J Algebr Comb (2014) 39:75–98
DOI 10.1007/s10801-013-0439-8

The absolute order of a permutation representation of a Coxeter group

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Received: 23 April 2012 / Accepted: 5 March 2013 / Published online: 27 March 2013

Abstract A permutation representation of a Coxeter group W naturally defines an absolute order. This family of partial orders (which includes the absolute order on W) is introduced and studied in this paper. Conditions under which the associated rank generating polynomial divides the rank generating polynomial of the absolute order on W are investigated when W is finite. Several examples, including a symmetric group action on perfect matchings, are discussed. As an application, a well-behaved absolute order on the alternating subgroup of W is defined.

Keywords Coxeter group · Group action · Absolute order · Rank generating polynomial · Reflection arrangement · Modular element · Perfect matching · Alternating subgroup

1 Introduction

The Bruhat order on a Coxeter group W is a key ingredient in understanding the structure of W. This order involves both the set of simple reflections $\mathtt{S}$ and the set of all reflections $\mathtt{T}$ of W: it may be defined by the condition that $u \in W$ is covered by $v \in W$ if there exists $t \in \mathtt{T}$ such that $v = tu$ and $\ell_{\mathtt{S}}(v) = \ell_{\mathtt{S}}(u) + 1$, where $\ell_{\mathtt{S}} : W \to \mathbb{N}$ is the length function with respect to the generating set $\mathtt{S}$. There are two "more coherent" closely related concepts. Replacing the role of $\mathtt{T}$ by $\mathtt{S}$ determines an order which was extensively studied in the past three decades, namely the weak

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order on W. Replacing the role of $\mathtt{S}$ by $\mathtt{T}$ determines the absolute order. The study of maximal chains in the absolute order on the symmetric group is traced at least back to Hurwitz [15]; see also [11, 28]. However, the growing interest in the absolute order is relatively recent and followed the discovery [6, 9] that distinguished intervals in the absolute order, known as the noncrossing partition lattices, are objects of importance in the theory of finite-type Artin groups. For further information on the absolute order, the reader is referred to [1, Sect. 2.4; 2, 16].

Consider a transitive action of W on a set X. Motivated by recent work of Rains and Vazirani [20], which introduces and studies the Bruhat order on X, a naturally defined absolute order on X is introduced in this paper. Our goal is to find conditions under which important enumerative and structural properties of the absolute order on the acting group W carry over to the absolute order on X; in particular, conditions under which the associated rank generating polynomial divides the rank generating polynomial of the absolute order on W. Several examples, including the symmetric group action on ordered tuples and its conjugation action on fixed point free involutions, are discussed. As an application, a well-behaved absolute order on the alternating subgroup of W is defined and studied.

2 Basic concepts

Let W be a Coxeter group with set of reflections $\mathtt{T}$ (for background on Coxeter groups the reader is referred to [7, 8, 14]). The minimum length of a $\mathtt{T}$-word for an element $w \in W$ is denoted by $\ell_{\mathtt{T}}(w)$ and called the absolute length of w. The absolute order on W, denoted by $\mathrm{Abs}(W)$, is the partial order $(W, \leq_{\mathtt{T}})$ defined by letting $u \leq_{\mathtt{T}} v$ if $\ell_{\mathtt{T}}(vu^{-1}) = \ell_{\mathtt{T}}(v) - \ell_{\mathtt{T}}(u)$, for $u, v \in W$. Equivalently, $\leq_{\mathtt{T}}$ is the reflexive and transitive closure of the relation on W consisting of the pairs (u, v) of elements of W for which $\ell_{\mathtt{T}}(u) < \ell_{\mathtt{T}}(v)$ and $v = tu$ for some $t \in \mathtt{T}$. For basic properties of $\mathrm{Abs}(W)$, see [1, Sect. 2.4].

We will be concerned with the following generalization of the absolute order on W. Consider a transitive action ρ of W on a set X. We will write wx for the result $\rho(w)(x)$ of the action of $w \in W$ on an element $x \in X$.

Definition 2.1 Fix an arbitrary element $x_0 \in X$.

(a) The absolute length of $x \in X$ is defined as $\ell_{\mathtt{T}}(x) := \min\{\ell_{\mathtt{T}}(w) : x = wx_0\}$.
(b) The absolute order on X, denoted $\mathrm{Abs}(X)$, associated to ρ is the partial order $(X, \leq_{\mathtt{T}})$ defined by letting $x \leq_{\mathtt{T}} y$ if there exists $w \in W$ such that $y = wx$ and $\ell_{\mathtt{T}}(w) = \ell_{\mathtt{T}}(y) - \ell_{\mathtt{T}}(x)$, for $x, y \in X$. Equivalently, $\leq_{\mathtt{T}}$ is the reflexive and transitive closure of the relation on X consisting of the pairs (x, y) of elements of X for which $\ell_{\mathtt{T}}(x) < \ell_{\mathtt{T}}(y)$ and $y = tx$ for some $t \in \mathtt{T}$.

The present section discusses elementary properties and examples of $\mathrm{Abs}(X)$. We begin with some comments on Definition 2.1.

Remark 2.2 (a) A different way to describe the relation $\leq_{\mathtt{T}}$ on X is the following. Let $x_0 \in X$ be fixed, as before, and consider the simple graph $\Gamma = \Gamma(W, \rho)$ on the vertex set X whose (undirected) edges are the sets of the form $\{x, tx\}$ for $t \in \mathrm{T}$ and $x \in X$. Then for every $x \in X$, the absolute length $\ell_{\mathrm{T}}(x)$ is equal to the distance between x_0 and x in the graph Γ and for $x, y \in X$, we have $x \leq_{\mathrm{T}} y$ if and only if x lies in a geodesic path in Γ with endpoints x_0 and y. This description implies that $\leq_{\mathrm{T}}$ is indeed a partial order on X and that it coincides with the reflexive and transitive closure of the relation on X described in Definition 2.1(b).

(b) The isomorphism type of $\mathrm{Abs}(X)$ is independent of the choice of $x_0 \in X$. Indeed, consider another base point $y_0 \in X$ and let $\mathrm{Abs}(X, x_0)$ and $\mathrm{Abs}(X, y_0)$ denote the absolute orders on X with respect to x_0 and y_0, respectively. Choose $w_0 \in W$ so that $y_0 = w_0 x_0$ and define a map $f : X \mapsto X$ by letting $f(x) = w_0 x$ for $x \in X$. Clearly, f is a bijection and satisfies $f(x_0) = y_0$. Moreover, since T is closed under conjugation, the map f is an automorphism of the graph Γ considered in part (a). These properties imply that $f : \mathrm{Abs}(X, x_0) \mapsto \mathrm{Abs}(X, y_0)$ is an isomorphism of partially ordered sets.

(c) The order $\mathrm{Abs}(X)$ has minimum element x_0.

(d) As an easy consequence of the definition of absolute length, we have $\ell_{\mathrm{T}}(wx) \leq \ell_{\mathrm{T}}(w) + \ell_{\mathrm{T}}(x)$ for all $w \in W$ and $x \in X$.

Since the action ρ is transitive, the set X may be identified with the set of left cosets of the stabilizer of $x_0 \in X$ in W. This identification leads to the following reformulation of Definition 2.1, which we will often find convenient (the role of the base point x_0 in Definition 2.1 will be played by the subgroup H).

Definition 2.3 Let H be a subgroup of W and let $X = W/H$ be the set of left cosets of H in W.

(a) The absolute length of $x \in X$ is defined as $\ell_{\mathrm{T}}(x) := \min\{\ell_{\mathrm{T}}(w) : w \in x\}$.
(b) The absolute order on X, denoted $\mathrm{Abs}(X)$, is the partial order $(X, \leq_{\mathrm{T}})$ defined by letting $x \leq_{\mathrm{T}} y$ if there exists $w \in W$ such that $y = wx$ and $\ell_{\mathrm{T}}(w) = \ell_{\mathrm{T}}(y) - \ell_{\mathrm{T}}(x)$, for $x, y \in X$.

We recall that a partially ordered set (poset) P with a minimum element $\hat{0}$ is said to be locally graded with rank function $\mathrm{rk} : P \to \mathbb{N}$ if for each $x \in P$, every maximal chain in the closed interval $[\hat{0}, x]$ of P has exactly $\mathrm{rk}(x) + 1$ elements (for background and terminology on posets we refer to [26, Chapter 3]). We note the following elementary property of $\mathrm{Abs}(X)$.

Proposition 2.4 *The absolute order* $\mathrm{Abs}(X)$ *is locally graded, with minimum element* $\hat{0} = x_0$ *and rank function given by the absolute length.*

Proof We have already noted that x_0 is the minimum element of $\mathrm{Abs}(X)$. Thus, it suffices to show that $\ell_{\mathrm{T}}(y) = \ell_{\mathrm{T}}(x) + 1$ whenever y covers x in $\mathrm{Abs}(X)$. This is an easy consequence of Definition 2.1. □

We recall (see, for instance, [1, Theorem 2.7.3; 14, Sect. 3.9] and the references given there) that when W is finite, the rank (or length) generating polynomial of

$\mathrm{Abs}(W)$ satisfies

$$W_{\mathrm{T}}(q) := \sum_{w \in W} q^{\ell_{\mathrm{T}}(w)} = \prod_{i=1}^{d} (1 + e_i q), \tag{1}$$

where d is the Coxeter rank of W and $e_1, e_2, \ldots, e_d$ are its exponents. The rank generating polynomial

$$X_{\mathrm{T}}(q) := \sum_{x \in X} q^{\ell_{\mathrm{T}}(x)} \tag{2}$$

of $\mathrm{Abs}(X)$ is well-defined when X is finite. The following question provided much of the motivation for this paper.

Question 2.5 *For which W-actions ρ does $X_{\mathrm{T}}(q)$ divide $W_{\mathrm{T}}(q)$?*

We now list examples, some of which will be studied in detail in later sections.

Example 2.6 (a) The order $\mathrm{Abs}(W)$ occurs by letting ρ be the left multiplication action of W on itself and choosing x_0 as the identity element $e \in W$ in Definition 2.1, or by choosing H as the trivial subgroup $\{e\}$ of W in Definition 2.3.

(b) Let H be the subgroup of W generated by a given reflection $t_0 \in \mathrm{T}$. The set $X = W/H$ of left cosets of H in W is in bijection with the alternating subgroup W^+ of W and hence $\mathrm{Abs}(X)$ gives rise to an absolute order on W^+. This order will be studied in Sect. 5.

(c) Let λ be an integer partition of m and let X consist of the set partitions of $\{1, 2, \ldots, m\}$ whose block sizes are the parts of λ. The symmetric group S_m acts transitively on X and thus defines an absolute order. This order will be studied in Sect. 4.3 in the motivating special case in which $m = 2n$ is even and all parts of λ are equal to 2. The resulting absolute order is a partial order on the set of perfect matchings of $\{1, 2, \ldots, 2n\}$. The stabilizer of this action is the natural embedding of the hyperoctahedral group B_n in S_{2n}.

(d) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition of n and let X consist of the ordered set partitions (meaning, set partitions in which the order of the blocks matters) of $\{1, 2, \ldots, n\}$ whose block sizes are $\lambda_1, \lambda_2, \ldots, \lambda_r$, in this order. The symmetric group S_n acts transitively on X and the stabilizer is a Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ of S_n. The resulting absolute order will be discussed in Sect. 6 in the special case in which $\lambda = (n-k, 1, \ldots, 1)$, where $k \in \{1, 2, \ldots, n-1\}$. Then X can be identified with the set of k-tuples of pairwise distinct elements of $\{1, 2, \ldots, n\}$.

(e) Consider the special case $n = 4$, $\lambda = (2, 2)$ and $x_0 = (\{1, 2\}, \{3, 4\})$ of the example of part (d). Equivalently, let W be the symmetric group S_4 and let H be the four element subgroup generated by the commuting reflections $(1\ 2)$ and $(3\ 4)$. Then $X = W/H$ has six elements. The Hasse diagram of $\mathrm{Abs}(X)$ is shown on Fig. 1.

Remark 2.7 It is possible that not all edges of the graph $\Gamma = \Gamma(W, \rho)$, defined in Remark 2.2(a), are edges of the Hasse diagram of $\mathrm{Abs}(X)$. For instance, consider the action of S_4 on the set X of perfect matchings of $\{1, 2, 3, 4\}$, discussed in Example 2.6(c). Then X has three elements and Γ is the complete graph on these three

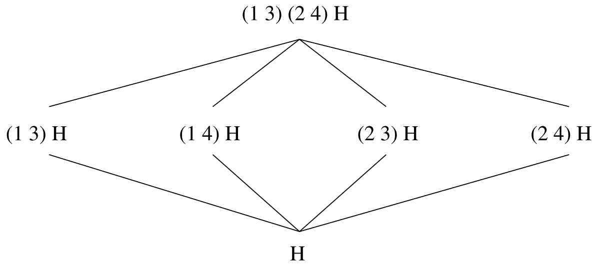

Fig. 1 An absolute order of S_4

vertices. On the other hand, $\mathrm{Abs}(X)$ has a minimum element x_0 which is covered by the other two elements of X. Thus exactly one of the edges of Γ is not an edge of the Hasse diagram of $\mathrm{Abs}(X)$.

3 Modular subgroups

This section investigates a natural condition on a subgroup of a Coxeter group, called modularity, and shows that under this condition, the corresponding absolute order is well-behaved in several ways. Enumerative (Proposition 3.4) and order-theoretic (Theorem 3.9) characterizations, as well as examples, of modularity are given. Throughout this section, W is a Coxeter group with identity element e, $\mathtt{T}$ is the set of reflections, H is a subgroup of W and $X = W/H$ is the set of left cosets of H in W. The Coxeter rank of W will be denoted by $\mathrm{rank}(W)$.

The following properties of absolute length will be frequently used throughout this paper.

Fact 3.1 *For $u, v, w \in W$ we have*:

(a) $\ell_{\mathtt{T}}(w) = 0 \Leftrightarrow w = e$,
(b) $\ell_{\mathtt{T}}(w) = 1 \Leftrightarrow w \in \mathtt{T}$,
(c) $\ell_{\mathtt{T}}(w^{-1}) = \ell_{\mathtt{T}}(w)$,
(d) $\ell_{\mathtt{T}}(uv) \leq \ell_{\mathtt{T}}(u) + \ell_{\mathtt{T}}(v)$,
(e) $\ell_{\mathtt{T}}(wuw^{-1}) = \ell_{\mathtt{T}}(u)$,
(f) $\ell_{\mathtt{T}}(w) = \mathrm{codim}(\mathrm{Fix}(w))$, *if W is finite*,

where $\mathrm{Fix}(w)$ *is the fixed space of w when W is realized as a group generated by reflections in Euclidean space* (*see the relevant discussions after Remark* 3.8).

The main definition of this section is as follows.

Definition 3.2 We say that H is a *modular subgroup* of W if every left coset of H in W has a minimum in $\mathrm{Abs}(W)$.

We note that for $x \in X$ and $w_\circ \in x$, the element $w_\circ$ is the minimum of x in $\mathrm{Abs}(W)$ if and only if we have $\ell_{\mathtt{T}}(w_\circ h) = \ell_{\mathtt{T}}(w_\circ) + \ell_{\mathtt{T}}(h)$ for every $h \in H$. We also note that if H is a modular subgroup of W, then so are its conjugate subgroups.

Example 3.3 (a) Let H be a subgroup of W generated by a single reflection $t \in \mathrm{T}$. Then every left coset $x \in X$ consists of two elements w and wt, which are comparable in $\mathrm{Abs}(W)$. This implies that H is a modular subgroup of W.

(b) Let H be the symmetric group S_{n-1}, naturally embedded in S_n. It will be shown in Example 3.19 (and can be verified directly) that H is a modular subgroup of S_n. The corresponding absolute order consists of the minimum element H and the left cosets $(i\ n)H$ for $i \in \{1, 2, \dots, n-1\}$, each of which covers H.

(c) The subgroup H of S_4 in part (e) of Example 2.6 is not modular. Indeed, there is a single left coset $wH \in X$, that with $w = (1\ 3)(2\ 4)$, which does not have a minimum in $\mathrm{Abs}(S_4)$. As an induced subposet of $\mathrm{Abs}(W)$, this coset has w and $(1\ 4)(2\ 3)$ as minimal elements, $(1\ 4\ 2\ 3)$ and $(1\ 3\ 2\ 4)$ as maximal elements and all four possible Hasse edges among these elements.

(d) It is possible for a subgroup H of a finite Coxeter group W to have a left coset which has a unique element of minimum absolute length but no minimum in $\mathrm{Abs}(W)$ (clearly, such a subgroup H cannot be modular). Consider, for instance, the hyperoctahedral group B_n for some $n \geq 4$ and write $((a\ b))$ for the reflection in W with cycle form $(a\ b)(-a\ \ -b)$. Let H be the subgroup of order 16 generated by the pairwise commuting reflections $t_1 = ((1\ 2))$, $t_2 = ((1\ \ -2))$, $t_3 = ((3\ 4))$ and $t_4 = ((3\ \ -4))$ and let $t = ((1\ 3))$ and $h = t_1t_2t_3t_4 \in H$. Then tH contains a unique reflection, namely t, but has no minimum element in $\mathrm{Abs}(W)$, since t is not comparable to th.

The following proposition explains the significance of modularity with respect to Question 2.5. It should be compared to [7, Lemma 7.1.2; 14, Sect. 5.2; 20, Theorem 8.1].

Proposition 3.4 *Assume that W is finite. Then the subgroup H is modular if and only if $W_{\mathrm{T}}(q) = H_{\mathrm{T}}(q) \cdot X_{\mathrm{T}}(q)$.*

Proof Let $w_x \in x$ be an element of minimum absolute length in $x \in X$. Thus, we have $\ell_{\mathrm{T}}(w_x) = \ell_{\mathrm{T}}(x)$ for every $x \in X$ and hence $\ell_{\mathrm{T}}(w_x h) \leq \ell_{\mathrm{T}}(w_x) + \ell_{\mathrm{T}}(h) = \ell_{\mathrm{T}}(x) + \ell_{\mathrm{T}}(h)$ for all $x \in X$ and $h \in H$. As a result, we find that

$$\begin{aligned} W_{\mathrm{T}}(q) &= \sum_{w \in W} q^{\ell_{\mathrm{T}}(w)} = \sum_{x \in X} \sum_{h \in H} q^{\ell_{\mathrm{T}}(w_x h)} \\ &\preceq \sum_{x \in X} \sum_{h \in H} q^{\ell_{\mathrm{T}}(x) + \ell_{\mathrm{T}}(h)} = X_{\mathrm{T}}(q) \cdot H_{\mathrm{T}}(q), \end{aligned}$$

where $\preceq$ stands for the reverse lexicographic order on the set of polynomials with nonnegative integer coefficients, i.e., for $f(q), g(q) \in \mathbb{N}[q]$ we write $f(q) \prec g(q)$ if the highest term of $g(q) - f(q)$ has positive coefficient. Equality holds if and only if $\ell_{\mathrm{T}}(w_x h) = \ell_{\mathrm{T}}(w_x) + \ell_{\mathrm{T}}(h)$, that is $w_x \leq_{\mathrm{T}} w_x h$, for all $x \in X$ and $h \in H$. The latter holds if and only if w_x is the minimum element of x in $\mathrm{Abs}(W)$ for every coset $x \in X$ and the proof follows. □

A subgroup of W generated by reflections is called a *reflection subgroup*. The absolute length function on such a subgroup K is defined with respect to the set of reflections $\mathrm{T} \cap K$. When W is finite, this function coincides with the restriction of ℓ_{T} : $W \to \mathbb{N}$ on K (this follows from part (f) of Fact 3.1). As a result, the corresponding absolute order on K coincides with the induced order from $\mathrm{Abs}(W)$ on K.

Proposition 3.5 *Assume that W is finite. If K is a modular reflection subgroup of W and H is a modular subgroup of K, then H is a modular subgroup of W.*

Proof Let x be any left coset of H in W. Clearly, x is contained in a left coset y of K in W. Since K is modular in W, the coset y has a minimum element $w_\circ$ in $\mathrm{Abs}(W)$. We leave it to the reader to check that the map $f : K \mapsto w_\circ K = y$, defined by $f(w) = w_\circ w$ for $w \in K$, is a poset isomorphism, where K and y are considered as induced subposets of $\mathrm{Abs}(W)$. Thus x is isomorphic to its preimage $f^{-1}(x)$ in K under f, which is a left coset of H in K. Since H is modular in K, this preimage has a minimum element in $\mathrm{Abs}(K)$, therefore in $\mathrm{Abs}(W)$, and hence so does x. It follows that H is modular in W. $\square$

Remark 3.6 The absolute length function on K with respect to $\mathrm{T} \cap K$ coincides with the restriction of $\ell_{\mathrm{T}} : W \to \mathbb{N}$ on K even if W is infinite, provided K is a parabolic reflection subgroup of W (meaning that K is conjugate to a subgroup generated by simple reflections) [12, Corollary 1.4]. Thus, the transitivity property of modularity in Proposition 3.5 holds in this situation as well.

Proposition 3.7 *Assume H is modular in W and let $\sigma(x)$ be the minimum element of $x \in X$ in* $\mathrm{Abs}(W)$. *Then the map $\sigma : X \mapsto W$ induces a poset isomorphism from* $\mathrm{Abs}(X)$ *onto an order ideal of* $\mathrm{Abs}(W)$.

Proof We need to show that (i) $x \le_{\mathrm{T}} y \Leftrightarrow \sigma(x) \le_{\mathrm{T}} \sigma(y)$ for all $x, y \in X$ and that (ii) $\sigma(X)$ is an order ideal of $\mathrm{Abs}(W)$. For $x, y \in X$ we have

$$\begin{aligned}
&x \le_{\mathrm{T}} y \\
&\quad\Leftrightarrow\quad y = wx \quad \text{for some } w \in W \text{ with } \ell_{\mathrm{T}}(w) = \ell_{\mathrm{T}}(y) - \ell_{\mathrm{T}}(x) \\
&\quad\Leftrightarrow\quad w\sigma(x) \in \sigma(y)H \quad \text{for some } w \in W \text{ with } \ell_{\mathrm{T}}(w) = \ell_{\mathrm{T}}\big(\sigma(y)\big) - \ell_{\mathrm{T}}\big(\sigma(x)\big) \\
&\quad\Leftrightarrow\quad w\sigma(x) = \sigma(y) \quad \text{for some } w \in W \text{ with } \ell_{\mathrm{T}}(w) = \ell_{\mathrm{T}}\big(\sigma(y)\big) - \ell_{\mathrm{T}}\big(\sigma(x)\big) \\
&\quad\Leftrightarrow\quad \sigma(x) \le_{\mathrm{T}} \sigma(y),
\end{aligned}$$

where the third equivalence is because $\sigma(y)$ is the unique element of minimum absolute length in its coset and $\ell_{\mathrm{T}}(w\sigma(x)) \le \ell_{\mathrm{T}}(w) + \ell_T(\sigma(x)) = \ell_{\mathrm{T}}(\sigma(y))$. This proves (i).

For (ii), given elements $u, w \in W$ with $u \le_{\mathrm{T}} w$ and $w \in \sigma(X)$, we need to show that $u \in \sigma(X)$. We set $v = u^{-1}w$, so that $uv = w$ and $\ell_{\mathrm{T}}(w) = \ell_{\mathrm{T}}(u) + \ell_{\mathrm{T}}(v)$. Since w is the minimum element of wH in $\mathrm{Abs}(W)$, we have $\ell_{\mathrm{T}}(wh) = \ell_{\mathrm{T}}(w) + \ell_{\mathrm{T}}(h)$ for every $h \in H$. Thus, for $h \in H$ we have

$$\ell_{\mathrm{T}}(uvh) = \ell_{\mathrm{T}}(wh) = \ell_{\mathrm{T}}(w) + \ell_{\mathrm{T}}(h) = \ell_{\mathrm{T}}(u) + \ell_{\mathrm{T}}(v) + \ell_{\mathrm{T}}(h),$$

$$\ell_{\mathrm{T}}(uvh) = \ell_{\mathrm{T}}\big(uh \cdot h^{-1}vh\big) \le \ell_{\mathrm{T}}(uh) + \ell_{\mathrm{T}}\big(h^{-1}vh\big) = \ell_{\mathrm{T}}(uh) + \ell_{\mathrm{T}}(v).$$

We conclude that $\ell_{\mathrm{T}}(uh) \ge \ell_{\mathrm{T}}(u) + \ell_{\mathrm{T}}(h)$, hence that $\ell_{\mathrm{T}}(uh) = \ell_{\mathrm{T}}(u) + \ell_{\mathrm{T}}(h)$, for every $h \in H$. This means that u is the minimum element of uH in $\mathrm{Abs}(W)$, so that $u \in \sigma(X)$, and the proof follows. □

Remark 3.8 Part (i) of the proof of Proposition 3.7 shows that $\mathrm{Abs}(X)$ is isomorphic to an induced subposet of $\mathrm{Abs}(W)$ (moreover, covering relations are preserved). For that we only needed that each left coset of H in W has a unique element of minimum absolute length.

Next we give a characterization of modularity (which explains our choice of terminology) for the class of parabolic reflection subgroups of W.

First we need to recall some background and notation on finite Coxeter groups. Such a group W acts faithfully on a finite-dimensional Euclidean space V by its standard geometric representation [8, §V.4; 14, §V.3]. This representation realizes W as a group of orthogonal transformations on V generated by reflections. Let Φ be a corresponding root system. For $\alpha \in \Phi$, we denote by $\mathcal{H}_\alpha$ the linear hyperplane in V which is orthogonal to α and by t_α the orthogonal reflection in $\mathcal{H}_\alpha$, so that $\mathrm{T} = \{t_\alpha : \alpha \in \Phi\}$. We denote by $\mathcal{L}_{\mathcal{A}}$ the intersection lattice [19, §2.1; 27, §1.2] of the Coxeter arrangement $\mathcal{A} = \{\mathcal{H}_\alpha : \alpha \in \Phi\}$ and by $\mathcal{L}_W$ the geometric lattice of all linear subspaces of V (flats) spanned by subsets of Φ, partially ordered by inclusion. Thus $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_W$ are isomorphic as posets and the map which sends an element of $\mathcal{L}_{\mathcal{A}}$ to its orthogonal complement in V is a poset isomorphism from $\mathcal{L}_{\mathcal{A}}$ onto $\mathcal{L}_W$.

Given a reflection subgroup H of W, we will denote by V_H the linear span of all roots $\alpha \in \Phi$ for which $t_\alpha \in H$, so that $V_H \in \mathcal{L}_W$. Then H is parabolic if and only if $t_\alpha \in H$ for every $\alpha \in \Phi \cap V_H$ (see, for instance, [4]). Finally, we recall that an element Z of a geometric lattice $\mathcal{L}$ is called *modular* [19, Definition 2.25; 25; 27, Definition 4.12] if we have

$$\mathrm{rk}(Y) + \mathrm{rk}(Z) = \mathrm{rk}(Y \wedge Z) + \mathrm{rk}(Y \vee Z)$$

for every $Y \in \mathcal{L}$, where $\mathrm{rk} : \mathcal{L} \mapsto \mathbb{N}$ denotes the rank function of $\mathcal{L}$ and $Y \wedge Z$ (respectively, $Y \vee Z$) stands for the greatest lower bound (respectively, least upper bound) of Y and Z in $\mathcal{L}$.

Theorem 3.9 *Assume that W is finite and that H is a parabolic reflection subgroup of W. Then H is a modular subgroup of W if and only if V_H is a modular element of the geometric lattice $\mathcal{L}_W$.*

We will give two proofs of Theorem 3.9. We first need to establish two crucial lemmas. We recall [1, §2.4] that to any $w \in W$ are associated the spaces $\mathrm{Fix}(w) \in \mathcal{L}_{\mathcal{A}}$ and $\mathrm{Mov}(w) \in \mathcal{L}_W$, where $\mathrm{Fix}(w)$ is the set of points in V which are fixed by the action of w and $\mathrm{Mov}(w)$ is the orthogonal complement of $\mathrm{Fix}(w)$ in V. For instance, for every $\alpha \in \Phi$ the space $\mathrm{Mov}(t_\alpha)$ is the one-dimensional subspace of V spanned by α. The maps $\mathrm{Fix} : W \mapsto \mathcal{L}_{\mathcal{A}}$ and $\mathrm{Mov} : W \mapsto \mathcal{L}_W$ are surjective and we have $\dim \mathrm{Mov}(w) = \ell_{\mathrm{T}}(w)$ for every $w \in W$. Moreover (see the proof of [1, Theorem 2.4.7]), if $w = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k}$ is a reduced T-word for w, then $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is an $\mathbb{R}$-basis of $\mathrm{Mov}(w)$. In particular, $u \leq_{\mathrm{T}} v \Rightarrow \mathrm{Mov}(u) \subseteq \mathrm{Mov}(v)$ for $u, v \in W$.

Lemma 3.10 *Assume that W is finite and that H is a reflection subgroup of W and let $w_\circ \in W$. Then $w_\circ$ is the minimum of $w_\circ H$ in* $\mathrm{Abs}(W)$ *if and only if* $\mathrm{Mov}(w_\circ) \cap V_H = \{0\}$.

Proof Let $w_\circ = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k}$ be a reduced T-word for $w_\circ$. Thus $\ell_{\mathrm{T}}(w_\circ) = k$ and $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is an $\mathbb{R}$-basis of $\mathrm{Mov}(w_\circ)$.

Suppose first that $\mathrm{Mov}(w_\circ) \cap V_H = \{0\}$. We need to show that $w_\circ \leq_{\mathrm{T}} w_\circ h$ for every $h \in H$. Let $h = t_{\beta_1} t_{\beta_2} \cdots t_{\beta_\ell}$ be a reduced T-word for $h \in H$. Then $\ell_{\mathrm{T}}(h) = \ell$ and $\{\beta_1, \beta_2, \dots, \beta_\ell\}$ is an $\mathbb{R}$-basis of $\mathrm{Mov}(h)$. Since h is a product of reflections in H, its fixed space contains the orthogonal complement of V_H and hence $\mathrm{Mov}(h) \subseteq V_H$. We conclude that $\{\beta_1, \beta_2, \dots, \beta_\ell\}$ is a linearly independent subset of V_H. Our hypothesis implies that $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell\}$ is a linearly independent subset of V. We may infer from Carter's lemma [1, Lemma 2.4.5] that $t_{\alpha_1} \cdots t_{\alpha_k} t_{\beta_1} \cdots t_{\beta_\ell}$ is a reduced T-word for $w_\circ h$. Therefore $\ell_{\mathrm{T}}(w_\circ h) = \ell_{\mathrm{T}}(w_\circ) + \ell_{\mathrm{T}}(h)$, which means that $w_\circ \leq_{\mathrm{T}} w_\circ h$.

Conversely, suppose that $w_\circ$ is the minimum of $w_\circ H$ in $\mathrm{Abs}(W)$. We choose an $\mathbb{R}$-basis $\{\beta_1, \beta_2, \dots, \beta_\ell\}$ of V_H consisting of roots β_i with $t_{\beta_i} \in H$ and set $h = t_{\beta_1} t_{\beta_2} \cdots t_{\beta_\ell} \in H$. By assumption, we have $\ell_{\mathrm{T}}(w_\circ h) = \ell_{\mathrm{T}}(w_\circ) + \ell_{\mathrm{T}}(h)$. This equation and Carter's lemma imply that $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell\}$ is linearly independent or, equivalently, that $\mathrm{Mov}(w_\circ) \cap V_H = \{0\}$. □

Lemma 3.11 *Assume that W is finite and that H is a parabolic reflection subgroup of W and let $w \in W$. Then w is a minimal element of wH in* $\mathrm{Abs}(W)$ *if and only if* $\mathrm{Mov}(w) \wedge V_H = \{0\}$ *holds in $\mathcal{L}_W$.*

Proof We recall that every element of $\mathcal{L}_W$ is of the form $\mathrm{Mov}(u)$ for some $u \in W$ and that $\mathrm{Mov}(u)$ is nonzero if and only if it contains $\mathrm{Mov}(t)$ for some $t \in \mathrm{T}$. Moreover, we have $\mathrm{Mov}(t) \subseteq \mathrm{Mov}(u) \Leftrightarrow t \leq_{\mathrm{T}} u$ [1, Theorem 2.4.7] for $t \in \mathrm{T}$ and since H is parabolic, we have $t \in H$ for every reflection $t \in \mathrm{T}$ for which $\mathrm{Mov}(t) \subseteq V_H$. From these facts we conclude that $\mathrm{Mov}(w) \wedge V_H \neq \{0\}$ holds in $\mathcal{L}_W$ if and only if there exists $t \in H \cap \mathrm{T}$ such that $t \leq_{\mathrm{T}} w$. The latter holds if and only if $wt <_{\mathrm{T}} w$ for some $t \in H \cap \mathrm{T}$ or, equivalently, if and only if w is not a minimal element of wH in $\mathrm{Abs}(W)$. □

First proof of Theorem 3.9 We will use the following characterization of modularity in $\mathcal{L}_W$: An element $Z \in \mathcal{L}_W$ is modular if and only if $Y \cap Z \in \mathcal{L}_W$ for every $Y \in \mathcal{L}_W$. This statement follows directly from [19, Lemma 2.24], which implies that an element $Z \in \mathcal{L}_{\mathcal{A}}$ is modular if and only if $Y + Z \in \mathcal{L}_{\mathcal{A}}$ for every $Y \in \mathcal{L}_{\mathcal{A}}$.

We first assume that H is modular in W and consider any element $Y \in \mathcal{L}_W$. We need to show that $Y \cap V_H \in \mathcal{L}_W$. Since $Y \in \mathcal{L}_W$, we have $Y = \mathrm{Mov}(w)$ for some $w \in W$. By our assumption, the coset wH has a minimum element, say $w_\circ$, in $\mathrm{Abs}(W)$. We claim that $Y \cap V_H = \mathrm{Mov}(w_\circ^{-1} w)$. Since $\mathrm{Mov}(w_\circ^{-1} w) \in \mathcal{L}_W$, it suffices to prove the claim. Indeed, since $w_\circ \leq_{\mathrm{T}} w$, we also have $w_\circ^{-1} w \leq_{\mathrm{T}} w$ and hence

$\mathrm{Mov}(w_\circ^{-1}w) \subseteq \mathrm{Mov}(w) = Y$. Similarly, since $w \in w_\circ H$, we have $w_\circ^{-1}w \in H$ and hence $\mathrm{Mov}(w_\circ^{-1}w) \subseteq V_H$, so we may conclude that $\mathrm{Mov}(w_\circ^{-1}w) \subseteq Y \cap V_H$. For the reverse inclusion, we recall [1, p. 25] that

$$Y = \mathrm{Mov}(w) = \mathrm{Mov}(w_\circ) \oplus \mathrm{Mov}\big(w_\circ^{-1}w\big).$$

By our choice of $w_\circ$ and Lemma 3.10 we have $\mathrm{Mov}(w_\circ) \cap V_H = \{0\}$. As we already know that $Y \cap V_H \supseteq \mathrm{Mov}(w_\circ^{-1}w)$, it follows that $Y \cap V_H = \mathrm{Mov}(w_\circ^{-1}w)$.

Suppose now that V_H is a modular element of $\mathcal{L}_W$ and consider any left coset x of H in W. We need to show that x has a minimum in $\mathrm{Abs}(W)$. Let $w_\circ$ be any minimal element of x in $\mathrm{Abs}(W)$. Since $\mathrm{Mov}(w_\circ) \cap V_H \in \mathcal{L}_W$, by modularity of V_H, the greatest lower bound $\mathrm{Mov}(w_\circ) \wedge V_H$ of $\mathrm{Mov}(w_\circ)$ and V_H in $\mathcal{L}_W$ must be equal to $\mathrm{Mov}(w_\circ) \cap V_H$. This statement and Lemmas 3.10 and 3.11 imply that $w_\circ$ is the minimum element of x in $\mathrm{Abs}(W)$ and the proof follows. □

Remark 3.12 The assumption in Theorem 3.9 that the reflection subgroup H is parabolic was not used in the proof of the *only if* direction of the theorem. However, it is essential for the other direction. Indeed, let W be the dihedral group of symmetries of a square Q and let H be the subgroup of order 4 generated by the reflections on the lines through the center of Q which are parallel to the sides. The unique left coset of H in W, other than H, has no minimum element in $\mathrm{Abs}(W)$ and hence H is not modular in W. On the other hand, $V_H = V$ is trivially a modular element of the lattice $\mathcal{L}_W$.

For the second proof of Theorem 3.9 we recall the following definition. Let $\mathcal{L}$ be a geometric lattice of rank d, with rank function $\mathrm{rk} : \mathcal{L} \mapsto \mathbb{N}$. The *characteristic polynomial* of $\mathcal{L}$ is defined by the formula

$$\chi_{\mathcal{L}}(q) := \sum_{Y \in \mathcal{L}} \mu_{\mathcal{L}}(\hat{0}, Y) q^{d-\mathrm{rk}(Y)}, \tag{3}$$

where $\mu_{\mathcal{L}}$ stands for the Möbius function [26, §3.7] of $\mathcal{L}$ and $\hat{0}$ is the minimum element of $\mathcal{L}$. We now let $\mathcal{L} = \mathcal{L}_W$ and recall that $\mathrm{rk}(Y) = \dim(Y)$ and (see, for instance, [18, Lemma 4.7])

$$(-1)^{\mathrm{rk}(Y)} \mu_{\mathcal{L}}(\hat{0}, Y) = \#\big\{w \in W : \mathrm{Mov}(w) = Y\big\} \tag{4}$$

for $Y \in \mathcal{L}$ and that $\dim \mathrm{Mov}(w) = \ell_{\mathrm{T}}(w)$ for $w \in W$. As a result, the characteristic polynomial of $\mathcal{L}_W$ is related to the rank generating polynomial of $\mathrm{Abs}(W)$ by the well known equality

$$W_{\mathrm{T}}(q) = (-q)^d \chi_{\mathcal{L}}(-1/q). \tag{5}$$

Second proof of Theorem 3.9 Let us write $\mathcal{L} = \mathcal{L}_W$, as before, and set $Z = V_H \in \mathcal{L}$. By the Modular Factorization Theorem for geometric lattices [25; 27, Theorem 4.13] and its converse (see [17, Sect. 8]) we see that Z is a modular element of $\mathcal{L}$ if and only if

$$\chi_{\mathcal{L}}(q) = \chi_{[\hat{0},Z]}(q)\Bigg(\sum_{Y\in\mathcal{L}:Y\wedge Z=\hat{0}} \mu_{\mathcal{L}}(\hat{0},Y)q^{d-\mathrm{rk}(Y)-\mathrm{rk}(Z)}\Bigg), \tag{6}$$

where $[\hat{0}, Z]$ denotes a closed interval in $\mathcal{L}$ and $\hat{0} = \{0\}$ is the minimum element of $\mathcal{L}$. Replacing q by $-1/q$ and taking (4) and (5) into account, we see that (6) can be rewritten as

$$W_{\mathrm{T}}(q) = H_{\mathrm{T}}(q)\Bigg(\sum_{\mathrm{Mov}(w)\wedge Z=\hat{0}} q^{\ell_{\mathrm{T}}(w)}\Bigg). \tag{7}$$

We recall that every finite partially ordered set has at least one minimal element. Assume first that Z is a modular element of $\mathcal{L}$. Setting $q = 1$ in (7) and using Lemma 3.11 we conclude that every left coset of H in W has exactly one minimal (and hence a minimum) element in $\mathrm{Abs}(W)$. By definition, this means that H is a modular subgroup of W. Conversely, suppose that H is a modular subgroup of W. Then, by Lemma 3.11, the sum in the right-hand side of (7) is equal to $X_{\mathrm{T}}(q)$ and hence (7) holds by Proposition 3.4. Thus Z is a modular element of $\mathcal{L}$ and the proof follows. □

Proposition 3.13 *Assume that W is finite. Then every modular reflection subgroup of W is a parabolic reflection subgroup.*

Proof Let H be a modular reflection subgroup of W and let K be the unique parabolic reflection subgroup of W with $V_K = V_H$. Thus K is generated by all reflections $t \in \mathrm{T}$ with $\mathrm{Mov}(t) \subseteq V_H$ and contains H as a reflection subgroup. We need to show that $H = K$. Since H is modular in W, it is also modular in K. Thus, without loss of generality we may assume that $K = W$, so that $\mathrm{rank}(H) = \mathrm{rank}(W)$. We note that $H_{\mathrm{T}}(q)$ and $W_{\mathrm{T}}(q)$ are both polynomials of degree $\mathrm{rank}(W)$. Therefore, Proposition 3.4 implies that $X_{\mathrm{T}}(q)$ is a constant. Since this can only happen if X is a singleton, we conclude that $H = W$ and the proof follows. □

Question 3.14 *Does there exist a modular subgroup of a Coxeter group which is not a reflection subgroup?*

We recall that a poset P is said to be graded of rank d if every maximal chain in P has exactly $d + 1$ elements. The following proposition generalizes the fact that $\mathrm{Abs}(W)$ is graded with rank equal to $\mathrm{rank}(W)$.

Proposition 3.15 *The order* $\mathrm{Abs}(X)$ *is graded of rank* $\mathrm{rank}(W) - \mathrm{rank}(H)$ *for every finite Coxeter group W and every modular reflection subgroup H of W.*

Proof Since $\mathrm{Abs}(X)$ has a minimum element and is locally graded with rank function given by absolute length (Proposition 3.7), it suffices to show that for every element $x \in X$ there exists $y \in X$ of absolute length $\mathrm{rank}(W) - \mathrm{rank}(H)$ such that $x \le_{\mathrm{T}} y$.

Consider any $x \in X$ and let $u_\circ$ be the minimum element of x in $\mathrm{Abs}(W)$. Thus we have $\mathrm{Mov}(u_\circ) \cap V_H = \{0\}$ by Lemma 3.10 and $\ell_{\mathtt{T}}(x) = \ell_{\mathtt{T}}(u_\circ) = \dim \mathrm{Mov}(u_\circ)$. Let $u_\circ = t_{\alpha_k} \cdots t_{\alpha_2} t_{\alpha_1}$ be a reduced $\mathtt{T}$-word for $u_\circ$, so that $\ell_{\mathtt{T}}(x) = k$. We extend $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ to a maximal linearly independent set of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ whose linear span intersects V_H trivially and set $w_\circ = t_{\alpha_r} \cdots t_{\alpha_2} t_{\alpha_1}$ and $y = w_\circ H \in X$. Clearly, we have $r = \dim(V) - \dim(V_H) = \mathrm{rank}(W) - \mathrm{rank}(H)$. Since $\mathrm{Mov}(w_\circ)$ is the linear span of $\alpha_1, \alpha_2, \ldots, \alpha_r$, we have $\mathrm{Mov}(w_\circ) \cap V_H = \{0\}$ by construction. Lemma 3.10 implies that $w_\circ$ is the minimum element of y in $\mathrm{Abs}(W)$ and hence that $\ell_{\mathtt{T}}(y) = \ell_{\mathtt{T}}(w_\circ) = r$. Finally, setting $v = w_\circ u_\circ^{-1} = t_{\alpha_r} \cdots t_{\alpha_{k+1}}$ we have $w_\circ = v u_\circ$ and hence $y = vx$. By Carter's lemma [1, Lemma 2.4.5] we also have $\ell_{\mathtt{T}}(v) = r - k = \ell_{\mathtt{T}}(y) - \ell_{\mathtt{T}}(x)$. Definition 2.1 implies that $x \leq_{\mathtt{T}} y$ and the proof follows. □

Question 3.16 *Does there exist a subgroup H of a Coxeter group W for which* $\mathrm{Abs}(X)$ *is not graded?*

A reflection subgroup H of W is said to be of *almost maximal rank* if $\mathrm{rank}(H) = \mathrm{rank}(W) - 1$. Modular parabolic reflection subgroups of this kind can be characterized as follows.

Proposition 3.17 *Assume that W is finite and that H is a parabolic reflection subgroup of W, other than W. The following are equivalent*:

(i) *H is a modular subgroup of W of almost maximal rank.*
(ii) *Every left coset of H, other than H, contains a reflection.*
(iii) *Every left coset of H, other than H, contains a unique reflection.*

Proof Suppose that (i) holds. We then have $W_{\mathtt{T}}(q) = H_{\mathtt{T}}(q) X_{\mathtt{T}}(q)$ by Proposition 3.4. Since the degrees of $W_{\mathtt{T}}(q)$ and $H_{\mathtt{T}}(q)$ are equal to the Coxeter ranks of W and H, respectively, it follows that the degree of $X_{\mathtt{T}}(q)$ is equal to one. This means that every left coset $x \in X$ of H, other than H, contains an element of absolute length one, so that (ii) is satisfied. We have shown that (i) implies (ii).

Suppose that (ii) holds and let $x \in X$ be a left coset of H in W, other than H. Choose a reflection $t \in x$. Since H is parabolic and does not contain t, we have $\mathrm{Mov}(t) \cap V_H = \{0\}$. Lemma 3.10 implies that t is the minimum element of x in $\mathrm{Abs}(W)$. In particular, x contains a unique reflection. We conclude that (ii) implies both (i) and (iii). The implication (iii) ⇒ (ii) is trivial. □

Question 3.18 *Does there exist a non-parabolic* (*necessarily non-modular*) *reflection subgroup H of a finite Coxeter group such that every left coset*, *other than H*, *contains a unique reflection?*

Example 3.19 For $k \leq n$ and under the natural embedding, the symmetric and hyperoctahedral groups S_k and B_k are modular subgroups of S_n and B_n, respectively. This follows from Theorem 3.9 and known facts on the modular elements of the geometric lattice $\mathcal{L}_W$ in these cases; see, for instance, [3, Theorem 2.2]. Alternatively, one can check directly that for $1 \leq i \leq n-1$, the transpositions $(i\ n)$ are representatives of the left cosets of S_{n-1} in S_n, other than S_{n-1}. Proposition 3.17 implies that S_{n-1} is modular in S_n. The transitivity property of Proposition 3.5 implies that S_k is modular in S_n for each $k \le n$. A similar argument works for the hyperoctahedral groups.

We end this section with two more open questions.

Question 3.20 *Do infinite modular subgroups exist?*

Question 3.21 *For which subgroups H of W does* $\mathrm{Abs}(X)$ *have a maximum element?*

4 Quasi-modular subgroups

This section introduces a condition on a subgroup of a Coxeter group, termed quasi-modularity, which is broader than modularity and guarantees an affirmative answer to Question 2.5. Examples of quasi-modular subgroups which are not modular are discussed. Throughout this section, the set of reflections of a Coxeter group H will be denoted by $\mathtt{T}(H)$.

4.1 Quasi-modularity

The main definition of this section is as follows.

Definition 4.1 A subgroup H of a finite Coxeter group W is *quasi-modular* if H is isomorphic to a Coxeter group and

$$W_{\mathtt{T}}(q) = H_{\mathtt{T}(H)}(q) \cdot X_{\mathtt{T}}(q), \tag{8}$$

where $\mathtt{T} = \mathtt{T}(W)$ and $\mathtt{T}(H)$ is the subset of H which corresponds to the set of reflections of this Coxeter group.

Proposition 3.4 implies that for reflection subgroups of W, quasi-modularity is equivalent to modularity. However, this is not the case for general subgroups as $\mathtt{T}(H)$ may not be equal to $H \cap \mathtt{T}(W)$.

Example 4.2 We list two families of examples of quasi-modular subgroups which are not modular.

(a) Let W be the Weyl group of type D_n, considered as a group of signed permutations of $\{1, 2, \dots, n\}$ with an even number of sign changes. Let H be the subgroup consisting of all $w \in W$ satisfying $w(n) \in \{n, -n\}$. Then H is isomorphic to the hyperoctahedral group B_{n-1} and the identity element $e \in W$ together with the reflections $((i\ n))$ for $1 \le i \le n-1$ (where the notation is as in Example 3.3(d)) form a complete list of coset representatives of H in W. As a result, we have $X_{\mathtt{T}}(q) = 1 + (n-1)q$, where $X = W/H$ and $\mathtt{T} = \mathtt{T}(W)$. Using this fact and (1), it can be easily verified that (8) holds in this situation and hence that H is a quasi-modular subgroup of W.

On the other hand, it is also easy to verify that $H_{\mathrm{T}}(q)$ has degree n, as does $W_{\mathrm{T}}(q)$. Thus H is not a modular subgroup of W by Proposition 3.4.

(b) Consider the symmetric group S_{2n} as the group of all permutations of the set $\Omega_n := \{1, -1, 2, -2, \ldots, n, -n\}$ and the natural embedding of the hyperoctahedral group B_n in S_{2n}, mapping the Coxeter generators of B_n to the transposition $(n\ \ -n)$ and the products $(i\ \ i+1)(-i\ \ -i-1)$ for $1 \le i \le n-1$. Clearly, this embedded copy of B_n is not a reflection subgroup of S_{2n}. Several combinatorial interpretations to the poset $\mathrm{Abs}(S_{2n}/B_n)$ will be given in Sect. 4.3, where the following statement will also be proved.

Theorem 4.3 *The group B_n is a non-modular, quasi-modular subgroup of S_{2n} for every $n \ge 2$.*

4.2 Balanced complex reflections

Before proving Theorem 4.3 we introduce an absolute order on balanced complex reflections. Recall that the wreath product of the cyclic group $\mathbb{Z}_r$ by the symmetric group S_n is defined as

$$G(r,n) = \mathbb{Z}_r \wr S_n := \bigl\{\bigl[(c_1, \ldots, c_n); \pi\bigr] : c_i \in \mathbb{Z}_r, \pi \in S_n\bigr\}$$

with group operation

$$\bigl[(c_1, \ldots, c_n); \pi\bigr] \cdot \bigl[\bigl(c'_1, \ldots, c'_n\bigr); \pi'\bigr] := \bigl[\bigl(c_1 + c'_{\pi^{-1}(1)}, \ldots, c_n + c'_{\pi^{-1}(n)}\bigr); \pi\pi'\bigr].$$

We think of the elements of $\mathbb{Z}_r$ as colors and denote by $\psi : G(r,n) \to S_n$ the canonical map, defined by $\psi([\bar{c}; \pi]) := \pi$. Via this map, the elements of $G(r,n)$ inherit a cycle structure from those of S_n.

Definition 4.4 A cycle of an element of $G(r,n)$ is *balanced* if the sum of the colors of its elements is zero modulo r. An element $w \in G(r,n)$ is *balanced* if all cycles of w are balanced. We denote by $C(r,n)$ the set of balanced elements of $G(r,n)$.

For example, there are three balanced elements in $G(2,2) \cong B_2$, namely the identity and the reflections $[(0,0); (1\ 2)] = (1\ 2)(-1\ \ -2)$ and $[(1,1); (1\ 2)] = (1\ \ -2)(-1\ 2)$. Note that $C(2,2)$ is not a subgroup of $G(2,2)$.

Remark 4.5 To motivate the notion of balanced, we note that balanced cycles generalize the notion of *paired cycles*, introduced by Brady and Watt [9] in the study of the absolute order of types B and D and further studied in [16]. Moreover, conjugacy classes in $G(r,n)$ are parametrized by cycle type and sum of colors (modulo r) in each cycle (so that $C(r,n)$ is the union of conjugacy classes in $G(r,n)$).

The wreath product $G(r,n)$ acts on the vector space $V = \mathbb{C}^n$ by permuting coordinates and multiplying them by suitable rth roots of unity, in a standard way. The set of pseudoreflections $\mathrm{T}(r,n) \subseteq G(r,n)$ consists of all elements fixing a hyperplane (codimension one subspace). The absolute length function $\ell_{\mathrm{T}(r,n)} : G(r,n) \to \mathbb{N}$ is defined with respect to the generating set $\mathrm{T}(r,n)$.

Remark 4.6 (a) We have $\psi(\mathtt{T}(r,n)) = \mathtt{T}(S_n) \cup \{e\}$, where $e \in S_n$ stands for the identity element.

(b) The set $\mathtt{T}(r,n) \cap C(r,n)$ of balanced pseudoreflections in $G(r,n)$ consists of all elements of the form $\tau = [\bar{c}; t]$, where $t = (a\ b) \in \mathtt{T}(S_n)$ and $\bar{c}$ assigns opposite colors to a and b and the zero color to all other elements of $\{1, 2, \dots, n\}$. As a result, we have $\psi(\mathtt{T}(r,n) \cap C(r,n)) = \mathtt{T}(S_n)$.

(c) The canonical map ψ has the following crucial property: given $w \in C(r,n)$ and $t \in \mathtt{T}(S_n)$ such that $t\psi(w)$ is covered by $\psi(w)$ in $\mathrm{Abs}(S_n)$, there is a unique (necessarily balanced) pseudoreflection $\tau \in \psi^{-1}(t)$ such that $\tau w \in C(r,n)$.

Definition 4.7 The absolute order on $C(r,n)$, denoted $\mathrm{Abs}(C(r,n))$, is the reflexive and transitive closure of the relation consisting of the pairs (u, v) of elements of $C(r,n)$ for which $v = \tau u$ for some $\tau \in \mathtt{T}(r,n) \cap C(r,n)$ and $\ell_{\mathtt{T}(r,n)}(u) < \ell_{\mathtt{T}(r,n)}(v)$.

The partial order $\mathrm{Abs}(C(r,n))$ is the subposet induced on $C(r,n)$ from Shi's absolute order on $G(r,n)$; see [22, 23]. We will focus on this subposet since it will be useful (in the special case $r = 2$) in our proof of Theorem 4.3.

Proposition 4.8

(a) *The canonical map* $\psi : G(r,n) \to S_n$ *induces a rank preserving poset epimorphism from the order* $\mathrm{Abs}(C(r,n))$ *onto* $\mathrm{Abs}(S_n)$.
(b) *Every maximal interval in* $\mathrm{Abs}(C(r,n))$ *is mapped isomorphically by* ψ *onto a maximal interval in* $\mathrm{Abs}(S_n)$.

Proof (a) The map ψ is a group epimorphism, by its definition, and $\psi(\mathtt{T}(r,n)) = \mathtt{T}(S_n) \cup \{e\}$ by Remark 4.6(a). Hence we have $\ell_{\mathtt{T}(r,n)}(w) \ge \ell_{\mathtt{T}}(\psi(w))$ for every $w \in G(r,n)$. For $w \in C(r,n)$ the reverse inequality $\ell_{\mathtt{T}(r,n)}(w) \le \ell_{\mathtt{T}}(\psi(w))$ follows from Remark 4.6(c). Thus we have $\ell_{\mathtt{T}(r,n)}(w) = \ell_{\mathtt{T}}(\psi(w))$ for every $w \in C(r,n)$. Furthermore, this fact and parts (b) and (c) of Remark 4.6 imply that for $u, v \in C(r,n)$, we have $v = \tau u$ for some $\tau \in \mathtt{T}(r,n) \cap C(r,n)$ and $\ell_{\mathtt{T}(r,n)}(u) < \ell_{\mathtt{T}(r,n)}(v)$ if and only if $\psi(v) = t\psi(u)$ for some $t \in \mathtt{T}(S_n)$ and $\ell_{\mathtt{T}}(\psi(u)) < \ell_{\mathtt{T}}(\psi(v))$. In other words, u is covered by v in $\mathrm{Abs}(C(r,n))$ if and only if $\psi(u)$ is covered by $\psi(w)$ in $\mathrm{Abs}(S_n)$.

(b) We first check that ψ maps maximal elements of $\mathrm{Abs}(C(r,n))$ to maximal elements of $\mathrm{Abs}(S_n)$. Indeed, since ψ is rank preserving, the rank of an element w in $\mathrm{Abs}(C(r,n))$ is equal to $n - k$, where k is the number of cycles. Thus, if $\psi(w)$ is not maximal in $\mathrm{Abs}(S_n)$, then $\psi(w)$ has at least two cycles and one can check that there exists $\tau \in \mathtt{T}(r,n) \cap C(r,n)$ such that τw has fewer cycles than w, so w is not maximal either. We next observe that by Remark 4.6(c), for every $w \in C(r,n)$ the map ψ induces a bijection between elements covered by w in $\mathrm{Abs}(C(r,n))$ and those covered by $\psi(w)$ in $\mathrm{Abs}(S_n)$. By induction on the rank of the top element, it follows that intervals in $\mathrm{Abs}(C(r,n))$ are mapped isomorphically by ψ to intervals in $\mathrm{Abs}(S_n)$. In particular, every maximal interval in $\mathrm{Abs}(C(r,n))$ is mapped isomorphically by ψ onto a maximal interval in $\mathrm{Abs}(S_n)$. $\square$

Corollary 4.9

$$\sum_{w\in C(r,n)} q^{\ell_{\mathrm{T}(r,n)}(w)} = \prod_{i=1}^{n-1}(1+riq).$$

Proof Let $\psi_0 : C(r,n) \to S_n$ be the restriction of ψ to $C(r,n)$. By Proposition 4.8 we have $\ell_{\mathrm{T}(r,n)}(w) = \ell_{\mathrm{T}}(\pi) = n-k$ for every $w \in C(r,n)$, where k is the number of cycles of $\pi := \psi_0(w)$. Since all elements in the preimage $\psi_0^{-1}(\pi)$ are balanced, we have

$$\left|\psi_0^{-1}(\pi)\right| = r^{n-k} = r^{\ell_T(\pi)}$$

and thus

$$\begin{aligned}\sum_{w\in C(r,n)} q^{\ell_{\mathrm{T}(r,n)}(w)} &= \sum_{\pi\in S_n} \left|\psi_0^{-1}(\pi)\right| q^{\ell_{\mathrm{T}}(\pi)} = \sum_{\pi\in S_n} r^{\ell_T(\pi)} q^{\ell_{\mathrm{T}}(\pi)} \\ &= \sum_{\pi\in S_n} (rq)^{\ell_{\mathrm{T}}(\pi)} = \prod_{i=1}^{n-1}(1+riq).\end{aligned}$$

□

4.3 Perfect matchings

A partition of a set Ω into two-element subsets is called a *perfect matching*. Throughout this section we will denote by $\mathcal{M}_n$ the set of perfect matchings of $\Omega_n = \{1,-1,2,-2,\dots,n,-n\}$. Consider the simple graph Δ_n, introduced in [13], on the set of nodes $\mathcal{M}_n$ in which two perfect matchings are adjacent if their symmetric difference is a cycle of length 4. The diameter and the enumeration of geodesics of this graph were studied in [5]; the induced subgraph on noncrossing perfect matchings was studied earlier in [13].

Definition 4.10 Fix an arbitrary element $x_0 \in \mathcal{M}_n$. The *absolute order* on $\mathcal{M}_n$, denoted $\mathrm{Abs}(\mathcal{M}_n)$, is the poset $(\mathcal{M}_n, \preceq)$ defined by letting $x \preceq y$ if x lies in a geodesic path in Δ_n with endpoints x_0 and y, for $x, y \in \mathcal{M}_n$.

The symmetric group S_{2n} of permutations of Ω_n acts naturally on $\mathcal{M}_n$ (this action may be identified with the conjugation action of S_{2n} on the set of fixed point free involutions on a $2n$-element set). The stabilizer of $x_0 = \{\{-1,1\},\{-2,2\},\dots,\{-n,n\}\}$ is the natural embedding of the hyperoctahedral group B_n in S_{2n} and hence we get the following statement.

Observation 4.11 *The poset* $\mathrm{Abs}(\mathcal{M}_n)$ *is isomorphic to* $\mathrm{Abs}(S_{2n}/B_n)$.

In particular, the isomorphism type of $\mathrm{Abs}(\mathcal{M}_n)$ is independent of the choice of $x_0 \in \mathcal{M}_n$. Without loss of generality, for the remainder of this section we will assume that $x_0 = \{\{-1,1\},\{-2,2\},\dots,\{-n,n\}\}$.

Proposition 4.12 *The poset* $\mathrm{Abs}(\mathcal{M}_n)$ *is isomorphic to* $\mathrm{Abs}(C(2,n))$.

Proof The proof generalizes a construction from [13].

Given a perfect matching $x \in \mathcal{M}_n$, consider the union $x \cup x_0$, consisting of the arcs of x and $\{-i, i\}$ for $1 \le i \le n$. This is a disjoint union of nontrivial cycles and isolated arcs. We orient the nontrivial cycles in the following way: Given any such cycle C, we let k be the minimum positive integer such that $\{-k, k\}$ is an arc of C and choose the cyclic orientation of C in which this edge is directed from $-k$ to k. We associate to x a signed permutation $f(x) = \pi \in B_n$ as follows. For $i \in \Omega_n$, we set $\pi(i) = i$ if $\{-i, i\} \in x$. Otherwise we set $\pi(i) = -j$ if either (i, j) or $(-i, -j)$ is a directed edge in the above orientation, and $\pi(i) = j$ if either $(-i, j)$ or $(i, -j)$ is a directed edge in the orientation. We will show that $f : \mathcal{M}_n \to C(2, n)$ is a well-defined map which is an isomorphism of the corresponding absolute orders.

We first observe that the map $f : \mathcal{M}_n \to B_n$ is well-defined. Indeed, this holds since $\{-i, i\} \in x \cup x_0$ for $1 \le i \le n$ and hence at most one of i and $-i$ can be the initial vertex of a directed arc in the above orientation. Moreover, since the number of arcs of any nontrivial cycle of $x \cup x_0$ is even, the number of arcs with vertices of same sign in such a cycle must also be even. This implies that every nontrivial cycle of the signed permutation $f(x)$ is balanced and hence we have a well-defined map $f : \mathcal{M}_n \to C(2, n)$.

To show that $f : \mathcal{M}_n \to C(2, n)$ is a bijection, it suffices to describe the inverse map $g : C(2, n) \to \mathcal{M}_n$. Given a balanced signed permutation $\pi \in C(2, n)$, we construct $g(\pi) \in \mathcal{M}_n$ as follows. First, we include in $g(\pi)$ the arc $\{-i, i\}$ for each $i \in \Omega_n$ with $\pi(i) = i$. Second, let $(a_1\ a_2\ \cdots\ a_k)$ be any nontrivial cycle of π and assume that a_1 is the minimum of the absolute values of the element of this cycle. We then include in $g(\pi)$ the arcs $\{a_1, -a_2\}, \{a_2, -a_3\}, \ldots, \{a_k, -a_1\}$. We leave it to the reader to verify that g is the inverse map of f.

Finally we prove that $f : \mathcal{M}_n \to C(2, n)$ induces an isomorphism of absolute orders. We consider the simple graph Γ_n on the node set $C(2, n)$ in which two permutations $\pi, \sigma \in C(2, n)$ are adjacent if $\pi^{-1}\sigma \in \mathtt{T}(2, n)$. Since f maps x_0 to the identity element of $C(2, n)$, it suffices to show that f induces a graph isomorphism from Δ_n to Γ_n. Indeed, two matchings $x_1, x_2 \in \mathcal{M}_n$ are adjacent in Δ_n if and only if there exist four distinct elements $i, j, k, l \in \Omega_n$ such that $x_1 \setminus \{\{i, j\}, \{k, l\}\} = x_2 \setminus \{\{i, k\}, \{j, l\}\}$. Without loss of generality, we may assume that (i, j) and (k, l) are directed edges in the orientation of $x_1 \cup x_0$. By considering the eight cases determined by the signs of i, j, k, l, one can verify that this happens if and only if there exists a reflection $\tau \in \{(j, k), (-j, -k)\} \subseteq \mathtt{T}(2, n)$ such that $f(x_2) = \tau f(x_1)$ and the proof follows. $\square$

Corollary 4.13 *There is a poset epimorphism from* Abs$(\mathcal{M}_n)$ *to* Abs(S_n) *which maps every maximal interval in* Abs$(\mathcal{M}_n)$ *isomorphically onto a noncrossing partition lattice of type A_{n-1}.*

Proof This follows from Propositions 4.12 and 4.8 and the fact that every maximal interval in Abs(S_n) is isomorphic to the lattice of noncrossing partitions of the set $\{1, 2, \ldots, n\}$. $\square$

The previous corollary implies [13, Corollaries 1.6 and 2.2] and [5, Theorem 3.20].

Corollary 4.14 *For every $n \geq 1$ we have*

$$(\mathcal{M}_n)_{\mathtt{T}}(q) = \prod_{i=0}^{n-1}(1+2iq).$$

Proof This follows from Proposition 4.12 and Corollary 4.9. □

Proof of Theorem 4.3 That B_n is a quasi-modular subgroup of S_{2n} follows from Observation 4.11, Corollary 4.14 and the known formulas for the rank generating functions of $\mathrm{Abs}(S_{2n})$ and $\mathrm{Abs}(B_n)$.

Suppose that B_n were a modular subgroup of S_{2n} for some $n \geq 2$. Then, according to Proposition 3.4 and Observation 4.11 we should have $(S_{2n})_{\mathtt{T}}(q) = (B_n)_{\mathtt{T}}(q) \cdot (\mathcal{M}_n)_{\mathtt{T}}(q)$, where $\mathtt{T} := \mathtt{T}(S_{2n})$, and hence $(B_n)_{\mathtt{T}}(q)$ should have degree n. This is not correct, since there exist elements of the natural embedding of B_n in S_{2n} which are cycles in S_{2n} of absolute length $2n-1$. □

Remark 4.15 By Corollary 4.14, the S_{2n} conjugation action on fixed point free involutions of Ω_n has a quasi-modular stabilizer. For $1 \leq k < n$ such that $n-k$ is even, however, the S_n conjugation action on involutions of $\{1, 2, \ldots, n\}$ with k fixed points has a nicely factorized rank generating function even though its stabilizer is not quasi-modular.

Question 4.16 *For $r \geq 3$, is there a Coxeter group action whose associated absolute order is isomorphic to* $\mathrm{Abs}(C(r,n))$*?*

The cardinality of $C(r,n)$ is equal to the product $\prod_{i=1}^{n-1}(1+ri)$ (by Corollary 4.9) and hence to the number of $(r+1)$-ary increasing trees of order n; see, for instance, [24].

Question 4.17 *For $r \geq 1$, is there a* (*natural*) *Coxeter group action on these trees whose associated absolute order is isomorphic to* $\mathrm{Abs}(C(r,n))$*?*

5 An application to alternating subgroups

Throughout this section (W,S) will be a Coxeter system with set of reflections $\mathtt{T} = \{wsw^{-1} : w \in W, s \in S\}$. The *alternating subgroup* W^+ is defined as the kernel of the *sign character* on W, which maps every element of S to -1. We will show that a natural absolute order on W^+ can be defined in a way which is compatible with the general construction of Sect. 2.

Choose any element $s_0 \in S$. Then $S_0 := \{s_0 s : s \in S\}$ is a generating set for W^+ which carries a simple presentation [8, §IV.1, Ex. 9] and a Coxeter-like structure [10]. Let us write

$$\mathtt{T}_0 := \{s_0 t \colon t \in \mathtt{T}\}.$$

Given a pair (G, A) of a group G and generating set A, we say that an element $g \in G$ is an *odd palindrome* if there is an $(A \cup A^{-1})^*$-word $(a_1, \dots, a_\ell)$ for g such that ℓ is odd and $a_i = a_{\ell-i+1}$ for every index i. For example, the set of odd palindromes for (W, S) is equal to $\mathtt{T}$.

Claim 5.1 *The set of odd palindromes for (W^+, S_0) is equal to $\mathtt{T}_0$.*

Proof Let w be an odd palindrome in (W^+, S_0). Then $s_0 w$ is an odd palindrome in (W, S) and hence a reflection in $\mathtt{T}$. Conversely, since s_0 is an involution, for every reflection $t = s_{i_1} s_{i_2} s_{i_3} s_{i_4} s_{i_5} \cdots s_{i_4} s_{i_3} s_{i_2} s_{i_1} \in \mathtt{T}$,

$$s_0 t = (s_0 s_{i_1})(s_{i_2} s_0)(s_0 s_{i_3})(s_{i_4} s_0)(s_0 s_{i_5}) \cdots (s_{i_4} s_0)(s_0 s_{i_3})(s_{i_2} s_0)(s_0 s_{i_1})$$

is an odd palindrome in (W^+, S_0). □

Odd palindromes in alternating subgroups play a role which is analogous to that played by reflections in Coxeter groups [10, §2.5, §3.5]. This leads to the following definition of absolute order to alternating subgroups.

Definition 5.2 Given a simple reflection $s_0 \in S$, the *(left) absolute order* $\leq_{\mathtt{T}_0}$ on the alternating subgroup W^+ of W is defined as the reflexive and transitive closure of the relation consisting of the pairs (u, v) of elements of W^+ for which $\ell_{\mathtt{T}_0}(u) < \ell_{\mathtt{T}_0}(v)$ and $v = \tau u$ for some $\tau \in \mathtt{T}_0$.

The absolute order on W^+, which we will denote by $\mathrm{Abs}_0(W^+)$, depends on the choice of s_0: non-conjugate simple reflections determine non-isomorphic absolute orders on W^+. For example, the absolute order on B_3^+, which is determined by the choice of the adjacent transposition $s_0 = (1\ 2)(-1\ \ -2)$, is not isomorphic to the one determined by the choice $s_0 = (1\ \ -1)$. However, the rank generating function is independent of the choice of s_0. This will be proved by considering the action of W on cosets of $\langle s_0 \rangle$, the subgroup generated by s_0.

Here are some basic lemmas on the absolute lengths $\ell_{\mathtt{T}}$ and $\ell_{\mathtt{T}_0}$ which will be used in the proof. For $w \in W$ we set $w^{s_0} := s_0 w s_0$.

Lemma 5.3 *For every $w \in W^+$ we have*

$$\ell_{\mathtt{T}_0}(w) = \begin{cases} \ell_{\mathtt{T}}(w), & \textit{if } \ell_{\mathtt{T}_0}(w) \textit{ is even} \\ \ell_{\mathtt{T}}(w) - 1, & \textit{if } \ell_{\mathtt{T}_0}(w) \textit{ is odd.} \end{cases}$$

Proof Let $w = t_1 \cdots t_\ell$ be a $\mathtt{T}$-word for w of length $\ell := \ell_{\mathtt{T}}(w)$. Since $w \in W^+$, the number ℓ is even and we may write

$$w = t_1 s_0 s_0 t_2 s_0 s_0 t_3 \cdots t_{\ell-1} s_0 s_0 t_\ell = s_0 t_1^{s_0} s_0 t_2 \cdots s_0 t_{\ell-1}^{s_0} s_0 t_\ell.$$

This proves that

$$\ell_{\mathtt{T}_0}(w) \leq \ell = \ell_{\mathtt{T}}(w). \tag{9}$$

Suppose that $\ell_{\mathrm{T}_0}(w)=2m$ is even. Then we may write

$$w=s_0t_1\cdots s_0t_{2m}=\prod_{i=1}^{m}t_{2i-1}^{s_0}t_{2i},$$

with $t_i\in\mathrm{T}$ for each index i. Thus $\ell_{\mathrm{T}}(w)\le 2m=\ell_{\mathrm{T}_0}(w)$ and the proof follows in this case. Finally, if $\ell_{\mathrm{T}_0}(w)=2m+1$ is odd, then we may write

$$w=s_0t_1\cdots s_0t_{2m+1}=s_0t_1\prod_{i=1}^{m}t_{2i}^{s_0}t_{2i+1},$$

with $t_i\in\mathrm{T}$ for each i. This shows that

$$\ell_{\mathrm{T}}(w)\le 2m+2=\ell_{\mathrm{T}_0}(w)+1. \tag{10}$$

Combining (9) with (10) yields $\ell_{\mathrm{T}_0}(w)\le\ell_{\mathrm{T}}(w)\le\ell_{\mathrm{T}_0}(w)+1$. Since $\ell_{\mathrm{T}}(w)$ and $\ell_{\mathrm{T}_0}(w)$ have distinct parities, we conclude that $\ell_{\mathrm{T}}(w)=\ell_{\mathrm{T}_0}(w)+1$ and the proof follows in this case too. □

Lemma 5.4 *For every $w\in W^+$, the following conditions are equivalent*:

(i) $\ell_{\mathrm{T}_0}(w)$ *is even.*
(ii) $\ell_{\mathrm{T}}(w)<\ell_{\mathrm{T}}(s_0w)$.
(iii) $\ell_{\mathrm{T}}(w)<\ell_{\mathrm{T}}(ws_0)$.

Proof Since the absolute length is invariant under conjugation (Fact 3.1(e)), we have $\ell_{\mathrm{T}}(s_0w)=\ell_{\mathrm{T}}(ws_0)$ and hence it suffices to prove that (i) $\Leftrightarrow$ (ii).

Suppose first that $\ell_{\mathrm{T}}(w)>\ell_{\mathrm{T}}(s_0w)$. We note that $\ell_{\mathrm{T}}(s_0w)$ is an odd number, since $w\in W^+$, say $\ell_{\mathrm{T}}(s_0w)=2m+1$, and let $t_1\cdots t_{2m+1}$ be a reduced T-word for s_0w. Then $s_0t_1\cdots t_{2m+1}$ is a reduced T-word for w and $\ell_{\mathrm{T}}(w)=2m+2$. Since

$$w=s_0t_1\prod_{i=1}^{m}\bigl(s_0t_{2i}^{s_0}\bigr)(s_0t_{2i+1}),$$

we have $\ell_{\mathrm{T}_0}(w)\le 2m+1$. On the other hand, we have $\ell_{\mathrm{T}_0}(w)\ge\ell_{\mathrm{T}}(w)-1=2m+1$ by Lemma 5.3. Thus $\ell_{\mathrm{T}_0}(w)=2m+1$ and, in particular, $\ell_{\mathrm{T}_0}(w)$ is odd. This proves the implication (i) $\Rightarrow$ (ii).

Conversely, suppose that $\ell_{\mathrm{T}_0}(w)$ is odd. Then the proof of Lemma 5.3 shows that there is a reduced T-word for w which starts with s_0. This implies that $\ell_{\mathrm{T}}(w)>\ell_{\mathrm{T}}(s_0w)$ and hence (ii) $\Rightarrow$ (i). □

Let us denote by $\langle s_0\rangle$ the two-element subgroup of W generated by s_0. We recall that the absolute length function on $W/\langle s_0\rangle$ is determined by Definition 2.3.

Corollary 5.5 *We have $\ell_{\mathrm{T}}(w\langle s_0\rangle)=\ell_{\mathrm{T}_0}(w)$ for every $w\in W^+$.*

Proof By definition of $\ell_{\mathrm{T}}(w\langle s_0\rangle)$ and Lemma 5.4 we have

$$\ell_{\mathrm{T}}\big(w\langle s_0\rangle\big) = \min\big\{\ell_{\mathrm{T}}(w), \ell_{\mathrm{T}}(ws_0)\big\} = \begin{cases} \ell_{\mathrm{T}}(w), & \text{if } \ell_{\mathrm{T}_0}(w) \text{ is even} \\ \ell_{\mathrm{T}}(w) - 1, & \text{if } \ell_{\mathrm{T}_0}(w) \text{ is odd} \end{cases}$$

and the result follows from Lemma 5.3. □

Proposition 5.6 *The orders* $\mathrm{Abs}_0(W^+)$ *and* $\mathrm{Abs}(W/\langle s_0\rangle)$ *are isomorphic.*

Proof We consider the map $\varphi : W^+ \longrightarrow W/\langle s_0\rangle$ defined by

$$\varphi(w) := \begin{cases} w\langle s_0\rangle, & \text{if } \ell_{\mathrm{T}_0}(w) \text{ is even} \\ w^{s_0}\langle s_0\rangle, & \text{if } \ell_{\mathrm{T}_0}(w) \text{ is odd} \end{cases}$$

for $w \in W^+$. We will show that φ is the required isomorphism of absolute orders.

We first note that conjugation by s_0 is an automorphism on both W and W^+ which preserves the lengths ℓ_{T} and ℓ_{T_0}, respectively. Corollary 5.5 then implies that

$$\ell_{\mathrm{T}}\big(\varphi(w)\big) = \ell_{\mathrm{T}_0}(w) \tag{11}$$

for every $w \in W^+$. Since the map $\pi : W^+ \longrightarrow W/\langle s_0\rangle$ defined by $\pi(w) = w\langle s_0\rangle$ is a bijection, we may conclude that φ is a bijection as well. Thus, it remains to show that the following conditions are equivalent for $u, v \in W^+$:

(a) u is covered by v in $\mathrm{Abs}_0(W^+)$,
(b) $\varphi(u)$ is covered by $\varphi(v)$ in $\mathrm{Abs}(W/\langle s_0\rangle)$.

Using the definitions of the relevant absolute orders, we find that

$$\begin{aligned} \text{(a)} \quad &\Leftrightarrow v = \tau u \text{ for some } \tau \in \mathrm{T}_0 \text{ and } \ell_{\mathrm{T}_0}(u) < \ell_{\mathrm{T}_0}(v) \\ &\Leftrightarrow v = s_0 t u \text{ for some } t \in \mathrm{T} \text{ and } \ell_{\mathrm{T}_0}(u) < \ell_{\mathrm{T}_0}(v) \\ &\Leftrightarrow v^{s_0}\langle s_0\rangle = tu\langle s_0\rangle \text{ for some } t \in \mathrm{T} \text{ and } \ell_{\mathrm{T}_0}(u) < \ell_{\mathrm{T}_0}(v) \\ &\Leftrightarrow v\langle s_0\rangle = t^{s_0}u^{s_0}\langle s_0\rangle \text{ for some } t \in \mathrm{T} \text{ and } \ell_{\mathrm{T}_0}(u) < \ell_{\mathrm{T}_0}(v) \end{aligned}$$

and

(b) $\Leftrightarrow \varphi(v) = t\varphi(u)$ for some $t \in \mathrm{T}$ and $\ell_{\mathrm{T}}(\varphi(u)) < \ell_{\mathrm{T}}(\varphi(v))$.

The claim that (a) $\Leftrightarrow$ (b) follows from the previous equivalences, (11) and the definition of the map φ. □

The following statement extends [21, Theorem 7.2] from the case of symmetric groups to that of all finite Coxeter groups.

Corollary 5.7 *For every finite Coxeter group* W *we have*

$$\sum_{w \in W^+} q^{\ell_{\mathrm{T}_0}(w)} = \frac{W_{\mathrm{T}}(q)}{1+q} = \prod_{i=2}^{d}(1 + e_i q), \tag{12}$$

where d *is the Coxeter rank and* $1 = e_1, e_2, \ldots, e_d$ *are the exponents of* W.

Proof Proposition 5.6 implies that

$$\sum_{w\in W^+} q^{\ell_{\mathrm{T}_0}(w)} = \big(W/\langle s_0\rangle\big)_{\mathrm{T}}(q).$$

Since $\langle s_0\rangle$ is a modular subgroup of W (see Example 3.3(a)), we have

$$\big(W/\langle s_0\rangle\big)_{\mathrm{T}}(q) = \frac{W_{\mathrm{T}}(q)}{\langle s_0\rangle_{\mathrm{T}}(q)} = \frac{W_{\mathrm{T}}(q)}{1+q}$$

by Proposition 3.4 and the first equality in (12) follows. The second equality is a restatement of (1). □

Another description of $\mathrm{Abs}_0(W^+)$ can be given as follows. Let us write $R_0 := \{w\in W : \ell_{\mathrm{T}}(ws_0) > \ell_{\mathrm{T}}(w)\}$. The proof of Proposition 3.7 shows that R_0 is an order ideal of $\mathrm{Abs}(W)$.

Corollary 5.8 *The absolute order* $\mathrm{Abs}_0(W^+)$ *is isomorphic to* $(R_0, \le_{\mathrm{T}})$.

Proof The proof of Proposition 3.7 shows that $\mathrm{Abs}(W/\langle s_0\rangle)$ is isomorphic to $(R_0, \le_{\mathrm{T}})$. The result follows from this statement and Proposition 5.6. □

6 Remarks on ordered tuples

This section briefly discusses the action of the symmetric group S_n on the set $X_{n,k}$ of ordered k-tuples of pairwise distinct elements of $\{1,2,\dots,n\}$, as well as a generalization. The stabilizer S_{n-k} of this action is a modular reflection subgroup of S_n (see Example 3.19). Therefore, by Proposition 3.4 we have

$$(X_{n,k})_{\mathrm{T}}(q) = \frac{(S_n)_{\mathrm{T}}(q)}{(S_{n-k})_{\mathrm{T}}(q)} = \prod_{i=n-k}^{n-1}(1+iq).$$

By a classical result of Hurwitz [15] (see also [11, 28]), there is a one-to-one correspondence between the maximal chains of any maximal interval of $\mathrm{Abs}(S_n)$ and labeled trees of order n. The following generalization of this statement on the enumeration of maximal chains of $X_{n,k}$ is possible. We will denote by $d_\Gamma(v)$ the valency (i.e., number of neighbors) of a node v of a labeled tree Γ of order n.

Proposition 6.1 *For all integers* $1\le k<n$, *the number of maximal chains of* $\mathrm{Abs}(X_{n,k})$ *is equal to*

$$k!\sum_{\Gamma}(n-k)^{d_\Gamma(v_0)},$$

where the sum runs over all trees Γ *on the node set* $\{v_0, v_1, \dots, v_k\}$.

The proof of this statement will be given elsewhere. The special case $k = n - 1$ is equivalent to Hurwitz's theorem.

Combining Propositions 5.6 and 6.1, we get the following statement.

Corollary 6.2 *The number of maximal chains of the absolute order on the alternating group of S_n is equal to*

$$(n-2)! \sum_{\Gamma} 2^{d_\Gamma(v_0)},$$

where the sum runs over all trees Γ on the node set $\{v_0, v_1, \dots, v_{n-2}\}$.

The previous setting has a natural extension to wreath product actions on ordered colored tuples. Recall from [22, 23] the absolute order on the complex reflection group $G(r,n) = \mathbb{Z}_r \wr S_n$; absolute length and order are naturally defined with respect to the set $\mathtt{T}$, consisting of all elements (pseudoreflections) of finite order fixing a hyperplane. Let $X_{r,n,k} := \{(a_1, \dots, a_k) \colon \forall i \ a_i \in \mathbb{Z}_r \times \mathbb{Z}_n\}$ be the set of ordered k-tuples of letters in an alphabet of size n which are r-colored. Then $G(r,n)$ acts naturally on $X_{r,n,k}$, with stabilizer $G(r, n-k)$. By extending Propositions 3.5 and 3.17, one can prove that the subgroup $G(r,n-k)$ is a modular subgroup of $G(r,n)$ for $1 \le k \le n$. Hence, by (the extension of) Proposition 3.4 we have

$$X_{r,n,k\,\mathtt{T}}(q) = \frac{G(r,n)_{\mathtt{T}}(q)}{G(r,n-k)_{\mathtt{T}}(q)} = \prod_{i=n-k}^{n-1} (1 + riq).$$

Acknowledgements The authors wish to thank the anonymous referees for several helpful comments.

References

1. Armstrong, D.: Generalized noncrossing partitions and combinatorics of Coxeter groups. Mem. Am. Math. Soc. **202**(949) (2009)
2. Athanasiadis, C.A., Kallipoliti, M.: The absolute order on the symmetric group, constructible partially ordered sets and Cohen–Macaulay complexes. J. Comb. Theory, Ser. A **115**, 1286–1295 (2008)
3. Barcelo, H., Ihrig, E.: Modular elements in the lattice $L(A)$ when A is a real reflection arrangement. Discrete Math. **193**, 61–68 (1998). Selected papers in honor of Adriano Garsia (Taormina, 1994)
4. Barcelo, H., Ihrig, E.: Lattices of parabolic subgroups in connection with hyperplane arrangements. J. Algebr. Comb. **9**, 5–24 (1999)
5. Ben-Ari, R.: Combinatorial parameters on matchings in complete graphs. M. Sc. Thesis, Bar-Ilan University (2010)
6. Bessis, D.: The dual braid monoid. Ann. Sci. Éc. Norm. Super. **36**, 647–683 (2003)
7. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups. Graduate Texts in Mathematics, vol. 231. Springer, New York (2005)
8. Bourbaki, N.: Lie Groups and Lie Algebras. Springer, Berlin (2002)
9. Brady, T., Watt, C.: $K(\pi,1)$'s for Artin groups of finite type. In: Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I, Haifa, 2000. Geom. Dedicata, vol. 94, pp. 225–250 (2002)
10. Brenti, F., Reiner, V., Roichman, Y.: Alternating subgroups of Coxeter groups. J. Comb. Theory, Ser. A **115**, 845–877 (2008)
11. Dénes, J.: The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs. Publ. Math. Inst. Hung. Acad. Sci. **4**, 63–71 (1959)

12. Dyer, M.J.: On minimal lengths of expressions of Coxeter group elements as products of reflections. Proc. Am. Math. Soc. **129**, 2591–2595 (2001)
13. Hernando, C., Hurtado, F., Noy, M.: Graphs of non-crossing perfect matchings. Graphs Comb. **18**, 517–532 (2002)
14. Humphreys, J.E.: Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics, vol. 29. Cambridge University Press, Cambridge (1990)
15. Hurwitz, A.: Üeber Riemannsche Flächen mit gegebenen Verzweigungspunkten (German). Math. Ann. **39**, 1–60 (1891)
16. Kallipoliti, M.: The absolute order on the hyperoctahedral group. J. Algebr. Comb. **34**, 183–211 (2011)
17. Kung, J.P.S.: Sign-coherent identities for characteristic polynomials of matroids. Comb. Probab. Comput. **2**, 33–51 (1993)
18. Orlik, P., Solomon, L.: Unitary reflection groups and cohomology. Invent. Math. **59**, 77–94 (1980)
19. Orlik, P., Terao, H.: Arrangements of Hyperplanes. Springer, New York (1992)
20. Rains, E.M., Vazirani, M.J.: Deformations of permutation representations of Coxeter groups. J. Algebr. Comb. **37**, 455–502 (2013)
21. Rotbart, A.: Generator sets for the alternating group. Sémin. Lothar. Comb. **65**, 16 (2011). Article B65b (electronic)
22. Shi, J.-Y.: Formula for the reflection length of elements in the group $G(m,p,n)$. J. Algebra **316**, 284–296 (2007)
23. Shi, J.-Y.: Reflection ordering on the group $G(m,p,n)$. J. Algebra **319**, 4646–4661 (2008)
24. Sloane, N.J.A., Plouffe, S.: The Encyclopedia of Integer Sequences. Academic Press, San Diego (1995). Sequences: A007559, A007696
25. Stanley, R.P.: Modular elements of geometric lattices. Algebra Univers. **1**, 214–217 (1971)
26. Stanley, R.P.: Enumerative Combinatorics, vol. 1. Wadsworth & Brooks/Cole, Pacific Grove (1986). Second printing, Cambridge Studies in Advanced Mathematics, vol. 49. Cambridge University Press, Cambridge (1997)
27. Stanley, R.P.: An introduction to hyperplane arrangements. In: Geometric Combinatorics. IAS/Park City Math., vol. 13, pp. 389–496. Am. Math. Soc., Providence (2007)
28. Strehl, V.: Minimal transitive products of transpositions—the reconstruction of a proof of A. Hurwitz. Sémin. Lothar. Comb. **37**, 12 (1996). Article S37c (electronic)