.2 *For a building set B of a simplicial complex C , the following statements are equivalent:*

- (1) $N \in \tilde{\mathcal{N}}(C, B)$;
- (2) $N \subseteq B$, $\bigcup N \in C$, and every N -antichain misses B ;
- (3) there is $\gamma \in C$ such that $N \in \tilde{\mathcal{N}}(P(\gamma), B_\gamma)$;
- (4) there is a basis α of C such that $N \in \tilde{\mathcal{N}}(P(\alpha), B_\alpha)$.

(Note that for (3) and (4) above, we do not require that B_γ contains γ and that B_α contains α as in the definition of $\tilde{\mathcal{N}}$ given in Sect. 2.1.)

The following proposition is a corollary of Propositions 2.3 and 3.2.

Proposition 3.3 *We have that $N \in \tilde{\mathcal{N}}(C, B)$ iff there is $\gamma \in C$ such that N is a subset of a construction of the hypergraph B_γ . Moreover, $N \in \tilde{\mathcal{N}}(C, B)$ is a basis of $\tilde{\mathcal{N}}(C, B)$ iff there is a basis α of C such that N is a construction of the hypergraph B_α .*

The following proposition defines $\mathcal{N}$ of Sect. 2.1 in terms of $\tilde{\mathcal{N}}$ from above.

Proposition 3.4 *If B is a building set of $P(\alpha)$, containing α , then*

$$\mathcal{N}(P(\alpha), B) = \tilde{\mathcal{N}}(P(\alpha) - \{\alpha\}, B - \{\alpha\}).$$

Proof Since $\alpha \in B$, we have that α is nonempty. By (2) of Proposition 3.1, $B - \{\alpha\}$ is a building set of the simplicial complex $P(\alpha) - \{\alpha\}$.

For the proof of $\subseteq$ -direction, let $N \in \tilde{\mathcal{N}}(P(\alpha), B)$. By the definition of $\mathcal{N}$ (cf. Sect. 2.1), $N \in \tilde{\mathcal{N}}(P(\alpha), B)$, and N is in the link of α . Hence, $N \subseteq B - \{\alpha\}$. We have that $\bigcup N$ is either empty, or it is equal to β for some $\beta \in N$, or it is equal to $\beta_1 \cup \dots \cup \beta_t$ for some N -antichain $\{\beta_1, \dots, \beta_t\}$. In all the three cases it follows that $\bigcup N \in P(\alpha) - \{\alpha\}$ since $\alpha \notin N$, $\alpha \in B$, and $N \in \tilde{\mathcal{N}}(P(\alpha), B)$. Let $\{\beta_1, \dots, \beta_t\}$ be an N -antichain. From $N \in \tilde{\mathcal{N}}(P(\alpha), B)$ we conclude that $\beta_1 \cup \dots \cup \beta_t \notin B$, and hence $\beta_1 \cup \dots \cup \beta_t \notin B - \{\alpha\}$. So, by (2) of Proposition 3.2, we have that $N \in \tilde{\mathcal{N}}(P(\alpha) - \{\alpha\}, B - \{\alpha\})$.

For the proof of $\supseteq$ -direction, let $N \in \tilde{\mathcal{N}}(P(\alpha) - \{\alpha\}, B - \{\alpha\})$. So, $N \subseteq B - \{\alpha\}$, and hence $N \subseteq B$. If $\{\beta_1, \dots, \beta_t\}$ is an N -antichain, then

$$\beta_1 \cup \dots \cup \beta_t \in (P(\alpha) - \{\alpha\}) - (B - \{\alpha\}) = P(\alpha) - B.$$

Hence, $N \in \tilde{\mathcal{N}}(P(\alpha), B)$ and analogously, since every $(N \cup \{\alpha\})$ -antichain is an N -antichain, we have that $N \cup \{\alpha\} \in \tilde{\mathcal{N}}(P(\alpha), B)$. From this and from the fact that $\alpha \notin N$ we conclude that N belongs to the link of α in $\tilde{\mathcal{N}}(P(\alpha), B)$. Hence, $N \in \mathcal{N}(P(\alpha), B)$. $\square$

This proposition sheds a new light on the interval simplex-permutohedron, and it points out a way how to change the boundaries of this interval. We will discuss this later in Sect. 10.

4 Commutative semigroup notation

We show how to use the elements of commutative semigroups freely generated by sets to interpret the families of families... of sets. We deal with “formal sums” of elements of some set and with functions mapping finite nonempty subsets of this set to the formal sums, or mapping families of finite nonempty subsets to families of formal sums, etc. For example, if we take the set $\{x, y, z, u\}$, then we are interested in mappings of the following forms:

$$\begin{aligned} \{x, y, z\} &\mapsto x + y + z, \\ \{\{x\}, \{x, y\}, \{z\}\} &\mapsto \{x, x + y, z\}, \quad \text{and} \\ \{\{\{x\}\}, \{\{x\}, \{y, u\}\}\} &\mapsto \{\{x\}, \{x, y + u\}\}. \end{aligned}$$

For the sake of precision, we use some basic notions from category theory, which all may be found in [23]. However, no result from category theory is needed in the sequel. We try to illustrate all the notions we use by examples so that a reader not familiar with category theory may deal with these notions intuitively.

Let $\mathbf{Cs}$ be the category of commutative semigroups, and let $G : \mathbf{Cs} \rightarrow \mathbf{Set}$ be the forgetful functor that maps a commutative semigroup to its underlying set. Let $F : \mathbf{Set} \rightarrow \mathbf{Cs}$ be its left adjoint assigning to every set the free commutative semigroup generated by this set. (If X is empty, then FX is the empty semigroup.) Consider

the monad $\langle T, \eta, \mu \rangle$ defined by this adjunction ($T = G \circ F$, η is the unit of this adjunction, and $\mu = G\varepsilon F$, where ε is the counit of this adjunction).

For every set X , we have that TX is the underlying set of the free commutative semigroup generated by X . We can take the elements of TX to be the formal sums of elements of X , so that if $X = \{x, y, z, u\}$, then

$$TX = \{x, y, z, u, 2x, x + y, \dots, 3x + y + 5z + u, \dots\}.$$

We use $+^2$ to denote the formal addition of $T^2X = TT X$. Analogously, $+^3$ denotes the formal addition of T^3X , etc. So, if X is as above, then

$$T^2X = \{x, \dots, 3x + y + 5z + u, \dots, x +^2 2x, \dots, z +^2 (3x + y + 5z + u), \dots\}.$$

For the natural transformations η and μ of the monad $\langle T, \eta, \mu \rangle$, we have that $\eta_X : X \rightarrow TX$ is such that $\eta_X(x) = x$ for every $x \in X$ and $\mu_X : T^2X \rightarrow TX$ evaluates $+^2$ as $+$, so that, for example, $\mu_X(z +^2 (3x + y + 5z + u)) = 3x + y + 6z + u$.

Besides the endofunctor $T : \mathbf{Set} \rightarrow \mathbf{Set}$, we are interested in the power set functor $P : \mathbf{Set} \rightarrow \mathbf{Set}$ (see [23], Section I.3) and its modification $P_F : \mathbf{Set} \rightarrow \mathbf{Set}$, which assigns to a set X the set of finite nonempty subsets of X . On arrows, P_F is defined as the power set functor, so that for $f : X \rightarrow Y$ and $A \in P_F X$, we have that $P_F f(A) = f[A]$.

For every set X , let $\sigma_X : P_F X \rightarrow TX$ be the function such that for $A = \{a_1, \dots, a_n\} \subseteq X$, where $n \geq 1$, we have $\sigma_X(A) = a_1 + \dots + a_n$. It follows that σ_X is one-to-one. (Note that σ is not a natural transformation from P_F to T .) For example, if X is as above, then $\sigma_X(\{x\}) = x$, and for $N = \{\{x\}, \{x, y\}, \{z\}\}$, we have that $P\sigma_X(N) = \{x, x + y, z\}$.

5 Iterating $\tilde{\mathcal{N}}$

If B_0 is a building set of a simplicial complex C , then $\tilde{\mathcal{N}}(C, B_0)$ is a simplicial complex, and if B_1 is a building set of $\tilde{\mathcal{N}}(C, B_0)$, then $\tilde{\mathcal{N}}(\tilde{\mathcal{N}}(C, B_0), B_1)$ is again a simplicial complex, and we may iterate this procedure. So, for $n \geq 0$, B_0 a building set of C , and B_{i+1} , for $0 \leq i \leq n$ a building set of $\tilde{\mathcal{N}}(C, B_0, \dots, B_i)$, we define

$$\tilde{\mathcal{N}}(C, B_0, \dots, B_{n+1}) =_{df} \tilde{\mathcal{N}}(\tilde{\mathcal{N}}(C, B_0, \dots, B_n), B_{n+1}).$$

Example 5.1 Let C be a simplicial complex whose bases are $\{x, y, z\}$, $\{x, y, u\}$, $\{x, z, u\}$, and $\{y, z, u\}$, i.e., $C = P(X) - \{X\}$ for $X = \{x, y, z, u\}$, and let $B_0 = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}\}$. By Proposition 3.3, the bases of $\tilde{\mathcal{N}}(C, B_0)$ are the constructions of $(B_0)_\alpha$ for all the bases α of C . So, $\tilde{\mathcal{N}}(C, B_0)$ has the following six bases:

$$\begin{array}{llll} \{\{x\}, \{x, y\}, \{z\}\} & \text{and} & \{\{y\}, \{x, y\}, \{z\}\} & \text{derived from } \{x, y, z\}, \\ \{\{x\}, \{x, y\}, \{u\}\} & \text{and} & \{\{y\}, \{x, y\}, \{u\}\} & \text{derived from } \{x, y, u\}, \\ & & \{\{x\}, \{z\}, \{u\}\} & \text{derived from } \{x, z, u\}, \\ & & \{\{y\}, \{z\}, \{u\}\} & \text{derived from } \{y, z, u\}. \end{array}$$

For $B_1 = \{\{\{x\}\}, \{\{y\}\}, \{\{z\}\}, \{\{u\}\}, \{\{x, y\}\}, \{\{x\}, \{x, y\}\}\}$, again by Proposition 3.3, we have that $\tilde{\mathcal{N}}(C, B_0, B_1)$ has the following eight bases:

$$\begin{aligned} & \{\{\{x\}\}, \{\{x\}, \{x, y\}\}, \{\{z\}\}\}, \quad \{\{\{x, y\}\}, \{\{x\}, \{x, y\}\}, \{\{z\}\}\}, \\ & \{\{\{x\}\}, \{\{x\}, \{x, y\}\}, \{\{u\}\}\}, \quad \{\{\{x, y\}\}, \{\{x\}, \{x, y\}\}, \{\{u\}\}\}, \\ & \{\{\{y\}\}, \{\{x, y\}\}, \{\{z\}\}\}, \quad \{\{\{y\}\}, \{\{x, y\}\}, \{\{u\}\}\}, \\ & \{\{\{x\}\}, \{\{z\}\}, \{\{u\}\}\}, \quad \{\{\{y\}\}, \{\{z\}\}, \{\{u\}\}\}. \end{aligned}$$

This example illustrates how the iterated application of $\tilde{\mathcal{N}}$ increases complexity of the structure of nested curly braces in the elements of the resulting simplicial complex. Our intention is to make these matters simpler.

For a simplicial complex C such that $\bigcup C \subseteq X$, a building set B of C , and $\sigma_X : P_F X \rightarrow T X$ defined in the preceding section, let

$$\tilde{\Sigma}_X(C, B) =_{df} \{P\sigma_X(N) \mid N \in \tilde{\mathcal{N}}(C, B)\}.$$

It is easy to see that $\tilde{\Sigma}_X(C, B)$ is indeed a simplicial complex. For C , X , and B_0 as in Example 5.1, we have that $\tilde{\Sigma}_X(C, B_0)$ is a simplicial complex with the bases $\{x, x + y, z\}$, $\{y, x + y, z\}$, $\{x, x + y, u\}$, $\{y, x + y, u\}$, $\{x, z, u\}$, and $\{y, z, u\}$. We have the following proposition.

Proposition 5.2 *The function σ_X underlies an isomorphism between $\tilde{\mathcal{N}}(C, B)$ and $\tilde{\Sigma}_X(C, B)$.*

Proof Since σ_X is one-to-one, we just apply Remark 2.1. $\square$

Let $\sigma_X^1 =_{df} \sigma_X$, and for $n \geq 1$, let $\sigma_X^{n+1} =_{df} \sigma_{T^n X} \circ P_F \sigma_X^n$. It is easy to see that $\sigma_X^n : P_F^n X \rightarrow T^n X$ satisfies the following inductive clauses:

$$\begin{aligned} \sigma_X^1(\{a_1, \dots, a_k\}) &= a_1 + \dots + a_k, \\ \sigma_X^{n+1}(\{\alpha_1, \dots, \alpha_m\}) &= \sigma_X^n(\alpha_1) +^{n+1} \dots +^{n+1} \sigma_X^n(\alpha_m). \end{aligned}$$

For example, we have that $\sigma_X^2(\{\{x\}\}) = x$, $\sigma_X^2(\{\{x\}, \{x, y\}\}) = x +^2 (x + y)$, and $\sigma_X^3(\{\{\{x\}\}, \{\{x\}, \{x, y\}\}\}) = x +^3 (x +^2 (x + y))$.

For $n \geq 1$, a simplicial complex C such that $\bigcup C \subseteq X$, a building set B_0 of C , and B_i , for $1 \leq i \leq n$ a building set of $\tilde{\mathcal{N}}(C, B_0, \dots, B_{i-1})$, let

$$\tilde{\Sigma}_X(C, B_0, \dots, B_n) =_{df} \{P\sigma_X^{n+1}(N) \mid N \in \tilde{\mathcal{N}}(C, B_0, \dots, B_n)\}.$$

We easily compute that $\tilde{\Sigma}_X(C, B_0, B_1)$, for C , X , B_0 , and B_1 as in Example 5.1, has the following eight bases: $\{x, x +^2 (x + y), z\}$, $\{x + y, x +^2 (x + y), z\}$, $\{x, x +^2 (x + y), u\}$, $\{x + y, x +^2 (x + y), u\}$, $\{y, x + y, z\}$, $\{y, x + y, u\}$, $\{x, z, u\}$, $\{y, z, u\}$. The following proposition is a generalization of Proposition 5.2.

Proposition 5.3 *The function σ_X^{n+1} underlies an isomorphism between $\tilde{\mathcal{N}}(C, B_0, \dots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$.*

Proof Since σ_X is one-to-one and P_F preserves this property, we conclude that σ_X^{n+1} is one-to-one. Then we apply Remark 2.1. $\square$

Remark 5.4 If ψ is a function that underlies an isomorphism between simplicial complexes C and D , and B is a building set of C , then $P^2\psi(B)$ is a building set of D , and the function $P_F\psi$ underlies an isomorphism between $\tilde{N}(C, B)$ and $\tilde{N}(D, P^2\psi(B))$.

Proposition 5.5 *For $n \geq 1$, the function σ_X^{n+1} underlies an isomorphism between $\tilde{N}(C, B_0, \dots, B_n)$ and $\tilde{\Sigma}_{T^n X}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n))$.*

Proof By Proposition 5.3 and Remark 5.4 we have that $P_F\sigma_X^n$ underlies an isomorphism between $\tilde{N}(\tilde{N}(C, B_0, \dots, B_{n-1}), B_n)$, which is equal to $\tilde{N}(C, B_0, \dots, B_n)$ and $\tilde{N}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n))$. By Proposition 5.2, since $\bigcup \tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}) \subseteq T^n X$, we have that $\sigma_{T^n X}$ underlies an isomorphism between $\tilde{N}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n))$ and $\tilde{\Sigma}_{T^n X}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n))$. So, $\sigma_{T^n X} \circ P_F\sigma_X^n$, which is equal to σ_X^{n+1} , underlies the desired isomorphism. $\square$

For $n \geq 1$, by Propositions 5.3 and 5.5 and Remark 2.2 we have that

$$\tilde{\Sigma}_X(C, B_0, \dots, B_n) = \tilde{\Sigma}_{T^n X}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n)).$$

Proposition 5.3 enables us to simplify a bit the notation—we just codify complicated curly braces notation using $+$, $+$ ², $+$ ³, etc., leaving the original nested structure untouched. However, we will show that the necessity of such a nested structure for describing these simplicial complexes is an illusion.

Let again C , X , B_0 , and B_1 be as in Example 5.1. It is easy to check that $\tilde{\Sigma}_X(C, B_0, B_1)$ (and hence $\tilde{N}(C, B_0, B_1)$) is isomorphic to the simplicial complex K whose bases are $\{x, 2x + y, z\}$, $\{x + y, 2x + y, z\}$, $\{x, 2x + y, u\}$, $\{x + y, 2x + y, u\}$, $\{y, x + y, z\}$, $\{y, x + y, u\}$, $\{x, z, u\}$, and $\{y, z, u\}$. (To obtain K from $\tilde{\Sigma}_X(C, B_0, B_1)$, we have just “evaluated” $+$ ² as $+$ in the elements of its bases.) We show that this will always be the case.

6 Flattening the nested structure

For a component $\mu_X : T^2X \rightarrow TX$ of the multiplication of the monad $\langle T, \eta, \mu \rangle$ (cf. Sect. 4), we define inductively the function $\mu_X^n : T^{n+1}X \rightarrow TX$ in the following way. For $n = 0$, let $\mu_X^0 : TX \rightarrow TX$ be the identity function on TX , and for $n \geq 1$, let $\mu_X^n =_{df} \mu_X \circ T\mu_X^{n-1}$. (For $n \geq 1$, by the associativity law of the monad, we have that $\mu_X^n = \mu_X \circ T\mu_X \circ \dots \circ T^{n-1}\mu_X = \mu_X \circ \mu_{TX} \circ \dots \circ \mu_{T^{n-1}X}$.) For example, $\mu_X^1(x +^2(x + y)) = 2x + y$ and $\mu_X^2(x +^3(x +^2(x + y))) = 3x + y$.

For $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ defined in the preceding section, we define

$$\tilde{\Sigma}_X(C, B_0, \dots, B_n) =_{df} \{P\mu_X^n(\alpha) \mid \alpha \in \tilde{\Sigma}_X(C, B_0, \dots, B_n)\}.$$

In our example above, we have that $K = \tilde{\Sigma}_X(C, B_0, B_1)$ for C , X , B_0 , and B_1 as in Example 5.1. Since the function μ_X^n is not one-to-one when X is nonempty and $n \geq 1$, it is not so obvious that μ_X^n underlies an isomorphism between $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$. The rest of this section is devoted to a proof of this result.

For a simplicial complex C such that $\bigcup C \subseteq TX$ and a building set B of C , we formulate some conditions that guarantee that μ_X restricted to $\bigcup \tilde{\Sigma}_{TX}(C, B)$ is one-to-one. That this is not always the case is shown by the following example.

Let $X = \{x, y\}$ and $C = P(\{x, x + y, 2x + y\})$. For a building set $B = \{\{x\}, \{x + y\}, \{2x + y\}, \{x, x + y\}\}$ of C , we have that $\bigcup \tilde{\Sigma}_{TX}(C, B) = \{x, x + y, 2x + y, x + 2(x + y)\}$ and $\mu_X(2x + y) = \mu_X(x + 2(x + y)) = 2x + y$.

Let f be a function, and let C be a simplicial complex such that for every $a \in \bigcup C$, we have that $f(a) \in \mathbf{R}^n$. We say that f *faithfully realizes* C when for all $\alpha, \beta \in C$, every $\{k_a > 0 \mid a \in \alpha\}$, and every $\{l_b > 0 \mid b \in \beta\}$, we have that

$$\sum_{a \in \alpha} k_a \cdot f(a) = \sum_{b \in \beta} l_b \cdot f(b)$$

implies that $\alpha = \beta$ and $k_a = l_a$ for every $a \in \alpha$.

For f and C as above, let

$$f(C) =_{df} \{\text{cone}(\{f(a) \mid a \in \alpha\}) \mid \alpha \in C\}.$$

(See [27] for the notions of cone and simplicial fan.) It is easy to conclude that if f faithfully realizes C , then $f(C)$ is a simplicial fan whose face lattice is isomorphic to C . On the other hand, if $f(C)$ is a simplicial fan whose face lattice is isomorphic to C via the mapping that extends $\{a\} \mapsto \text{cone}(\{f(a)\})$ for $a \in \bigcup C$, then f faithfully realizes C . We can now prove the following:

Proposition 6.1 *Suppose that $f : A \rightarrow \mathbf{R}^n$ and $g : B \rightarrow \mathbf{R}^n$ faithfully realize C and D , respectively, and let $\bigcup g(D) \subseteq \bigcup f(C)$. If $L : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a linear transformation such that Lf faithfully realizes C , then Lg faithfully realizes D .*

Proof Suppose that $\gamma, \delta \in D$, $p_c, q_d > 0$ for $c \in \gamma$, $d \in \delta$, and

$$\sum_{c \in \gamma} p_c \cdot Lg(c) = \sum_{d \in \delta} q_d \cdot Lg(d).$$

For $x = \sum_{c \in \gamma} p_c \cdot g(c)$ and $y = \sum_{d \in \delta} q_d \cdot g(d)$, we have $x, y \in \bigcup g(D) \subseteq \bigcup f(C)$. So, there are $\alpha, \beta \in C$ and $k_a, l_b > 0$ for $a \in \alpha$ and $b \in \beta$ such that $x = \sum_{a \in \alpha} k_a \cdot f(a)$ and $y = \sum_{b \in \beta} l_b \cdot f(b)$.

From $L(x) = L(y)$ we obtain

$$\sum_{a \in \alpha} k_a \cdot Lf(a) = \sum_{b \in \beta} l_b \cdot Lf(b),$$

and since Lf faithfully realizes C , we have $\alpha = \beta$ and $k_a = l_a$ for every $a \in \alpha$. Hence, $x = y$ and since g faithfully realizes D , we obtain $\gamma = \delta$ and $p_c = q_c$ for every $c \in \gamma$. $\square$

Let X be a finite and nonempty set. We always assume that X is linearly ordered. In our examples, if $X = \{x, y, z, u\}$, then we assume that $x < y < z < u$. There is a natural identification of TX with $\mathbf{N}^{|X|} - \{(0, \dots, 0)\}$ when X is finite and linearly ordered. If X is as above, then we identify $2x$ with $(2, 0, 0, 0)$, $3x + y + 5z + u$ with $(3, 1, 5, 1)$, etc. Let $\kappa_X : TX \rightarrow \mathbf{R}^{|X|}$ be defined by this identification.

Lemma 6.2 *Let $A, B \subseteq TX$ be such that $\sigma_X(X) \in A \cap B$ and suppose that there are $x, y \in X$ (not necessarily distinct) such that $A - \{\sigma_X(X)\} \subseteq T(X - \{x\})$ and $B - \{\sigma_X(X)\} \subseteq T(X - \{y\})$. If*

$$(*) \quad \sum_{a \in A} k_a \cdot \kappa_X(a) = \sum_{b \in B} l_b \cdot \kappa_X(b) \quad \text{for } k_a, l_b > 0,$$

then $A - \{\sigma_X(X)\}, B - \{\sigma_X(X)\} \subseteq T(X - (\{x\} \cup \{y\}))$ and $k_{\sigma_X(X)} = l_{\sigma_X(X)}$.

Proof By restricting $(*)$ to the x th and y th coordinates, respectively, we obtain $k_{\sigma_X(X)} = l + l_{\sigma_X(X)}$ and $k + k_{\sigma_X(X)} = l_{\sigma_X(X)}$ for $l = \sum_{b \in B - \{\sigma_X(X)\}} l_b \cdot (\kappa_X(b))_x$, where $(\kappa_X(b))_x$ is the x th coordinate of $\kappa_X(b)$, and analogously, $k = \sum_{a \in A - \{\sigma_X(X)\}} k_a \cdot (\kappa_X(a))_y$.

From this we infer $l + k = 0$. Since $l, k \geq 0$, we obtain $l = k = 0$, which implies that $k_{\sigma_X(X)} = l_{\sigma_X(X)}$ and $(\kappa_X(b))_x = (\kappa_X(a))_y = 0$ for $b \in B - \{\sigma_X(X)\}$ and $a \in A - \{\sigma_X(X)\}$. So, x and y do not occur in the elements of $B - \{\sigma_X(X)\}$ and $A - \{\sigma_X(X)\}$. $\square$

Lemma 6.3 *If C is a simplicial complex such that $X = \bigcup C$, and B is a building set of C , then $\kappa_X : TX \rightarrow \mathbf{R}^{|X|}$ faithfully realizes C and $\tilde{\Sigma}_X(C, B)$. Moreover, we have $\bigcup \kappa_X(\tilde{\Sigma}_X(C, B)) \subseteq \bigcup \kappa_X(C)$.*

Proof It is obvious that κ_X faithfully realizes C since the set $\{\kappa_X(a) \mid a \in X\}$ is linearly independent. For the rest of the proof, we may rely on comments given in [14] (Sect. 5) immediately after the definition of $\Sigma(\mathcal{L}, \mathcal{G})$. Here is how we prove these results.

Suppose $(*) \sum_{a \in \alpha} k_a \cdot \kappa_X(a) = \sum_{b \in \beta} l_b \cdot \kappa_X(b)$ for $\alpha, \beta \in \tilde{\Sigma}_X(C, B)$ and $k_a, l_b > 0$. Let $\alpha', \beta' \in \tilde{\mathcal{N}}(C, B)$ be such that $\alpha = P\sigma_X(\alpha')$ and $\beta = P\sigma_X(\beta')$. From $(*)$ it follows that $x \in X$ occurs in some $a \in \alpha$ iff it occurs in some $b \in \beta$. Hence, by Proposition 3.2 we conclude that for

$$\gamma = \{x \in X \mid x \text{ occurs in some } a \in \alpha\} = \{x \in X \mid x \text{ occurs in some } b \in \beta\},$$

we have that $\alpha', \beta' \in \tilde{\mathcal{N}}(P(\gamma), B_\gamma)$, and hence $\alpha, \beta \in \tilde{\Sigma}_X(P(\gamma), B_\gamma)$.

To prove that $(*)$ implies that $\alpha = \beta$ and that $k_a = l_a$ for every $a \in \alpha$, we proceed by induction on the cardinality $|\gamma|$ of γ by relying on the inductive definition of a construction given in Sect. 2.3. (By Proposition 3.3 we have that α' and β' are subsets of some constructions of B_γ .) If $|\gamma| = 0$, then we are done.

Suppose that $|\gamma| \geq 1$ and B_γ is connected and hence $\gamma \in B_\gamma$. By the choice of γ , we have $\bigcup \alpha' = \bigcup \beta' = \gamma$, and since every α' -antichain and every β' -antichain miss B_γ , we must have that $\gamma \in \alpha' \cap \beta'$, and hence $\sigma_X(\gamma) \in \alpha \cap \beta$. By the inductive definition of a construction we have that $\alpha - \{\sigma_X(\gamma)\} \in \tilde{\Sigma}_X(P(\gamma - \{x\}), B_{\gamma - \{x\}})$ and $\beta - \{\sigma_X(\gamma)\} \in \tilde{\Sigma}_X(P(\gamma - \{y\}), B_{\gamma - \{y\}})$ for some $x, y \in \gamma$.

From Lemma 6.2 and Proposition 3.2, it follows that

$$\alpha - \{\sigma_X(\gamma)\}, \beta - \{\sigma_X(\gamma)\} \in \tilde{\Sigma}_X(P(\gamma - (\{x\} \cup \{y\})), B_{\gamma - (\{x\} \cup \{y\})})$$

and that $k_{\sigma_X(\gamma)} = l_{\sigma_X(\gamma)}$. Hence, we may cancel $k_{\sigma_X(\gamma)} \cdot \kappa_X(\sigma_X(\gamma))$ on both sides of $(*)$ and apply the induction hypothesis to $\alpha - \{\sigma_X(\gamma)\}$, $\beta - \{\sigma_X(\gamma)\}$ and $\gamma - (\{x\} \cup \{y\})$. (Note that $|\gamma - (\{x\} \cup \{y\})| \leq |\gamma| - 1$.)

Suppose that $|\gamma| \geq 2$ and B_γ is not connected. Then for $\{B_{\gamma_1}, \dots, B_{\gamma_n}\}$, $n \geq 2$, being the finest hypergraph partition of B_γ , we have that $\alpha = \alpha_1 \cup \dots \cup \alpha_n$ and $\beta = \beta_1 \cup \dots \cup \beta_n$, where $\emptyset \neq \alpha_i, \beta_i \subseteq B_{\gamma_i}$. Since $\{\kappa_X(x) \mid x \in \gamma\}$ is linearly independent, we have that $(*)$ breaks into n equations, and we may apply the induction hypothesis to each α_i, β_i , and γ_i . This concludes the first part of the proof.

To prove that $\bigcup \kappa_X(\tilde{\Sigma}_X(C, B)) \subseteq \bigcup \kappa_X(C)$, by Proposition 3.2, it is sufficient to show that for every $\gamma \in C$, we have

$$\bigcup \kappa_X(\tilde{\Sigma}_X(P(\gamma), B_\gamma)) \subseteq \text{cone}(\{\kappa_X(x) \mid x \in \gamma\}),$$

which is trivial. Actually, we may prove that

$$\bigcup \kappa_X(\tilde{\Sigma}_X(P(\gamma), B_\gamma)) = \text{cone}(\{\kappa_X(x) \mid x \in \gamma\}),$$

relying on the inductive definition of a construction. This would deliver that $\bigcup \kappa_X(\tilde{\Sigma}_X(C, B)) = \bigcup \kappa_X(C)$, but we do not need this stronger result here. $\square$

Remark 6.4 If ψ underlies an isomorphism between C and D and $f \circ \psi$ faithfully realizes C , then f faithfully realizes D .

Let X be a finite nonempty set, and let C be a simplicial complex such that $\bigcup C \subseteq TX$. Let B be a building set of C . We define

$$\check{S}_X(C, B) =_{df} \{P\mu_X(\alpha) \mid \alpha \in \tilde{\Sigma}_{TX}(C, B)\}.$$

(It is easy to see that if $\bigcup C \subseteq X$, then $\check{S}_X(C, B) = \tilde{S}_X(C, B) = \tilde{S}_X(C, B)$.) The following proposition shows when μ_X underlies an isomorphism between $\tilde{\Sigma}_{TX}(C, B)$ and $\check{S}_X(C, B)$.

Proposition 6.5 *If κ_X faithfully realizes the simplicial complex C , then μ_X restricted to $\bigcup \tilde{\Sigma}_{TX}(C, B)$ is one-to-one, and κ_X faithfully realizes $\check{S}_X(C, B)$.*

Proof Let $Y = \bigcup C$ and assume that Y is linearly ordered in an arbitrary way. By Lemma 6.3 we have that κ_Y faithfully realizes C and $\tilde{\Sigma}_Y(C, B)$. To simplify the notation, we assume that $+^2$ is the formal addition of TY . Otherwise, we would carry an extra isomorphism converting $+^2$ into the formal addition of TY at some places of this proof. By this assumption we have that $\tilde{\Sigma}_Y(C, B) = \tilde{\Sigma}_{TX}(C, B)$, and hence κ_Y faithfully realizes $\tilde{\Sigma}_{TX}(C, B)$.

Let $L : \mathbf{R}^{|Y|} \rightarrow \mathbf{R}^{|X|}$ be a linear transformation defined by $L\kappa_Y(a) = \kappa_X(a)$ for $a \in Y$. Since κ_X faithfully realizes C , we have that $L\kappa_Y$ faithfully realizes C . By Lemma 6.3 it follows that $\bigcup \kappa_Y(\tilde{\Sigma}_{TX}(C, B)) \subseteq \bigcup \kappa_Y(C)$. Hence, the assumptions of Proposition 6.1 are satisfied for $f = g = \kappa_Y$, C , and $D = \tilde{\Sigma}_{TX}(C, B)$. So, we conclude that $L\kappa_Y$ faithfully realizes $\tilde{\Sigma}_{TX}(C, B)$.

By the definition of L it follows that $L\kappa_Y(c) = \kappa_X(\mu_X(c))$ for every $c \in TY$, and since $\bigcup \tilde{\Sigma}_{TX}(C, B) \subseteq TY$, this holds for every $c \in \bigcup \tilde{\Sigma}_{TX}(C, B)$. Hence, for $c, d \in \bigcup \tilde{\Sigma}_{TX}(C, B)$, we have that $\mu_X(c) = \mu_X(d)$ implies $L\kappa_Y(c) = L\kappa_Y(d)$, which implies $c = d$ since $L\kappa_Y$ faithfully realizes $\tilde{\Sigma}_{TX}(C, B)$. We conclude that μ_X restricted to $\bigcup \tilde{\Sigma}_{TX}(C, B)$ is one-to-one.

By the above we have that $L\kappa_Y = \kappa_X \circ \mu_X$ (restricted to TY), that μ_X underlies an isomorphism between $\tilde{\Sigma}_{TX}(C, B)$ and $\check{S}_X(C, B)$, and that $L\kappa_Y$ faithfully realizes $\tilde{\Sigma}_{TX}(C, B)$. Hence, by Remark 6.4, κ_X faithfully realizes $\check{S}_X(C, B)$. $\square$

The following remark is a corollary of Remark 5.4.

Remark 6.6 If $\psi : X \rightarrow Y$ is a function that underlies an isomorphism between simplicial complexes C and D , and B is a building set of C , then the function $T\psi$ underlies an isomorphism between $\tilde{\Sigma}_X(C, B)$ and $\tilde{\Sigma}_Y(D, P^2\psi(B))$.

For C a simplicial complex such that $\bigcup C \subseteq X$ and $B_0, \dots, B_n$ as in the definition of $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ (cf. Sect. 5), we have the following:

Proposition 6.7 *The restriction of μ_X^n to $\bigcup \tilde{\Sigma}_X(C, B_0, \dots, B_n)$ is one-to-one, and κ_X faithfully realizes $\check{S}_X(C, B_0, \dots, B_n)$.*

Proof We proceed by induction on n . In the basis, when $n = 0$, we have that μ_X^0 is the identity function on TX , and by Lemma 6.3, κ_X faithfully realizes $\tilde{\Sigma}_X(C, B_0)$, which is equal to $\check{S}_X(C, B_0)$. For the induction step, we use that

$$\tilde{\Sigma}_X(C, B_0, \dots, B_n) = \tilde{\Sigma}_{T^n X}(\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1}), P^2\sigma_X^n(B_n))$$

(cf. the end of the preceding section). By the induction hypothesis and Remark 2.1 we have that $\mu_X^{n-1} : T^n X \rightarrow TX$ underlies an isomorphism between $\tilde{\Sigma}_X(C, B_0, \dots, B_{n-1})$ and $C' = \check{S}_X(C, B_0, \dots, B_{n-1})$. Hence, by Remark 6.6, $T\mu_X^{n-1}$ underlies an isomorphism between $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ and $\tilde{\Sigma}_{TX}(C', B)$ for $B = P^2(\mu_X^{n-1} \circ \sigma_X^n)(B_n)$. Since by the induction hypothesis we have that κ_X faithfully realizes C' , we may apply Proposition 6.5 to conclude that μ_X restricted to $\bigcup \tilde{\Sigma}_{TX}(C', B)$ is one-to-one. So, the restriction of $\mu_X^n = \mu_X \circ T\mu_X^{n-1}$ to $\bigcup \tilde{\Sigma}_X(C, B_0, \dots, B_n)$ is one-to-one.

By Proposition 6.5 we have that κ_X faithfully realizes the simplicial complex $\check{S}_X(C', B)$. By Remark 2.1 we have that μ_X^n underlies an isomorphism between $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ and $\check{S}_X(C', B)$, and between $\tilde{\Sigma}_X(C, B_0, \dots, B_n)$ and $\check{S}_X(C, B_0, \dots, B_n)$. Hence, by Remark 2.2, we have that $\check{S}_X(C', B) = \check{S}_X(C, B_0, \dots, B_n)$. So, we obtain that κ_X faithfully realizes the simplicial complex $\check{S}_X(C, B_0, \dots, B_n)$. $\square$

As an immediate corollary of Propositions 5.3 and 6.7, we have the following for C, X , and $B_0, \dots, B_n$ as above.

Theorem 6.8 *The function $\mu_X^n \circ \sigma_X^{n+1}$ underlies an isomorphism between $\tilde{N}(C, B_0, \dots, B_n)$ and $\check{S}_X(C, B_0, \dots, B_n)$.*

7 An alternative approach

Although Theorem 6.8 provides us with an efficient notation for the result of the iteration of $\tilde{\mathcal{N}}$, we still have to calculate first $\tilde{\mathcal{N}}(C, B_0, \dots, B_n)$ in order to obtain $\tilde{\mathcal{S}}_X(C, B_0, \dots, B_n)$. Also, it is not easy to write down B_n for large n , and since we always have that $\{\{\beta\} \mid \beta \in B_i\} \subseteq B_{i+1}$, there is some superfluous information carried in this notation. The aim of this section is to formulate a direct procedure of calculating these simplicial complexes and to give a simple, more economical notation for building sets. We will just formulate here the results without going into the proofs, which are straightforward but tedious.

Let X be a finite and nonempty set. For $\beta \in P_F(TX)$, let $\beta^+ =_{df} \mu_X \sigma_{TX}(\beta)$. For example, if $\beta = \{x, x + y\}$, then $\beta^+ = 2x + y$. For $B \subseteq P_F(TX)$, let $[B]^+ = P(\mu_X \sigma_{TX})(B) = \{\beta^+ \mid \beta \in B\}$.

Let $\alpha \subseteq TX$ be a finite set such that κ_X faithfully realizes the simplicial complex $P(\alpha)$. This means that $\{\kappa_X(a) \mid a \in \alpha\}$ is linearly independent. Hence, $\mu_X \sigma_{TX}$ is one-to-one on $P_F(\alpha)$, and $P(\mu_X \sigma_{TX})$ is one-to-one on $P(P_F(\alpha))$.

We say that $D \subseteq [P_F(\alpha)]^+ = [P(\alpha) - \{\emptyset\}]^+$ is a *flat building set* of $P(\alpha)$ when the following conditions hold:

- (D1) If $\gamma, \delta \subseteq \alpha$, $\gamma \cap \delta \neq \emptyset$, and $\gamma^+, \delta^+ \in D$, then $(\gamma \cup \delta)^+ \in D$;
- (D2) $D \cap \alpha = \emptyset$.

This means that there is a building set B of $P(\alpha)$ such that $[B]^+ - \alpha = D$. For example, if $\alpha = \{x, x + y, x + y + z\}$, then $\{2x + y\}$ is a flat building set of $P(\alpha)$.

For a flat building set D of $P(\alpha)$ and $\beta \subseteq \alpha$, let $D_\beta = D \cap [P_F(\beta)]^+$ (cf. Sect. 2.3). If B is a building set of $P(\alpha)$ such that $[B]^+ - \alpha = D$, then, since $\mu_X \sigma_{TX}$ is one-to-one, we have that $D_\beta = [B_\beta]^+ - \beta$. So, D_β is a flat building set of $P(\beta)$. By induction on the cardinality $n \geq 0$ of D , we define the set $\mathcal{C}_X(\alpha, D) \subseteq P_F(TX)$ of *flat constructions* of a flat building set D of $P(\alpha)$ as follows:

- (0) if $n = 0$, i.e., $D = \emptyset$, then $\mathcal{C}_X(\alpha, D) = \{\alpha\}$;
- (1) if $n \geq 1$ and $\alpha^+ \in D$, then

$$\mathcal{C}_X(\alpha, D) = \{\{\alpha^+\} \cup \gamma \mid (\exists x \in \alpha) \gamma \in \mathcal{C}_X(\alpha - \{x\}, D_{\alpha - \{x\}})\};$$

- (2) if $n \geq 1$ and $\alpha^+ \notin D$, then, for $M = \max\{\beta \mid \beta \subseteq \alpha \text{ \& } \beta^+ \in D \cup \alpha\}$,

$$\mathcal{C}_X(\alpha, D) = \left\{ \gamma_1 \cup \dots \cup \gamma_n \mid (\gamma_1, \dots, \gamma_n) \in \prod_{\beta \in M} \mathcal{C}_X(\beta, D_\beta) \right\}.$$

If $\alpha = \{x, x + y, x + y + z\}$ and $D = \{2x + y\}$, then $\mathcal{C}_X(\alpha, D) = \{\{x, 2x + y, x + y + z\}, \{x + y, 2x + y, x + y + z\}\}$.

For a building set B of $P(\alpha)$, by comparing the definition of a construction given in Sect. 2.3 and the definition of a flat construction given above we have the following:

Proposition 7.1 *K is a construction of B iff $[K]^+$ is a flat construction of $[B]^+ - \alpha$.*

Let now C be a simplicial complex such that $\bigcup C \subseteq TX$ and suppose that κ_X faithfully realizes C . We say that $D \subseteq \bigcup \{[P_F(\alpha)]^+ \mid \alpha \in C\}$ is a *flat building set* of

C when for every $\alpha \in C$, we have that $D_\alpha = D \cap [P_F(\alpha)]^+$ is a flat building set of $P(\alpha)$. This means that there is a building set B of C such that $[B]^+ - \bigcup C = D$. Let $\tilde{\mathcal{E}}_X(C, D)$ be the simplicial complex based on

$$\bigcup \{C_X(\alpha, D_\alpha) \mid \alpha \text{ is a basis of } C\}.$$

For C and D as above, let B be a building set of C such that $[B]^+ - \bigcup C = D$. From the definition given after Remark 6.4 we conclude that $\check{\mathcal{S}}_X(C, B) = \{[N]^+ \mid N \in \tilde{\mathcal{N}}(C, B)\}$, and by relying on Proposition 7.1, we have the following:

Proposition 7.2 $\check{\mathcal{S}}_X(C, B) = \tilde{\mathcal{E}}_X(C, D)$.

From Proposition 6.5 we conclude that κ_X faithfully realizes $\tilde{\mathcal{E}}_X(C, D)$. Since $\bigcup \tilde{\mathcal{E}}_X(C, D) \subseteq TX$, we may iterate this procedure, and for $n \geq 0$, D_0 a flat building set of C , and D_{i+1} , for $0 \leq i \leq n$ a flat building set of $\tilde{\mathcal{E}}_X(C, D_0, \dots, D_i)$, we define

$$\tilde{\mathcal{E}}_X(C, D_0, \dots, D_{n+1}) =_{df} \tilde{\mathcal{E}}_X(\tilde{\mathcal{E}}_X(C, D_0, \dots, D_n), D_{n+1}).$$

Let C be a simplicial complex such that $\bigcup C \subseteq X$, and let $B_0, \dots, B_n$ be as in the definition of $\tilde{\mathcal{S}}(C, B_0, \dots, B_n)$. Let $G_0 = B_0$, and for $1 \leq i \leq n$, let $G_i = P^2(\mu_X^{i-1} \circ \sigma_X^i)(B_i)$. Let $D_0 = [G_0]^+ - \bigcup C$, and for $1 \leq i \leq n$, let $D_i = [G_i]^+ - [G_{i-1}]^+$. By using Proposition 7.2 and results of the preceding section, we have the following:

Proposition 7.3 $\check{\mathcal{S}}_X(C, B_0, \dots, B_n) = \tilde{\mathcal{E}}_X(C, D_0, \dots, D_n)$.

In Table 1 we give, as an example, the bases of simplicial complexes C , $\tilde{\mathcal{E}}_X(C, D_0)$, $\tilde{\mathcal{E}}_X(C, D_0, D_1)$, and $\tilde{\mathcal{E}}_X(C, D_0, D_1, D_2)$, where $X = \{x, y, z, u\}$, $C = P(X) - \{X\}$, $D_0 = \{x + y, x + y + z\}$, $D_1 = \{2x + y, 2x + y + z, 2x + 2y + z, 3x + 2y + z\}$, and $D_2 = \{6x + 5y + 3z, 3x + 3y + 2z\}$.

8 $\check{\mathcal{S}}_X$ and combinatorial blowups

Feichtner and Kozlov defined in [11] (Definition 3.1; see also [3], Definition 1.4) the poset $\text{Bl}_\alpha \mathcal{L}$, called the *combinatorial blowup* of a finite-meet semilattice $\mathcal{L}$ at its element α , in the following way. (Here the authors presumably assumed that α is not the bottom element of $\mathcal{L}$.) The set of elements of $\text{Bl}_\alpha \mathcal{L}$ is

$$\{\gamma \in \mathcal{L} \mid \gamma \not\geq \alpha\} \cup \{(\alpha, \gamma) \mid \gamma \in \mathcal{L} \text{ and } \gamma \not\geq \alpha \text{ and } (\gamma \vee \alpha)_\mathcal{L} \text{ exists}\},$$

while for $\beta, \gamma \not\geq \alpha$, the order relation is given by

- (1) $\beta > \gamma$ in $\text{Bl}_\alpha \mathcal{L}$ if $\beta > \gamma$ in $\mathcal{L}$;
- (2) $(\alpha, \beta) > (\alpha, \gamma)$ in $\text{Bl}_\alpha \mathcal{L}$ if $\beta > \gamma$ in $\mathcal{L}$;
- (3) $(\alpha, \beta) > \gamma$ in $\text{Bl}_\alpha \mathcal{L}$ if $\beta \geq \gamma$ in $\mathcal{L}$.

Let X be finite nonempty set, and let C be a simplicial complex such that $Y = \bigcup C \subseteq TX$ and κ_X faithfully realizes C . Note that for every $\alpha \in C$ that is neither

Table 1 Table of simplicial complexes and their bases

C	$\tilde{\mathcal{E}}_X(C, D_0)$	$\tilde{\mathcal{E}}_X(C, D_0, D_1)$	$\tilde{\mathcal{E}}_X(C, D_0, D_1, D_2)$
$\{x, y, z\}$	$\{x, x + y, x + y + z\}$	$\{x + y + z, 2x + 2y + z, 3x + 2y + z\}$	$\{x + y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$
			$\{2x + 2y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$
			$\{x + y + z, 3x + 2y + z, 6x + 5y + 3z\}$
			$\{2x + 2y + z, 3x + 2y + z, 6x + 5y + 3z\}$
		$\{x + y + z, 2x + y + z, 3x + 2y + z\}$	$\{x + y + z, 2x + y + z, 3x + 2y + z\}$
		$\{x + y, 2x + 2y + z, 3x + 2y + z\}$	$\{x + y, 2x + 2y + z, 3x + 2y + z\}$
		$\{x + y, 2x + y, 3x + 2y + z\}$	$\{x + y, 2x + y, 3x + 2y + z\}$
		$\{x, 2x + y + z, 3x + 2y + z\}$	$\{x, 2x + y + z, 3x + 2y + z\}$
		$\{x, 2x + y, 3x + 2y + z\}$	$\{x, 2x + y, 3x + 2y + z\}$
		$\{y, x + y + z, 2x + 2y + z\}$	$\{y, x + y + z, 3x + 3y + 2z\}$
			$\{y, 2x + 2y + z, 3x + 3y + 2z\}$
		$\{y, x + y, 2x + 2y + z\}$	$\{y, x + y, 2x + 2y + z\}$
$\{x, z, x + y + z\}$	$\{x, z, x + y + z\}$	$\{x, z, 2x + y + z\}$	$\{x, z, 2x + y + z\}$
		$\{z, x + y + z, 2x + y + z\}$	$\{z, x + y + z, 2x + y + z\}$
		$\{y, z, x + y + z\}$	$\{y, z, x + y + z\}$
$\{x, y, u\}$	$\{x, u, x + y\}$	$\{x, u, 2x + y\}$	$\{x, u, 2x + y\}$
		$\{u, x + y, 2x + y\}$	$\{u, x + y, 2x + y\}$
$\{y, u, x + y\}$	$\{y, u, x + y\}$	$\{y, u, x + y\}$	$\{y, u, x + y\}$
$\{x, z, u\}$	$\{x, z, u\}$	$\{x, z, u\}$	$\{x, z, u\}$
$\{y, z, u\}$	$\{y, z, u\}$	$\{y, z, u\}$	$\{y, z, u\}$

empty nor a singleton, we have that $\alpha^+ = \mu_X \sigma_{TX}(\alpha) \notin Y$. It is easy to see that for such C , we have that $\text{Bl}_\alpha C$ for $\alpha \neq \emptyset$ is isomorphic to the simplicial complex

$$\{\gamma \in C \mid \alpha \not\subseteq \gamma\} \cup \{\gamma \cup \{\alpha^+\} \mid \alpha \not\subseteq \gamma \text{ and } \alpha \cup \gamma \in C\},$$

which we will also denote by $\text{Bl}_\alpha C$ and call the *blowup* of C at α . Note that if α is a singleton, then $\text{Bl}_\alpha C = C$.

For X and C as above, we prove the following proposition, whose content may be derived from the proof of Proposition 2.4 of [3].

Proposition 8.1 *If B and $B \cup \{\beta\}$ are building sets of C , then for $\alpha = [\max B_\beta]^+$, we have*

$$\check{\mathcal{S}}_X(C, B \cup \{\beta\}) = \text{Bl}_\alpha \check{\mathcal{S}}_X(C, B).$$

Proof If the cardinality $|\max B_\beta|$ of $\max B_\beta$ is 1, then since B is a building set of C , we have that $\beta \in B$. Hence, $\check{S}_X(C, B \cup \{\beta\}) = \check{S}_X(C, B) = \text{Bl}_\alpha \check{S}(C, B)$ since α is a singleton. So, we may proceed with the assumption that $|\max B_\beta| \geq 2$, i.e. $\beta \notin B$.

For the proof of $\subseteq$ -direction, let $\gamma \in \check{S}_X(C, B \cup \{\beta\})$. Note that $\check{S}_X(C, B \cup \{\beta\}) = \{[N]^+ \mid N \in \tilde{\mathcal{N}}(C, B \cup \{\beta\})\}$. According to Proposition 3.2, we have that $\gamma = [N]^+$ for some N that satisfies $N \subseteq B \cup \{\beta\}$, $\bigcup N \in C$ and every N -antichain misses $B \cup \{\beta\}$. Since B is a building set of C , we have that the members of $\max B_\beta$ are mutually disjoint and $\bigcup(\max B_\beta) = \beta$. We may conclude that $\alpha^+ = \beta^+$ and $\alpha \not\subseteq \gamma$ (otherwise, since κ_X faithfully realizes C , $\max B_\beta$ would be an N -antichain that does not miss $B \cup \{\beta\}$).

Suppose that $\beta^+ \notin \gamma$, which is equivalent to $\beta \notin N$ since κ_X faithfully realizes C . We have that $N \subseteq B$ and every N -antichain misses B since it misses $B \cup \{\beta\}$. By Proposition 3.2, we have that $\gamma \in \check{S}_X(C, B)$, and since $\alpha \not\subseteq \gamma$, we conclude that $\gamma \in \text{Bl}_\alpha \check{S}_X(C, B)$.

Suppose that $\beta^+ \in \gamma$. We want to show that $\alpha \cup (\gamma - \{\alpha^+\}) \in \check{S}_X(C, B)$. Since $\alpha^+ = \beta^+$, we have that $\gamma - \{\alpha^+\} = \gamma - \{\beta^+\}$, and hence $\alpha \cup (\gamma - \{\alpha^+\}) = [M]^+$ for $M = \max B_\beta \cup (N - \{\beta\})$. We have that $M \subseteq B$ and $\bigcup M = \bigcup N \in C$. We have to show that every M -antichain misses B .

Suppose that S is an M -antichain that does not miss B . We have that $S \cap \max B_\beta \neq \emptyset$ since otherwise S would be an N -antichain that does not miss $B \cup \{\beta\}$. Let $\beta' \in S \cap \max B_\beta$, and let $S' = S - B_\beta$.

If $S' = \emptyset$, then $\bigcup S \subseteq \beta$. From $\bigcup S \in B$ we conclude that $\beta' \subseteq \bigcup S \in B_\beta$ and hence $\beta' = \bigcup S$, which, together with $\beta' \in S$, contradicts the assumption that S is an M -antichain.

If $S' \neq \emptyset$, then for every $\gamma \in S'$, we have $\beta \not\subseteq \gamma$. Otherwise, $\beta', \gamma \in S$ and $\beta' \subseteq \gamma$, which contradicts the assumption that S is an M -antichain. Since $\gamma \not\subseteq \beta$ holds by the definition of S' , we have that $S' \cup \{\beta\}$ is an N -antichain. We have that $(\bigcup S') \cup \beta = (\bigcup S) \cup \beta \subseteq \bigcup N \in C$, and from $\emptyset \neq \beta' \subseteq (\bigcup S) \cap \beta$, $\bigcup S \in B \cup \{\beta\}$, and $\beta \in B \cup \{\beta\}$, since $B \cup \{\beta\}$ is a building set of C , we have that $(\bigcup S) \cup \beta \in B \cup \{\beta\}$. So, $S' \cup \{\beta\}$ is an N -antichain that does not miss $B \cup \{\beta\}$, which is a contradiction.

By Proposition 3.2 we have that $\alpha \cup (\gamma - \{\alpha^+\}) \in \check{S}_X(C, B)$. Since $\alpha \not\subseteq \gamma$, we have that $\alpha \not\subseteq \gamma - \{\alpha^+\}$, and hence $(\gamma - \{\alpha^+\}) \cup \{\alpha^+\} = \gamma \in \text{Bl}_\alpha \check{S}_X(C, B)$.

For the proof of $\supseteq$ -direction, suppose first that $\gamma \in \text{Bl}_\alpha \check{S}_X(C, B)$ is such that $\gamma \in \check{S}_X(C, B)$ and $\alpha \not\subseteq \gamma$. By Proposition 3.2 we have that $\gamma = [N]^+$ for some N that satisfies $N \subseteq B$, $\bigcup N \in C$ and every N -antichain misses B .

We show that

(*) if S is an N -antichain, then $\bigcup S \neq \beta$.

Since $\alpha \not\subseteq \gamma$, we have that $\max B_\beta \not\subseteq N$. So, there is $\beta' \in \max B_\beta$ such that $\beta' \notin N$. Suppose that $\bigcup S = \beta$, then since all the elements of $\max B_\beta$ are mutually disjoint and $S \subseteq B_\beta$, there must be a subset S' of S such that $\bigcup S' = \beta'$. From $\beta' \notin N$ we conclude that $\beta' \notin S'$ and hence $|S'| \geq 2$. So, S' is an N -antichain that does not miss B , which is a contradiction. Hence, (*) holds, and this guarantees that every N -antichain misses $B \cup \{\beta\}$. By Proposition 3.2 we have that $\gamma \in \check{S}_X(C, B \cup \{\beta\})$.

Suppose now that $\gamma \cup \{\alpha^+\} \in \text{Bl}_\alpha \check{S}_X(C, B)$, where $\alpha \cup \gamma \in \check{S}_X(C, B)$ and $\alpha \not\subseteq \gamma$. Since $\check{S}_X(C, B)$ is a simplicial complex, we have $\gamma \in \check{S}_X(C, B)$. By Proposition 3.2

we have that $\gamma = [N]^+$ and $\alpha \cup \gamma = [M]^+$ for N and M such that $N, M \subseteq B$, $\bigcup N, \bigcup M \in C$ and every N -antichain and every M -antichain misses B .

Let $K = N \cup \{\beta\}$. We have that $\gamma \cup \{\alpha^+\} = [K]^+$, $K \subseteq B \cup \{\beta\}$, and $\bigcup K = (\bigcup N) \cup \beta = (\bigcup N) \cup \max B_\beta = \bigcup M \in C$. We have to show that every K -antichain misses $B \cup \{\beta\}$.

If S is a K -antichain such that $\beta \notin S$, then S is an N -antichain, and it misses B . We can prove that $\bigcup S \neq \beta$ as we proved $(*)$ above, and hence S is a K -antichain that misses $B \cup \{\beta\}$.

If S is a K -antichain such that $\beta \in S$, then $\bigcup S \neq \beta$ (otherwise, S is not a K -antichain). If $\bigcup S \in B$, then for $S' = (S - \{\beta\}) \cup \max B_\beta$, we have $\bigcup S' = \bigcup S \in B$. By eliminating from S' every member properly contained in some other member of S' , we obtain an M -antichain S'' such that $\bigcup S'' \in B$ (it is easy to see that $|S''| \geq 2$), which is a contradiction. So S is a K -antichain that misses $B \cup \{\beta\}$.

Hence, every K -antichain misses $B \cup \{\beta\}$, and by Proposition 3.2, we have that $\gamma \cup \{\alpha^+\} \in \check{S}_X(C, B \cup \{\beta\})$. $\square$

Let X be finite nonempty set, and let C be a simplicial complex such that $\emptyset \neq Y = \bigcup C \subseteq TX$ and κ_X faithfully realizes C . Let B be a building set of C . Then we have the following:

Proposition 8.2 *There exist $\beta_1, \dots, \beta_m \in B$ such that $\check{S}_X(C, B) = \text{Bl}_{\beta_1}(\dots \text{Bl}_{\beta_m} C)$.*

Proof We proceed by induction on the cardinality of $B - B_\perp$, where $B_\perp = \{a \mid a \in Y\}$. If $B = B_\perp$, then $\check{S}_X(C, B) = C = \text{Bl}_{\{a\}} C$ for arbitrary $a \in Y$. Otherwise, let $\beta_1 \in \min(B - B_\perp)$. By (5) of Proposition 3.1 we have that $B - \{\beta_1\}$ is a building set of C . By Proposition 8.1, since $[\max B_{\beta_1}]^+ = \beta_1$, we have that $\check{S}_X(C, B) = \text{Bl}_{\beta_1} \check{S}_X(C, B - \{\beta_1\})$, and we may apply the induction hypothesis. $\square$

For a simplicial complex C such that $\emptyset \neq \bigcup C \subseteq X$ and $B_0, \dots, B_n$ as in the definition of $\check{S}_X(C, B_0, \dots, B_n)$ (cf. Sect. 5), we have the following:

Proposition 8.3 *There exist $\beta_1, \dots, \beta_m \in P_F(TX)$ such that $\check{S}_X(C, B_0, \dots, B_n) = \text{Bl}_{\beta_1}(\dots \text{Bl}_{\beta_m} C)$.*

Proof We proceed by induction on n . If $n = 0$, then $\check{S}_X(C, B_0) = \check{S}_X(C, B_0)$, and we may apply Proposition 8.2 since κ_X faithfully realizes C . If $n \geq 1$, then as in the proof of Proposition 6.7, we have that

$$\check{S}_X(C, B_0, \dots, B_n) = \check{S}_X(\check{S}_X(C, B_0, \dots, B_{n-1}), B)$$

for $B = P^2(\mu_X^{n-1} \circ \sigma_X^n)(B_n)$, and κ_X faithfully realizes $\check{S}_X(C, B_0, \dots, B_{n-1})$. By the induction hypothesis, $\check{S}_X(C, B_0, \dots, B_{n-1}) = \text{Bl}_{\beta_1}(\dots \text{Bl}_{\beta_k} C)$, and it remains just to apply Proposition 8.2 to $\check{S}_X(\text{Bl}_{\beta_1}(\dots \text{Bl}_{\beta_k} C), B)$. $\square$

9 Stellar subdivision and truncation of polytopes

In this section we elaborate two operations on polytopes, one dual to the other, which realize the operation of combinatorial blowup on the face lattices of polytopes. For a general reference to the theory of polytopes, we refer the reader to [20] and [27]. We also try to keep to the notation used in these two books.

According to [21] (see also [22]), a stellar subdivision of a polytope P in a proper face F is a polytope $\text{conv}(P \cup \{x^F\})$, where x^F is a point of the form $y^F - \varepsilon(y^P - y^F)$, where y^P is in the interior of P , y^F is in the relative interior of F , and ε is small enough. We use $\text{st}_F P$ to denote a stellar subdivision of P in F . For example, if P is a cube $A_1 B_1 C_1 D_1 A_2 B_2 C_2 D_2$ given on the left-hand side, then $\text{st}_{B_1 B_2} P$ is illustrated on the right-hand side:

To show that the stellar subdivision is well defined, i.e., that the face lattice of $\text{st}_F P$ does not depend on the choice of x^F , we need a more precise definition of this notion. For this, we rely on some notions introduced in [20].

Let $P \subseteq \mathbf{R}^d$ be a d -polytope, H a hyperplane such that $H \cap \text{int } P = \emptyset$, and let $V \in \mathbf{R}^d$. Then V is *beneath* or *beyond* H (with respect to P), provided that V belongs to the open halfspace determined by H that contains $\text{int } P$ or does not meet P , respectively. If $V \in \mathbf{R}^d$ and F is a facet of the d -polytope $P \subseteq \mathbf{R}^d$, then V is *beneath* F or *beyond* F , provided that V is beneath or beyond $\text{aff } F$, respectively. A *stellar subdivision* $\text{st}_F P$ of a polytope P in a proper face F is a polytope $\text{conv}(P \cup \{x^F\})$, where x^F is a point beneath every facet not containing F and beyond every facet containing F .

Let $\mathcal{F}(P)$ denote the face lattice of a polytope P , and let $\mathcal{F}^-(P)$ denote the meet-semilattice obtained from $\mathcal{F}(P)$ by removing the top element P . The following proposition connects the operation of combinatorial blowup on finite-meet semilattices and the operation of stellar subdivision on polytopes.

Proposition 9.1 *For a proper face F of a polytope P , we have that*

$$\mathcal{F}^-(\text{st}_F P) \cong \text{Bl}_F \mathcal{F}^-(P).$$

Proof Apply Theorem 1 of [20] (Sect. 5.2). □

We define now an operation that is polar (dual) to stellar subdivision, which we will call truncation. This operation is mentioned in [20] under the name “cutting off” and appears, under different names, in some recent publications (“blow-up” in [2], “shaving construction” in [25], etc.)

Let $P \subseteq \mathbf{R}^d$ be a d -polytope, $V \in P$, and let π^+ be a halfspace. We say that π^+ is *beneath* V when V belongs to $\text{int } \pi^+$, and we say that π^+ is *beyond* V when V does not belong to π^+ .

A *truncation* $\text{tr}_F P$ of a polytope P in a proper face F is a polytope $P \cap \pi^+$, where π^+ is a halfspace beneath every vertex not contained in F and beyond every vertex contained in F . This defines an operation dual to stellar subdivision, and hence we have the following proposition.

Proposition 9.2 *For a proper face F of a polytope P , we have that (up to combinatorial equivalence)*

$$(\text{st}_F P)^\Delta = \text{tr}_F P^\Delta.$$

For example, if P is an octahedron $ABCDMN$ given on the left-hand side, then $\text{tr}_{AB} ABCDMN$ is illustrated on the right-hand side:

Let $\mathcal{F}_-(P)$ denote the join-semilattice obtained from $\mathcal{F}(P)$ by removing the bottom element $\emptyset$. Then we have the following proposition, which is analogous to Proposition 9.1 (the operation op reverses the order of a poset as in Sect. 2.4).

Proposition 9.3 *For a proper face F of a polytope P , we have that*

$$(\mathcal{F}_-(\text{tr}_F P))^{op} \cong \text{Bl}_F(\mathcal{F}_-(P))^{op}.$$

Roughly speaking, a stellar subdivision makes a polytope more simplicial, while truncation makes it more simple. This is essential for the fact that the family of all simplicial d -polytopes is dense in the family of all d -polytopes (see [20], Sect. 5.2, Theorem 5) and, analogously, that the family of all simple d -polytopes is dense in the family of all d -polytopes.

10 Stretching the interval

Let P be a simple polytope with X as the set of its facets. Then

$$C_P =_{df} \{G^* \mid G \text{ is a nonempty face of } P\},$$

where $G^* = \{F \in X \mid G \subseteq F\}$, is a simplicial complex isomorphic to $(\mathcal{F}_-(P))^{op}$. (Note that the bases of C_P correspond here to the vertices of P .) So, for a simple polytope P , our Proposition 9.3 reads

$$C_{\text{tr}_F P} \cong \text{Bl}_{F^*} C_P.$$

By combining this with Proposition 8.3, for B_0 a building set of C_P , and B_i , for $1 \leq i \leq n$ a building set of $\tilde{\mathcal{N}}(C_P, B_0, \dots, B_{i-1})$, we easily obtain that

$$\tilde{\mathcal{S}}_X(C_P, B_0, \dots, B_n) \cong C_{\text{tr}_{F_1}(\dots \text{tr}_{F_m} P)}$$

for some $F_1, \dots, F_m$ such that F_m is a proper face of P , and F_j , $1 \leq j \leq m-1$, is a proper face of $\text{tr}_{F_{j+1}}(\dots \text{tr}_{F_m} P)$. So, we have the following:

Proposition 10.1 *For $P, B_0, \dots, B_n$ as above, there is a simple polytope Q such that $\tilde{\mathcal{S}}_X(C_P, B_0, \dots, B_n) \cong C_Q$.*

Let Δ be an n -dimensional simplex with X as the set of its facets. If $B_\perp$ is the minimal building set of $C_\Delta = P(X) - \{X\}$, i.e., $B_\perp = \{\{a\} \mid a \in X\}$, then $\tilde{\mathcal{S}}_X(C_\Delta, B_\perp) = C_\Delta$. On the other hand, if $B_\top$ is the maximal building set of C_Δ , i.e., $B_\top = C_\Delta - \{\emptyset\}$, then $\tilde{\mathcal{S}}_X(C_\Delta, B_\top) \cong C_P$, where P is an n -dimensional permutohedron.

By varying B over all possible building sets of C_Δ , according to the definition given at the end of Sect. 2.4, by relying on Proposition 3.4, we obtain the whole interval simplex-permutohedron in dimension n . Proposition 9.10 of [10], together with our Proposition 3.4, shows that for every building set B of C_Δ , there is a simple n -dimensional polytope Q such that $\tilde{\mathcal{S}}_X(C_\Delta, B) \cong C_Q$. According to [10], Q may be presented explicitly by a finite set of inequalities (halfspaces), easily derived from B . However, if one is satisfied with a less explicit construction of Q , we suggest just to rely on Proposition 10.1, which we will always do in the sequel.

It should be clear how we can modify now the interval simplex-permutohedron. If we replace Δ in $\tilde{\mathcal{S}}_X(C_\Delta, B)$ by some other simple polytope P with X as the set of its facets, by varying B over all possible building sets of the simplicial complex C_P , we obtain a new family of simplicial complexes tied to a new interval of simple polytopes. For example, in dimension 2, the interval simplex-permutohedron is the interval triangle-hexagon (triangle, quadrilateral, pentagon, and hexagon). If we replace the triangle by a quadrilateral, then we obtain the interval quadrilateral-octagon.

For $B_\perp = \{\{a\} \mid a \in X\}$ being the minimal building set of C_P , we have, as before, that $\tilde{\mathcal{S}}_X(C_P, B_\perp) = C_P$. So, P is the initial polytope of the new interval. On the other hand, for $B_\top = C_P - \{\emptyset\}$ being the maximal building set of C_P , by Proposition 10.1 there is a simple polytope Q such that $\tilde{\mathcal{S}}_X(C_P, B_\top) \cong C_Q$. We may call Q

a P -based permutohedron, and in that case the ordinary permutohedron would be a Δ -based permutohedron. So, P -based permutohedron is the terminal polytope of the new interval.

If we want to stretch the interval simplex-permutohedron and all the other intervals obtained by replacing Δ by some other simple polytope P , then it is sufficient to allow the iterated application of $\tilde{\mathcal{S}}_X$. As an example, we may consider the following four polytopes, which correspond to the simplicial complexes described in Table 1, ($X = \{x, y, z, u\}$, $D_0 = \{x + y, x + y + z\}$, $D_1 = \{2x + y, 2x + y + z, 2x + 2y + z, 3x + 2y + z\}$, and $D_2 = \{6x + 5y + 3z, 3x + 3y + 2z\}$).

$$D_2 = \{6x + 5y + 3z, 3x + 3y + 2z\}.$$

$$C = P(X) - \{X\}_u$$

$$\tilde{\mathcal{E}}_X(C, D_0)$$

$$\tilde{\mathcal{E}}_X(C, D_0, D_1)$$

$$\tilde{\mathcal{E}}_X(C, D_0, D_1, D_2)$$

In dimension 2, if we start with a triangle, this procedure delivers every polygon, i.e., every simple two-dimensional polytope. However, in dimension 3, if we start with a tetrahedron, it is not the case that this procedure delivers every simple three-dimensional polytope. For example, dodecahedron would never correspond to $\tilde{\mathcal{S}}_X(C_P, B)$ unless P is itself a dodecahedron and $B = B_\perp$. This is because every truncation of a simple polytope in dimension 3 leaves at least one facet to be a triangle or a quadrilateral.

We have described above how to produce a P -based permutohedron for an arbitrary simple polytope P . We conclude this paper with an example of a family of polytopes, which we call permutohedron-based associahedra. The polytope PA_n corresponds to $\tilde{\mathcal{S}}_X(C_\Delta, B_\top, B)$, where $|X| = n + 1$, $C_\Delta = P(X) - \{X\}$, $B_\top = C_\Delta - \{\emptyset\}$, and B is

$$\left\{ \{a_1, \dots, a_k\}, \{a_1, \dots, a_k, a_{k+1}\}, \dots, \{a_1, \dots, a_k, a_{k+1}, \dots, a_l\} \mid a_1, \dots, a_l \text{ are different elements of } X, 1 \leq k \leq l \leq n \right\}.$$

Note that the bases of $\tilde{\mathcal{S}}_X(C_\Delta, B_\top)$ are of the form $\alpha = \{a_1, a_1 + a_2, \dots, a_1 + a_2 + \dots + a_n\}$ for $a_1, \dots, a_n$ different elements of X and $n = |X| - 1$. If we denote $a_1 + \dots + a_k$ by b_k for $1 \leq k \leq n$, then $\alpha = \{b_1, \dots, b_n\}$ and $(P^2\sigma_X(B))_\alpha$ is of the form $\{b_k, b_{k+1}, \dots, b_l\} \mid 1 \leq k \leq l \leq n\}$.

Sometimes it is much easier to present a building set by a graph whose saturated closure (see [10], Sect. 4) is this building set. For example,

$$\begin{array}{c} x \quad y \quad z \\ \hline \end{array}$$

is the graph whose saturated closure is the following building set of $P(\{x, y, z\})$:

$$\{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}.$$

(This building set of $P(\{x, y, z\})$ gives rise to a two-dimensional associahedron, i.e., pentagon K_4 .) In this sense, $(P^2\sigma_X(B))_\alpha$ may be presented by the following graph:

$$\begin{array}{c} b_1 \quad b_2 \quad \dots \quad b_n \\ \hline \end{array}$$

and, hence, $\tilde{\mathcal{S}}_X(P(\alpha), (P^2\sigma_X(B))_\alpha)$ corresponds to an $(n - 1)$ -dimensional associahedron K_{n+1} . This means that every vertex of the permutohedron that corresponds to $\tilde{\mathcal{S}}_X(C_\Delta, B_\top)$ expands into K_{n+1} in the polytope that corresponds to $\tilde{\mathcal{S}}_X(C_\Delta, B_\top, B) = \tilde{\mathcal{S}}(\tilde{\mathcal{S}}_X(C_\Delta, B_\top), P^2\sigma_X(B))$.

The bases of $\tilde{\mathcal{S}}_X(C_\Delta, B_\top)$ are in one-to-one correspondence with the permutations of X in such a way that $a_1 \dots a_n a_{n+1}$, for $\{a_{n+1}\} = X - \{a_1, \dots, a_n\}$, is the permutation that corresponds to the above α . The bases of $\tilde{\mathcal{S}}_X(C_\Delta, B_\top, B)$ derived from α may be interpreted as terms obtained from $a_1 \dots a_n a_{n+1}$ by putting $n - 2$ pairs of

brackets (the outermost brackets are omitted). This is done in such a way that, for example, if $n = 4$, the basis

$$\{b_1, b_4, b_3 + b_4, b_1 + b_2 + b_3 + b_4\}$$

of $\tilde{S}_X(C_\Delta, B_\top, B)$, which is derived from α , is interpreted as $(a_1 a_2)(a_3(a_4 a_5))$.

Hence, PA_n has the vertices, as an n -dimensional permutohedron $K\Pi_n$ (see [27], Sect. 9, Example 9.14) labeled by the terms built out of $n + 1$ different letters with the help of one binary operation. Some edges of PA_n correspond to associativity, i.e., they connect two terms such that one is obtained from the other by replacing a subterm of the form $A \cdot (B \cdot C)$ by $(A \cdot B) \cdot C$. The other edges correspond to transpositions of neighbors. These transpositions are such that in a term of the form $A \cdot B$, one may permute the rightmost letter in A with the leftmost letter in B . So, they correspond to “most unexpected transposition of neighbors”. Eventually, our family of permutohedron-based associahedra may be taken as an alternative presentation of the symmetric monoidal category freely generated by a set of objects. Here is a picture of the three-dimensional permutohedron-based associahedron.

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