Jack vertex operators and realization of Jack functions

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Abstract We give an iterative method to realize general Jack functions using vertex operators. We first prove some cases of Stanley’s conjecture on positivity of the Littlewood–Richardson coefficients, and then use this method to give a new realization of Jack functions. We also show in general that the images of coefficients of products of Jack vertex operators form a basis of symmetric functions. In particular, this gives a new proof of linear independence for the rectangular and marked rectangular Jack vertex operators. Finally, a generalized Frobenius formula for Jack functions is given and used for evaluation of Dyson integrals and even powers of Vandermonde determinants.

Keywords Symmetric functions · Jack polynomials · Vertex operators

1 Introduction

Jack symmetric functions play a fundamental role in the Calogero–Sutherland model in statistical mechanics [6] and they also appear as extremal vectors in the Fock space representations of the Virasoro algebra [3, 21] and W -algebras [1]. From algebraic and combinatorial viewpoints, explicit realization of Jack functions is a desirable task and an interesting question. In the special case of Schur functions, the explicit realization is the Jacobi–Trudi formula, which has a long list of applications ranging from Schubert calculus in algebraic geometry to various combinatorial problems.

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The Jacobi–Trudi formula for rectangular Jack functions is known [20], and it also reveals the connection with a special case of Dyson’s conjecture for constant terms and even powers of Vandermonde determinants [2]. Recently, Jack functions have also appeared as Whittaker vectors of certain modules of the Virasoro algebra [26].

In this paper, we first settle a long-standing problem of realizing Jack symmetric functions of all shapes using vertex operators. The realization is based upon our earlier work of reformulation of Jack functions for rectangular-shapes using vertex operators [3, 5, 21]. The key idea behind our approach is to realize a general Jack symmetric function by a sequence of rectangular shaped Jack functions and then introduce a procedure to encode the filtration in the Fock space. We would like to remark that this new vertex operator realization of classical symmetric functions is different from earlier realizations of Schur and Hall–Littlewood polynomials [10, 11].

As an application of the new iterative vertex operator realization of Jack symmetric functions, we derive a generalized Frobenius formula expressing a Jack function as a linear combination of power sum symmetric functions. We note that our formula seems to be new even in the case of zonal spherical functions (in both orthogonal and symplectic cases). Previous formulas of Jack polynomials have been given using multiple integrals [1, 22] and also combinatorially in [13]. Rodrigues type formula [15] of Jack polynomials also provides an explicit realization of Jack polynomials. It would be interesting to know the relationship of our formula with Lassalle’s conjectures [16, 17].

As our second goal, we study the connection between the contraction function of the Jack vertex operators for the rectangular shapes and Dyson’s constant term conjecture, and evaluate more terms in the generalized Vandermonde determinant or its even powers. This is done by writing more terms in the expansion using our earlier work [3] on Jack symmetric functions of rectangular shapes. Explicitly, vertex operators provide an effective tool to compute all coefficients of $z_1^{-\lambda_1} \cdots z_i^{-\lambda_i} z_{s-j+1}^{\mu_j} \cdots z_{s-1}^{\mu_2} z_s^{\mu_1}$ ($i + j \leq s$), in the product $\prod_{1 \leq i \neq j \leq s} (1 - z_i z_j^{-1})^\beta$, where $\lambda = (\lambda_1, \dots, \lambda_i)$, $\mu = (\mu_1, \dots, \mu_j)$ are two partitions of equal weight and β is a positive integer.

Our third goal is to prove that the products of Jack vertex operators form a basis in the Fock space associated to the Jack functions. This is deeply rooted to a basic question in the representation theory of higher level representations of affine Lie algebras. Lepowsky–Wilson’s work on Rogers–Ramanujan identities was partly related to linear independence of some products of vertex operators similar to our Jack vertex operators. Our case is simpler as we can prove a general result which is equivalent to a statement for products of vertex operators in higher level representations of the affine Lie algebra $\widehat{sl}_2$ [3, 18].

A by-product of our approach is that we can prove several non-trivial cases of Stanley’s positivity conjecture [24, Conj. 8.3] on the Littlewood–Richardson coefficients for Jack symmetric functions. We first give a formula on the product of two Jack functions of dual shapes, which can be viewed as a Jack case generalization of Schubert-like product in the cohomology of a Grassmannian. This turns out to be useful in computations related to Stanley’s conjecture. We then generalize the result to marked rectangular (rectangle with part of the last row/column removed) case.

This paper is organized as follows. In Sect. 2, we recall some basic notions about symmetric functions and collect a few results from [3] for later use. In Sect. 3, we first give a refinement and a generalization of the main result of [3] and a new method to compute the coefficients of the monomials in the generalized Vandermonde determinant. Then we prove that the coefficients of products of vertex operators form a basis in the Fock space of symmetric functions (Theorem 3.15), and their basic properties are also listed. Section 4 treats special cases of Stanley's conjecture, and then solves the problem of realizing Jack functions by vertex operators in general (Theorem 4.21), and finally a Frobenius type formula for Jack symmetric functions is given in Corollary 4.24.

2 The realization of Jack functions of rectangular partitions

We first recall basic notions about symmetric functions. A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is a sequence of non-negative integers in non-increasing order. Sometimes we also write $\lambda = (1^{m_1} 2^{m_2} \dots)$, where $m_i = m_i(\lambda)$ is the multiplicity of i occurring among the parts of λ . The length of λ , denoted as $l(\lambda)$, is the number of non-zero parts in λ , and the weight $|\lambda|$ is $\sum_i i m_i$. It is convenient to denote $m(\lambda)! = m_1! m_2! \dots$, $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$. A partition λ of weight n is denoted by $\lambda \vdash n$. The set of all partitions is denoted by $\mathcal{P}$, similarly $\mathcal{P}_n$ means the subset of weight n . For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ with equal weight, we say that $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i , this defines a partial order (the dominance order) in $\mathcal{P}$. If $m_i(\mu) \leq m_i(\lambda)$ for all i , we say that μ is *exponentially contained* in λ or *contained by multiplicity* and we use the notation $\mu \subset' \lambda$. Partition $\lambda \setminus \mu$ is then defined by $m_i(\lambda \setminus \mu) = m_i(\lambda) - m_i(\mu)$, and we also define $\binom{m(\lambda)}{m(\mu)} = \prod_i \binom{m_i(\lambda)}{m_i(\mu)}$. Sometimes we also identify λ with its Young or Ferrers diagram $Y(\lambda) = \{(i, j) | 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$ which is a left justified diagram made of $l(\lambda)$ rows of boxes (or squares as being called by some authors) with λ_i boxes in the i th row. A partition of the form $\lambda = (r^s)$ is called a rectangular partition or a partition of rectangular shape. If the Young diagram of μ is a subset of the Young diagram of λ , we write $\mu \subset \lambda$, define $\lambda - \mu$ to be the set difference $Y(\lambda) - Y(\mu)$. We call $\lambda - \mu$ a horizontal n -strip if there are n boxes in $\lambda - \mu$ and with no two boxes in the same column of λ . The conjugate of λ , denoted as λ' , is a partition whose diagram is the transpose of the diagram of λ , thus λ'_i is the number elements in the set $\{j | \lambda_j \geq i\}$. For a box $s = (i, j) \in \lambda$, the lower hook-length $h_\lambda^*(s)$ and the upper hook-length $h_\lambda^\lambda(s)$ are defined as the following polynomials in the rational field $\mathbb{Q}(\alpha)$ of indeterminate α [24]:

$$h_\lambda^*(s) = \alpha(\lambda_i - j) + (\lambda'_j - i + 1),$$

$$h_\lambda^\lambda(s) = \alpha(\lambda_i - j + 1) + (\lambda'_j - i).$$

Example 2.1 Figure 1(a) is partition $\lambda = (5, 3, 3, 1)$, Fig. 1(b) is the diagram of partition λ' . For $a = (1, 1), b = (2, 2) \in Y(\lambda)$, we have $h_\lambda^*(a) = 4\alpha + 4$, $h_\lambda^\lambda(b) = 2\alpha + 1$. As shown in Fig. 1(c), the coefficient of α (resp., constant) is the number of boxes the solid line passes horizontally (resp., vertically).

Fig. 1 Young diagrams and Jack hook lengths

We sometimes write partition $\lambda = (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s})$, with $m_{a_i}(\lambda) = n_i > 0$, and $a_1 > a_2 > \cdots > a_s > 0$. For each $i = 1, 2, \dots, s$, we replace the part a_i of λ by a_{i+1} , setting $a_{s+1} = 0$, therefore obtaining a new partition, denoted as $\lambda(i)$. Equivalently,

$$\lambda(i) = (a_1^{n_1} a_2^{n_2} \cdots a_{i-1}^{n_{i-1}} a_{i+1}^{n_i+n_{i+1}} a_{i+2}^{n_{i+2}} \cdots a_s^{n_s}).$$

Then we define the i th corner of λ as $C_i(\lambda) = \lambda - \lambda(i)$. For example, in Fig. 1(b) we mark the second corner of the partition $(4, 3, 3, 1, 1)$ with stars. $C_i(\lambda)$ is of the same shape as the rectangle $((a_i - a_{i+1})^{n_i})$. λ has s corners, we write $n_c(\lambda) = s$ as the number of corners of λ . $n_c(\lambda) = 1$ if and only if λ is a rectangular partition and in this case the unique corner is λ itself. For a partition $\nu = (\nu_1, \dots, \nu_l)$ with $\nu_1 \leq a_i - a_{i+1}$, $l \leq n_i$, we define the upside-down ν of the $C_i(\lambda)$ as the union of B_j for $j = 1, 2, \dots, l$, where B_j are the rightmost ν_j boxes on the $(n_i + 1 - j)$ th row of $C_i(\lambda)$. For example, we can write $\lambda' = (4, 3, 3, 1, 1)$ as $(4^1 3^2 1^2)$, $\lambda'(2) = (4^1 1^4)$ and the second corner of λ' is $C_2(\lambda') = \lambda' - \lambda'(2) = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$, i.e., the boxes marked by stars as in Fig. 1(b). An upside-down $(2, 1)$ of the second corner of λ is marked by the minus sign in Fig. 1(a).

Definition 2.2 For a subset S of the Young diagram of λ , define $h_*^\lambda(S)$ as $\prod_{s \in S} h_*^\lambda(s)$, and similarly for $h_\lambda^*(S)$. For a pair of partitions $\mu \subset \lambda$, we say that a square s is based if s is in a column which contains at least one square of $\lambda - \mu$, and it is un-based otherwise. We denote μ_b (resp., λ_b) as the set of the based squares of μ (resp., λ), and μ_u (resp., λ_u) as the set of the un-based squares of μ (resp., λ).

For a set (finite or infinite) of variables $z_1, z_2, \dots$ and a partition λ , we define $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots$. For a Laurent polynomial $F(z, w)$ in variables $z_1, z_2, \dots, w_1, w_2, \dots$, we denote the coefficient of monomial z^λ / w^μ as $C_{z^\lambda / w^\mu} F(z, w)$.

The ring Λ of symmetric functions in variables x_i 's is a $\mathbb{Z}$ -module with basis of monomial functions $m_\lambda = (m(\lambda)!)^{-1} \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, where $k = l(\lambda)$ and the multi-indices $(i_1, \dots, i_k)$ are pairwise distinct and run through $(\mathbb{Z}_{>0})^k$. We define $p_n = m_{(n)} = \sum_{i \geq 1} x_i^n$. The power sum symmetric functions $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, $\lambda \in \mathcal{P}$, form a basis in $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We also define $p_0 = 1$.

Let $F = \mathbb{Q}(\alpha)$. For two partitions $\lambda, \mu \in \mathcal{P}$ the Jack scalar product on $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ is defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} \alpha^{l(\lambda)} z_\lambda, \quad (1)$$

where δ is the Kronecker symbol.

The Jack symmetric functions $P_\lambda(\alpha) = P_\lambda(x_1, x_2, \dots; \alpha)$ for $\lambda \in \mathcal{P}$ are determined by the following conditions:

$$P_\lambda(\alpha) = \sum_{\lambda \geq \mu} c_{\lambda\mu}(\alpha) m_\mu,$$

$$\langle P_\lambda(\alpha), P_\mu(\alpha) \rangle = 0 \quad \text{for } \lambda \neq \mu,$$

where $c_{\lambda\mu}(\alpha) \in F$ ($\lambda, \mu \in \mathcal{P}$) and $c_{\lambda\lambda}(\alpha) = 1$. Note that symmetric functions m_λ , p_λ , $P_\lambda(\alpha)$, and the following $Q_\lambda(\alpha)$, $J_\lambda(\alpha)$ are all in variables $x_1, x_2, \dots$. Jack functions were first introduced in [9] and later defined via triangularity and orthogonality by Macdonald (cf. [19]) and Stanley [24]. Further properties of Jack functions were studied in [13] and [8]. In this work, we will also need two other normalizations. One is the dual Jack functions $Q_\lambda(\alpha) = \langle P_\lambda(\alpha), P_\lambda(\alpha) \rangle^{-1} P_\lambda(\alpha)$. Another is the integral Jack function $J_\lambda(\alpha)$, which is defined by $J_\lambda(\alpha) = |\lambda|! (c_{\lambda, (1^{|\lambda|})})^{-1} P_\lambda(\alpha)$. To simplify the notations, we write $J_{(n)}$ as J_n . Let us recall Pieri formula of Jack functions for later use.

Theorem 2.3 [24] *For partitions μ, λ and non-negative integer n , the scalar product $\langle J_\mu(\alpha) J_n(\alpha), J_\lambda(\alpha) \rangle \neq 0$ if and only if $\lambda - \mu$ is a horizontal n -strip. And in this case*

$$\langle J_\mu(\alpha) J_n(\alpha), J_\lambda(\alpha) \rangle = n! \alpha^n h_\lambda^*(\lambda_u) h_\mu^*(\lambda_b) h_\mu^*(\mu_b) h_\mu^*(\mu_u),$$

where we use the notations from Definition 2.2.

We now introduce some terminology. For a linear operator S on symmetric functions, if $\langle J_\mu, S.J_\lambda \rangle$ is non-zero, we say that S can remove $\lambda - \mu$ from λ ; otherwise we say that S can not remove $\lambda - \mu$ from λ .

For an operator S on V (or Λ_F), the conjugate of S , S^* is defined by $\langle S.u, v \rangle = \langle u, S^*.v \rangle$, $u, v \in V$. For a symmetric function f , we also denote the multiplication operator $g \rightarrow fg$ by the same symbol f . For example, $J_n(\alpha)^*$ can remove a horizontal n -strip from λ if $\lambda_1 \geq n$. Furthermore, if λ is of rectangular shape, then $J_n(\alpha)^*$ can only remove the n boxes on the right side of last row of λ .

The generalized complete symmetric functions of Λ_F are defined by

$$q_\lambda(\alpha) = Q_{\lambda_1}(\alpha) Q_{\lambda_2}(\alpha) \cdots Q_{\lambda_l}(\alpha), \quad (2)$$

where $Q_n(\alpha) = Q_{(n)}(\alpha)$ has a generating function given by

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{\alpha n} p_n\right) = \sum_{n \in \mathbb{Z}_+} Q_n(\alpha) z^n. \quad (3)$$

Thus $Q_0(\alpha) = 1$ and $Q_n(\alpha) = 0$ for $n < 0$. We list some useful properties of q_λ and J_λ as follows.

Lemma 2.4 [24] *For any partition λ, ν , one has*

$$\begin{aligned}\langle q_\lambda(\alpha), m_\nu \rangle &= \delta_{\lambda, \nu}, \\ Q_\lambda(\alpha) &= q_\lambda(\alpha) + \sum_{\mu > \lambda} a_{\lambda\mu}(\alpha) q_\mu(\alpha), \\ J_\lambda(\alpha) &= h_\lambda^*(\lambda) P_\lambda(\alpha), \\ J_\lambda(\alpha) &= h_\lambda^*(\lambda) Q_\lambda(\alpha),\end{aligned}$$

where $a_{\lambda\mu}(\alpha) \in F = \mathbb{Q}(\alpha)$.

2.1 Fock space V and rectangular Jack functions

Let us recall the realization of rectangular Jack functions on the Fock space of the rank one vertex operator algebra [4, 5] (see also [1, 23, 26]). Let

$$V = \text{Sym}(h_{-1}, h_{-2}, \dots) \otimes \mathbb{C}\left[\frac{1}{2}\mathbb{Z}\right],$$

where $V_0 = \text{Sym}(h_{-1}, h_{-2}, \dots)$ is the complex symmetric algebra generated by the variable h_{-n} ($n \in \mathbb{N}$) and $\mathbb{C}[\frac{1}{2}\mathbb{Z}]$ is the group algebra of $\frac{1}{2}\mathbb{Z}$ with generators $\{e^{n\eta} | n \in \frac{1}{2}\mathbb{Z}\}$ with multiplication given by $e^{m\eta}e^{n\eta} = e^{(m+n)\eta}$. The underlying Heisenberg algebra is the infinite dimensional Lie algebra generated by $h_n, n \in \mathbb{Z} \setminus \{0\}$ subject to relations

$$[h_m, h_n] = m\alpha\delta_{m, -n}I. \quad (4)$$

The associated scalar product of V is given by

$$\langle h_{-\lambda} \otimes e^{m\eta}, h_{-\mu} \otimes e^{n\eta} \rangle = z_\lambda \alpha^{l(\lambda)} \delta_{\lambda, \mu} \delta_{m, n}, \quad (5)$$

where for partition $\lambda = (\lambda_1, \lambda_2, \dots)$, $h_{-\lambda}$ is defined to be $h_{-\lambda_1} h_{-\lambda_2} \dots$. We caution the reader that the symbol h_{-n} has nothing to do with the complete symmetric function. In fact, h_{-n} will be identified with the power sum symmetric function p_n .

Note that $B = \{h_{-\lambda} \otimes e^{n\eta} | \lambda \in \mathcal{P}, n \in \frac{1}{2}\mathbb{Z}\}$ is a basis of V , we can define a linear map T from V to Λ_F by

$$T(h_{-\lambda} \otimes e^{n\eta}) = p_\lambda.$$

Comparing Eqs. (1) and (5), we know that T preserves the scalar product, i.e.,

$$\langle T(u), T(v) \rangle = \langle u, v \rangle,$$

for every pair $u, v \in V$.

We then define the vertex operators

$$X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{\alpha n} h_{-n}\right) A \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-\alpha n} 2h_n\right) = \sum_{n \in \mathbb{Z}} X_n z^{-n}, \quad (6)$$

$$Y(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{\alpha n} h_{-n}\right) = \sum_{n \in \mathbb{Z}} Y_n z^{-n}, \quad (7)$$

$$Y^*(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{\alpha n} h_n\right) = \sum_{n \in \mathbb{Z}} Y_n^* z^{-n}, \quad (8)$$

where

$$\begin{aligned} h_n \cdot v \otimes e^{s\eta} &= \left(\alpha n \frac{\partial}{\partial h_{-n}} v\right) \otimes e^{s\eta} \quad \text{if } n > 0, \\ h_n \cdot v \otimes e^{s\eta} &= (h_n v) \otimes e^{s\eta} \quad \text{if } n < 0, \end{aligned}$$

and

$$A \cdot v \otimes e^{s\eta} = z^{(s+\frac{1}{2})2/\alpha} v \otimes e^{(s+1)\eta}. \quad (9)$$

Remark 2.5 Comparing (3) with (7), we know that $T(Y_{-n}) = Q_n(\alpha)$ for $n \in \mathbb{Z}$. Moreover, because $h_{-n}^* = h_n$, we know that Y_n^* is actually the conjugate of Y_n for $n \in \mathbb{Z}$.

The following result provides a realization of Jack functions of rectangular shapes. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of length l , we denote $X_{-\lambda} = X_{-\lambda_1} X_{-\lambda_2} \cdots X_{-\lambda_l}$, and $Y_{-\lambda} = Y_{-\lambda_1} Y_{-\lambda_2} \cdots Y_{-\lambda_l}$. To simplify the notation, we also write $1 \otimes e^{n\eta}$ as $e^{n\eta}$.

Theorem 2.6 [3] For partition $\lambda = ((k+1)^s, (k)^t)$ with $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{>0}$, $t \in \{0, 1\}$ and $\alpha^{-1} \in \mathbb{Z}_{>0}$, we have

$$T(X_{-\lambda} e^{-(s+t)\eta/2}) = c_{s,t}(\alpha) Q_{\lambda}(\alpha),$$

where $c_{s,t}(\alpha)$ is a function of α , and $c_{s,t}(1) = (-1)^{s(s-1+2t)/2} s!$.

We remark that this result is also true formally for arbitrary real α as the identification is boiled down to an algebraic equation in α . We also need the following proposition for later use.

Proposition 2.7 [3] For a positive integer α^{-1} and partitions λ, μ of the same weight, we have

$$\begin{aligned} \langle X_{-\lambda} \cdot e^{-l(\lambda)\eta/2}, Y_{-\mu} \cdot e^{l(\mu)\eta/2} \rangle \\ = (-1)^{s(s-1)/(2\alpha)} C_{z^{\lambda}/w^{\mu}} H_{\alpha}(z_1, \dots, z_s; w_1, \dots, w_t), \end{aligned} \quad (10)$$

where $s = l(\lambda)$, $t \geq l(\mu)$ and

$$H_{\alpha}(z_1, \dots, z_s; w_1, \dots, w_t) = \prod_{s \geq i \neq j \geq 1} (1 - z_j z_i^{-1})^{\frac{1}{\alpha}} \prod_{j=1}^s \prod_{i=1}^t (1 - z_j w_i^{-1})^{-\frac{1}{\alpha}}. \quad (11)$$

Moreover, Eq. (10) is zero if $\mu_1 > \lambda_1$.

3 A basis for symmetric functions

3.1 Further results on Jack vertex operators

A special case of Dyson's constant term conjecture, proved by Wilson [25] and Gunson [7] independently, says that for a positive integer β , the constant term of

$$\Delta_s^\beta = \prod_{1 \leq i \neq j \leq s} \left(1 - \frac{D_i}{D_j}\right)^\beta$$

is $(s\beta)!(\beta!)^{-s}$, thus we have the following refinement of Theorem 2.6.

Corollary 3.1 *In Theorem 2.6, we have*

$$c_s(\alpha) = c_{s,0}(\alpha) = (-1)^{s(s-1)/(2\alpha)} (s\alpha^{-1})! (\alpha^{-1})!^{-s}$$

in the case of $t = 0$.

Remark 3.2 If $s \geq 2$, then $c_s(\alpha)$ is not a rational function of α .

Proof Recall that we have [3]

$$\begin{aligned} T(X_{-\lambda_1} X_{-\lambda_2} \cdots X_{-\lambda_s} e^{n\eta}) \\ = (-1)^{\frac{s(s-1)/2}{\alpha}} \prod_{1 \leq i \neq j \leq s} \left(1 - \frac{D_i}{D_j}\right)^{\frac{1}{\alpha}} \cdot Q_{\lambda_1}(\alpha) \cdots Q_{\lambda_s}(\alpha), \end{aligned} \quad (12)$$

where D_i is the lowering operator defined as

$$D_i \cdot Q_{\lambda_1}(\alpha) \cdots Q_{\lambda_s}(\alpha) = Q_{\lambda_1}(\alpha) \cdots Q_{\lambda_i-1}(\alpha) \cdots Q_{\lambda_s}(\alpha).$$

Note that for rectangular λ , only constant term contributes to q_λ , the statement then follows from Dyson's constant term conjecture. $\square$

From the proof we also have the following Eq. (13) which links even power of Vandermonde determinant to rectangular Jack functions:

$$\Delta_s^{\alpha^{-1}} \cdot q_{(k^s)}(\alpha) = \frac{(s/\alpha)!}{(\alpha^{-1})!^s} Q_{(k^s)}(\alpha). \quad (13)$$

In the context of Theorem 2.6, there is also a combinatorial formula in terms of p_μ 's for the products of vertex operators (Theorem 3.2 in [3]). Combining this with Corollary 3.1, we get a new expression for rectangular Jack functions. For clarity and latter use, let us first give the following:

Definition 3.3 For a rectangular partition $R = (k^s)$, α^{-1} being a positive integer and $\mu \vdash ks$, define

$$g_{R,\mu}(\alpha) = (-1)^{\alpha^{-1}s(s-1)/2} \frac{(\alpha^{-1}!)^s}{(s\alpha^{-1})!(-2)^{l(\mu)}} \sum_{\underline{\mu} \vdash \mu} \prod_{i=1}^s \frac{(-2\alpha^{-1})^{l(v^i)}}{z_{v^i}} \binom{m(\mu^{i-1})}{m(\mu^i \setminus v^i)}, \quad (14)$$

where the sum is over sequences of partitions $\underline{\mu} = (\mu^1, \mu^2, \dots, \mu^s)$ and $\underline{v} = (v^1, v^2, \dots, v^s)$ such that $\mu^0 = (0)$, $\mu^s = \mu$, $v^i \subset' \mu^i$, $|\mu^i| = i(k + (s-i)\alpha^{-1})$, and $\mu^i \setminus v^i \subset' \mu^{i-1}$.

Corollary 3.4 For positive integers k, s, α^{-1} , we have

$$Q_{(k^s)}(\alpha) = \sum_{\mu \vdash ks} g_{(k^s),\mu}(\alpha) p_{\mu}. \quad (15)$$

Note that we can expand Jack functions in terms of p_{λ} 's by other approaches, thus Corollary 3.4 also gives various different combinatorial identities correspondingly. As a simple example, we set $(k^s) = (1^2)$, and get the following non-trivial identity by considering the coefficients of p_1^2 on both sides of identity (15).

Corollary 3.5 Let β be a positive integer, one has

$$\sum_{i=0}^2 \left[\frac{(-2\beta)^{2-i}}{(2-i)!} \sum_{(1^i) \subset' v \vdash 1+\beta} \frac{(-2\beta)^{l(v)}}{z_v} \binom{m(v)}{m(1^i)} \right] = \frac{(2\beta)!}{(\beta!)^2} \frac{4\beta^2}{\beta+1} (-1)^{\beta}.$$

Since we realized the Jack functions of rectangular shapes in $t = 0$ case of Theorem 2.6, we can further generalize this result to the case of a rectangular tableau plus one row.

Corollary 3.6 For partition $\lambda = ((k+1)^s)$ with $k \in \mathbb{Z}_{\geq 0}$, $\alpha^{-1}, s \in \mathbb{Z}_{>0}$, and integer n with $0 \leq n \leq k+1$, we have

$$T(Y_{-n}^* X_{-\lambda} \cdot e^{-s\eta/2}) = d_{s,k}(\alpha) Q_{((k+1)^{s-1}, k+1-n)}(\alpha),$$

where $d_{s,k}(\alpha) \neq 0$ is a function of α .

Proof By Remark 2.5, it is immediate that

$$T(Y_{-n}^* X_{-\lambda} \cdot e^{-s\eta/2}) = Q_{-n}(\alpha)^* \cdot (c_{s,0}(\alpha) Q_{\lambda}(\alpha)),$$

where we used Theorem 2.6 and its notation. By Pieri formula (Theorem 2.3), $\langle Q_{-n}(\alpha)^* \cdot Q_{\lambda}(\alpha), Q_{\mu}(\alpha) \rangle = \langle Q_{\lambda}(\alpha), Q_{-n}(\alpha) Q_{\mu}(\alpha) \rangle$ is non-zero if and only if $\lambda - \mu$ is a horizontal n -strip. But $\lambda = ((k+1)^s)$ is of rectangular shape, there is a unique μ satisfying this property, i.e., $\mu = ((k+1)^{s-1}, k+1-n)$. $\square$

3.2 Even powers of the Vandermonde determinant

The lowering operator $\Delta_s^\beta = \prod_{1 \leq i \neq j \leq s} (1 - \frac{D_i}{D_j})^\beta$ is essentially an even power of the Vandermonde determinant in the sense that

$$\prod_{1 \leq j < i \leq s} (D_i - D_j)^{2\beta} = (D_1 \cdots D_s)^{(s-1)\beta} (-1)^{s(s-1)\beta/2} \Delta_s^\beta.$$

Thus the following corollary can be used to study the generalized Vandermonde determinant. Recall that by Lemma 2.4, the transition matrices between $q_\lambda(\alpha)$'s and $Q_\lambda(\alpha)$'s are triangular matrices if we arrange them by the partitions with any total order compatible with the dominance order.

Corollary 3.7 Set $Q_{(k^s)}(\alpha) = \sum_{\mu \geq (k^s)} a_\mu(\alpha) q_\mu(\alpha)$. For $b = (b_1, \dots, b_s) \in \mathbb{Z}^s$ such that $b_1 + \dots + b_s = 0$ and $b_1 \geq b_2 \geq \dots \geq b_s > -k$, we have

$$C_{D_1^{b_1} \dots D_s^{b_s}} \cdot \Delta_s^{\alpha^{-1}} = \frac{m(\mu_b)!}{s!} \frac{(s/\alpha)!}{(\alpha^{-1})!^s} a_{\mu_b}(\alpha), \quad (16)$$

where α^{-1} is a positive integer and $\mu_b = (k^s) + b$ is a partition.

Proof Consider the coefficient of $q_{\mu_b}(\alpha)$ on the two sides of (13), and notice that including $D_1^{b_1} \cdots D_s^{b_s}$, there are $s!/m(\mu_b)!$ terms contributing to $q_{\mu_b}(\alpha)$ when $\Delta_s^{-\alpha}$ acts on $q_{(k^s)}(\alpha)$. $\square$

Remark 3.8 Dyson's constant term corresponds to the special case of $b = 0 \in \mathbb{Z}^s$. Note that we have an iterative formula [4, 14] for the coefficients $a_\mu(\alpha)$'s. Hence these coefficients can be evaluated effectively using vertex operator calculus. The following gives some examples on this.

Proposition 3.9 For positive integers i, s, β with $2i \leq s$, the coefficient of $D_1^{-1} \cdots D_i^{-1} D_{s-i+1} D_{s-i+2} \cdots D_s$ in Laurent polynomial $\Delta_s^\beta = \prod_{1 \leq i \neq j \leq s} (1 - \frac{D_i}{D_j})^\beta$ is

$$C_{D_1^{-1} \dots D_i^{-1} D_{s-i+1} D_{s-i+2} \dots D_s} \cdot \Delta_s^\beta = \frac{(s\beta)!}{(\beta!)^s} i! (-\beta)^i \prod_{j=1}^i (1 + (s-j)\beta)^{-1}. \quad (17)$$

And we also have

$$C_{D_1^{-2} D_s^2} \cdot \Delta_s^\beta = \frac{(s\beta)!}{(\beta!)^s} \cdot \frac{2\beta^2 - (1 + (s-1)\beta)(1 + (s-2)\beta)\beta}{(1 + (s-1)\beta)(1 + (s-2)\beta)(2 + (s-1)\beta)},$$

$$C_{D_1^{-1} D_2^{-1} D_3^2} \cdot \Delta_s^\beta = \frac{(s\beta)!}{(\beta!)^s} \frac{2(2 + (s-1)\beta)\beta^2}{(1 + (s-1)\beta)(1 + (s-2)\beta)(3 + (2s-3)\beta)}.$$

Remark 3.10 In [12], Kadell evaluated the coefficients of $z_1 z_s^{-1}$, $z_1 z_2 z_{s-1}^{-1} z_s^{-1}$, and $z_1 z_2 z_s^{-2}$ in Dyson's Laurent polynomial $\prod_{1 \leq i \neq j \leq s} (1 - z_i z_j^{-1})^{\beta_i}$. Here we consider

the special case when the exponents β_i are identical, but we can compute the coefficient of $z_1^{-\lambda_1} \cdots z_i^{-\lambda_i} z_{s-j+1}^{\mu_j} \cdots z_{s-1}^{\mu_2} z_s^{\mu_1}$ for any pair of partitions $\lambda = (\lambda_1, \dots, \lambda_i)$, $\mu = (\mu_1, \dots, \mu_j)$, with $|\lambda| = |\mu|$ and $i + j \leq s$. Moreover, the steps of computation depend on λ and μ but not β and s .

3.3 The independence of the vertex operator products

For a partition λ , denote $T(X_{-\lambda} \cdot e^{-l(\lambda)\eta/2})$ as $X_{-\lambda}(\alpha)$, and set

$$X'_{-\lambda} = X'_{-\lambda}(\alpha) = c_{l(\lambda)}(\alpha)^{-1} X_{-\lambda}(\alpha), \quad (18)$$

where $c_s(\alpha) = (-1)^{s(s-1)/(2\alpha)} (s/\alpha)! (\alpha^{-1})^{-s}$ is defined in Corollary 3.1. Note that in general $c_s(\alpha)$ is *not* a rational function of α , and that is why we factor it out from $X_{-\lambda}(\alpha)$. We will prove that these X'_λ 's form a basis for the space of symmetric functions over $F = \mathbb{Q}(\alpha)$. First, Corollary 3.1 implies that $X'_{-\lambda}$ is in Λ_F for a rectangular partition λ . In fact, this is true for general partitions.

Lemma 3.11 *For any partition λ , $X'_{-\lambda}$ is a symmetric function over $\mathbb{Q}(\alpha)$, i.e., $X'_{-\lambda}(\alpha) \in \Lambda_F$.*

Proof In (12), the coefficient of the term $q_\mu(\alpha)$ for each μ is $c_s(\alpha)$ multiplied by some rational function (which depends on μ) of α by Corollary 3.7, thus we can factor out $c_s(\alpha)$ and rewrite $T(X_{-\lambda} \cdot e^{-l(\lambda)\eta/2})$ in the desired form. $\square$

Second, we can prove that the $X_{-\lambda}(1)$'s are linearly independent with the following detailed analysis about the specialization of $\alpha = 1$ in Proposition 2.7. For convenience, denote

$$H = H_1(z_1, \dots, z_s; w_1, \dots, w_t) = \prod_{1 \leq i \neq j \leq s} (1 - z_i z_j^{-1}) \prod_{i=1}^s \prod_{j=1}^t (1 - z_i w_j^{-1})^{-1}.$$

Lemma 3.12 *We have $H = \sum_{l=1}^s H_{1,l}$, where*

$$H_{1,l} = \left(1 - \frac{z_l}{w_1}\right)^{-1} \cdot \prod_{k \neq l} \left(1 - \frac{z_l}{z_k}\right) \cdot \prod_{j=2}^t \left(1 - \frac{z_l}{w_j}\right)^{-1} \\ \cdot H_1(z_1, \dots, \widehat{z_l}, \dots, z_s; w_2, \dots, w_t),$$

where the hat $\widehat{}$ means omission.

Proof Multiplying both sides of $H = \sum_{l=1}^s H_{1,l}$ with $\prod_{i=1}^s \prod_{j=1}^t (1 - \frac{z_i}{w_j})$, we need to prove the following:

$$\prod_{1 \leq i \neq j \leq s} \left(1 - \frac{z_i}{z_j}\right) = \sum_{l=1}^s \left[\prod_{j \neq l} \left(1 - \frac{z_j}{w_1}\right) \cdot \prod_{k \neq l} \left(1 - \frac{z_l}{z_k}\right) \cdot \prod_{i \neq j}^{i,j \neq l} \left(1 - \frac{z_i}{z_j}\right) \right]. \quad (19)$$

For each l , setting $w_l = z_l$ on right-hand side of Eq. (19), we see that only one term in the summation is left, and it is equal to the left-hand side of Eq. (19). It is clear now that the two sides of (19) are actually equal. $\square$

Lemma 3.13 *Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of length s , $\mu = (\mu_1, \dots, \mu_t)$ be a partition of length $\leq t$. Then $C_{z^\lambda/w^\mu} \cdot H \neq 0$ implies $\lambda \geq \mu$. Moreover, $C_{z^\lambda/w^\mu} \cdot H$ is a positive integer.*

Proof We use induction on s . The case of $s = 1$ or 2 is trivial by Proposition 2.7. Assume that it is true in the case of $s - 1$. In the case of s , we consider the decomposition $H = \sum_{l=1}^s H_{1,l}$ in Lemma 3.12. Assume that $A = z_1^{a_1} \cdots \widehat{z_l^{a_l}} \cdots z_s^{a_s} / (w_2^{b_2} \cdots w_t^{b_t})$ is a term from

$$H_{1,l,1} = H_1(z_1, \dots, \widehat{z_l}, \dots, z_s; w_2, \dots, w_t)$$

and $B = z_1^{c_1} \cdots z_s^{c_s} / (w_1^{d_1} \cdots w_t^{d_t})$ is a term from

$$H_{1,l}/H_{1,l,1} = \left(1 - \frac{z_l}{w_1}\right)^{-1} \cdot \prod_{k \neq l} \left(1 - \frac{z_l}{z_k}\right) \cdot \prod_{j=2}^t \left(1 - \frac{z_l}{w_j}\right)^{-1}.$$

Assume that

$$AB = z_1^{a_1+c_1} \cdots z_l^{c_l} \cdots z_s^{a_s+c_s} / (w_1^{d_1} w_2^{b_2+d_2} \cdots w_t^{b_t+d_t}) = z^\lambda / w^\mu$$

for partitions λ, μ . We are going to prove $\lambda \geq \mu$.

From the expression of $H_{1,l}/H_{1,l,1}$, we know that $c_1, \dots, \widehat{c_l}, \dots, c_s$ are all non-positive. But $\lambda = (a_1 + c_1, \dots, c_l, \dots, a_s + c_s)$ is a partition, thus $a_1, \dots, \widehat{a_l}, \dots, a_s$ are all non-negative. By induction assumption and the symmetry of $H_{1,l,1}$ with respect to $z_1, \dots, \widehat{z_l}, \dots, z_s$, we have

$$\sum_{1 \leq n \leq i} a_{j_n} \geq \sum_{1 \leq n \leq i} b_{1+n} \quad (i \geq 0), \quad (20)$$

where $j_1, \dots, j_{s-1}$ is a rearrangement of $1, \dots, \widehat{l}, \dots, s$ such that $(a_{j_1}, \dots, a_{j_{s-1}})$ is a partition.

From the expression of $H_{1,l}/H_{1,l,1}$, we also know that for $i \geq 0$,

$$c_l + \sum_{1 \leq n \leq i} c_{j_n} \geq d_1 + \sum_{1 \leq n \leq i} d_{1+n}, \quad (21)$$

because $c_k \leq 0$ for $k \neq l$, $d_k \geq 0$ and $\sum_k c_k = \sum_k d_k$.

Combining inequalities (20) and (21), we have

$$c_l + \sum_{1 \leq n \leq i} (a_{j_n} + c_{j_n}) \geq d_1 + \sum_{1 \leq n \leq i} (d_{1+n} + b_{1+n}),$$

i.e., $\lambda_l + \sum_{1 \leq n \leq i} \lambda_{j_n} \geq \mu_1 + \sum_{1 \leq n \leq i} \mu_{1+n}$ for $i \geq 0$, where $(l, j_1, \dots, j_{s-1})$ is a permutation of $(1, 2, \dots, s)$. But $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition, and for any permutation $(\lambda_{m_1}, \dots, \lambda_{m_s})$ of λ , we have $\sum_{n \leq i} \lambda_n \geq \sum_{j \leq i} \lambda_{m_j}$. Now we see that $\lambda \geq \mu$.

For the second statement, let $\lambda_1 = \cdots = \lambda_a > \lambda_{a+1}$, then the term z^λ/w^λ can only come from $H_{1,l}$ with $l = 1, \dots, a$. The only way it appears in $H_{1,l}$ is when we have $z_l^{\lambda_l}/w_1^{\lambda_1}$ from $(1 - z_l w_1^{-1})^{-1}$, with coefficient 1, and then $z_1^{\lambda_1} \cdots \widehat{z_l^{\lambda_l}} \cdots z_s^{\lambda_s}/(w_2^{\lambda_2} \cdots w_t^{\lambda_t})$ from $H_{1,l,1}$, with the coefficient being a positive integer by induction, we then multiply them together to get the result. $\square$

Corollary 3.14 *For positive integer n , $\{X_{-\lambda}(1)|\lambda \vdash n\}$ is linearly independent.*

Proof By the previous lemma, when expanded into combinations of monomials, $X_{-\lambda}(1)$ contains only terms m_μ with $\mu \leq \lambda$, and it contains m_λ . $\square$

Now we can prove the main theorem of this section.

Theorem 3.15 $\{X'_{-\lambda}(\alpha)|\lambda \in \mathcal{P}\}$ is a basis for Λ_F .

Proof We only need to show that for each positive integer n , the set $\{X'_{-\lambda}(\alpha)|\lambda \vdash n\}$ is linearly independent. Suppose it is linearly dependent, there are relatively prime polynomials $f_\lambda(\alpha) \in \mathbb{Q}[\alpha]$, $\lambda \vdash n$ such that $\sum_{\lambda \vdash n} f_\lambda(\alpha) X'_{-\lambda}(\alpha) = 0$. Setting $\alpha = 1$ it follows that $\sum_{\lambda \vdash n} f_\lambda(1) X'_{-\lambda}(1) = 0$, or $\sum_{\lambda \vdash n} f_\lambda(1) c_{l(\lambda)}(1)^{-1} X_{-\lambda}(1) = 0$ by Eq. (18). Note that the coefficients $f_\lambda(1)$ are not all zero since $\alpha - 1$ is not the common divisor of the $f_\lambda(\alpha)$'s. This contradicts to the fact that the $X_{-\lambda}(1)$'s are independent. $\square$

For convenience we list the properties of this basis (generated by vertex operators) in the following.

Proposition 3.16 *The basis $X'_{-\lambda}(\alpha)$'s has the following properties:*

- (a) $X'_{-\lambda}(\alpha) = Q_\lambda(\alpha)$, for $\lambda = ((k+1)^s) \in \mathcal{P}$;
- (b) $X'_{-\lambda}(\alpha) = \frac{-s}{2(\alpha^2+s)} Q_\lambda(\alpha)$, for $\lambda = ((k+1)^s, k) \in \mathcal{P}$;
- (c) $X'_{-\lambda}(\alpha)$ is a linear combination of the $q_\mu(\alpha)$'s with $l(\mu) \leq l(\lambda)$;
- (d) $X'_{-\lambda}(\alpha)$ is orthogonal to $q_\mu(\alpha)$ if $l(\mu') > l(\lambda')$.

Proof Equality (a) comes from Corollary 3.1. Equality (b) comes from $t = 1$ case of Theorem 2.6 with the coefficient fixed by using $i = 1$ case of Eq. (17) in Proposition 3.9. Property (c) is from Eq. (12) in the proof of Corollary 2.6. Property (d) comes from Proposition 2.7. $\square$

Remark 3.17 The vertex operators X_λ and X'_λ differ by the Dyson's constant term in $\prod_{1 \leq i \neq j \leq l(\lambda)} (1 - x_i x_j^{-1})^{\alpha^{-1}}$ up to a sign.

To end this section, we give a new vertex operator realization of Schur functions S_λ 's, when λ are of rectangular shapes with part of the last column removed (cf. [10]). Note that $l(\mu) \leq l(\lambda)$ is a necessary condition for q_μ to appear in the expansion of $X_{-\lambda}(\alpha)$. As a byproduct of Lemma 3.13, we have

Corollary 3.18 For partition $\lambda = ((k+1)^s, k^t)$ of length $s+t$, $X_{-\lambda}(1) = c_\lambda S_\lambda$ for some non-zero constant c_λ .

4 Stanley's positivity conjecture and filtration of rectangles

The following conjecture about Littlewood–Richardson coefficients of the Jack function was made by Stanley [24]. Denote by $\mathbb{Z}_{\geq 0}[\alpha]$ the set of polynomials in α with non-negative integer coefficients.

Conjecture 4.1 The LR-coefficient $C_{\mu\nu}^\lambda(\alpha) := \langle J_\mu J_\nu, J_\lambda \rangle$ is in $\mathbb{Z}_{\geq 0}[\alpha]$ for any triple of partitions (μ, ν, λ) .

Remark 4.2 Stanley's Pieri formula for Jack function implies his conjecture in the case of $l(\nu) = 1$. On the other hand, one knows that $C_{\mu\nu}^\lambda(\alpha) \in \mathbb{Z}[\alpha]$ using a main result of [13]. If the classical Littlewood–Richardson coefficient $C_{\mu\nu}^\lambda = 1$, Stanley further conjectured a form for $C_{\mu\nu}^\lambda(\alpha)$ which he verified under a technical condition, and this contains part of the following Theorem 4.7. Furthermore, Stanley showed that the conjecture is true for $\nu = (2, 1)$ under the condition $C_{\mu\nu}^\lambda = 1$. We will show that Conjecture 4.1 is true for $\nu = (2, 1)$ besides other cases.

4.1 Rectangular and marked rectangular cases

In the following, we prove two special cases of Conjecture 4.1.

Definition 4.3 For a rectangular partition $\lambda = (r^s)$, and a partition $\nu = (\nu_1, \nu_2, \dots, \nu_s) \subset \lambda$, define the complement partition of ν in λ as $\bar{\nu} = \lambda - \nu = (r - \nu_s, r - \nu_{s-1}, \dots, r - \nu_1)$.

Example 4.4 For $\lambda = (6^5)$, $\nu = (6, 5, 3, 3, 0)$, we have $\lambda - \nu = (6, 3, 3, 1, 0)$. As shown in Fig. 2(a), the boxes below the bold line form an upside-down ν inside λ . The partition above the bold line is $\lambda - \nu$.

Note that $\nu \subset (r^s)$ is equivalent to $l(\nu) \leq s$ and $\nu_1 \leq r$, and in this case, the Young diagram of $(r^s) - \nu$ is the set difference of $Y(\lambda)$ with the upside-down ν of λ . We have the following simple properties for the complement of a partition.

Fig. 2 Hook lengths and new factors in Stanley conjecture

Lemma 4.5 For a rectangular partition $\lambda = (r^s)$, and partitions $\nu, \omega \subset \lambda$, we have

$$\begin{aligned}\overline{\overline{\nu}} &= \nu, \\ \omega \geq \nu &\Rightarrow \overline{\omega} \geq \overline{\nu}, \\ \omega \supset \nu &\Rightarrow \overline{\omega} \subset \overline{\nu}.\end{aligned}$$

Note that in the second inequality, the dominance order is only defined for two partitions of the same weight.

Lemma 4.6 For $\nu \subset \lambda = (r^s)$, $\langle J_\omega q_\nu, J_\lambda \rangle \neq 0$ implies $\omega \geq \overline{\nu}$.

Proof Let $q_\nu^* J_\lambda = q_{\nu_s}^* \cdots q_{\nu_1}^* J_\lambda$, the action of $q_{\nu_s}^* \cdots q_{\nu_1}^*$ on J_λ has the effect of removing $\nu_1 + \cdots + \nu_i$ squares in the last i row(s) of λ . Here we mean that setting $q_{\nu_s}^* \cdots q_{\nu_1}^* q_\lambda = \sum_\omega a_\omega J_\omega$, if $a_\omega \neq 0$, then ω is coming from λ by removing $\nu_1 + \cdots + \nu_i$ squares in the last i row(s) of λ . Thus in the first $s - i$ row(s), there are at most $\nu_{i+1} + \cdots + \nu_s$ squares being removed. Subsequently ω satisfies $\omega_1 + \cdots + \omega_{s-i} \geq r(s-i) - (\nu_{i+1} + \cdots + \nu_s) = (r - \nu_s) + \cdots + (r - \nu_{i+1})$, which means that $\omega \geq \overline{\nu}$. $\square$

The following theorem asserts the truth of Conjecture 4.1 in the case of a rectangular λ .

Theorem 4.7 For a rectangular partition $\lambda = (r^s)$, $\langle J_\mu J_\nu, J_\lambda \rangle \neq 0$ if and only if $\nu \subset \lambda$ and $\mu = \overline{\nu}$. Furthermore, in this case we have $\langle J_\mu J_\nu, J_\lambda \rangle = h_1^\mu(\mu) h_1^*(\nu) \times h_1^\lambda(\mu) h_2^\lambda(\nu)$, where

$$\begin{aligned}h_1^\lambda(\mu) &= \prod_{(i,j) \in \mu} [(\mu'_j - i) + \alpha(r - j + 1)], \\ h_2^\lambda(\nu) &= \prod_{(i,j) \in \nu} [(s - i + 1) + \alpha(j - 1)].\end{aligned}$$

Remark 4.8 Note that except for the usual hook-length terms $h_1^\mu(\mu)$ and $h_1^*(\nu)$, there are two new terms $h_1^\lambda(\mu)$, $h_2^\lambda(\nu)$ in $C_{\mu\nu}^\lambda(\alpha)$. For instance, let $\lambda = (6^5)$, $\nu = (6, 5, 3, 3)$ as in Example 4.4. The partition $\mu = \lambda -' \nu$ lies above the bold line in Fig. 2(a). Figure 2(b) illustrates the hook-length terms, the box a in μ contributes $2 + \alpha$, and box b in $(6^5) - \mu$ contributes $3 + 4\alpha$. Figure 2(c) illustrates the new terms, where the box a contributes $1 + 5\alpha$ and the box b contributes $5 + 2\alpha$. Therefore, in (b) or (c), we associate a linear polynomial of α to each box, the number of the boxes that the solid line passes horizontally (resp., vertically) is the coefficient of α (resp., the constant).

Proof Set $J_\nu = \sum_{\omega \geq \nu} c_\omega q_\omega$, $\langle J_\mu J_\nu, J_\lambda \rangle \neq 0$ implies at least one of $\langle J_\mu q_\omega, J_\lambda \rangle \neq 0$ for some $\omega \geq \nu$, thus $\mu \geq \overline{\omega} \geq \overline{\nu}$ by Lemmas 4.6 and 4.5. Similarly, we have $\nu \geq \overline{\mu}$, which leads to $\overline{\nu} \geq \mu$ by Lemma 4.5. This forces $\mu = \overline{\nu}$.

Now we can set $J_\nu^* J_\lambda = C_{\lambda, \nu} J_{\overline{\nu}}$, and then $\langle J_\mu J_\nu, J_\lambda \rangle = \langle J_{\overline{\nu}}, J_\nu^* J_\lambda \rangle = C_{\lambda, \nu} \langle J_{\overline{\nu}}, J_{\overline{\nu}} \rangle = C_{\lambda, \nu} h_1^*(\overline{\nu}) h_1^{\overline{\nu}}(\overline{\nu})$. We need to determine the coefficient $C_{\lambda, \nu}$. Write $J_\nu^* =$

$\sum_{\omega \geq v} c_{\omega} q_{\omega}^*$, then $c_v = h_v^*(v)$ by Lemma 2.4. Thus the only term that matters is $c_v q_v^*$ by Lemma 4.6. The action of $q_{v_1}^*$ on J_{λ} removes v_1 squares in the last row of λ , this results in, say $C_{\lambda, v_1} J_{\overline{(v_1)}}$. Observe that C_{λ, v_1} is explicitly given by Stanley's Pieri formula for Jack function. Then we can apply the action $q_{v_2}^*$ on $J_{\overline{(v_1)}}$. Note there are more than one ways to remove v_2 squares horizontally from (v_1) , but the only way to get to $J_{\overline{v}}$ is by removing v_2 squares in the second last row. Then we arrive at $C_{\lambda, (v_1, v_2)} J_{\overline{(v_1, v_2)}}$, with $C_{\lambda, (v_1, v_2)}$ given again by Stanley's formula. Continuing in this way, we find that $C_{\lambda, v} = h_v^*(v) \cdot (C_{\lambda, v_1} C_{\lambda, (v_1, v_2)} \cdots) = h_v^*(v) \cdot (h_1^{\lambda}(\mu) h_2^{\lambda}(v) / h_v^*(\overline{v}))$. $\square$

Remark 4.9

1. It is interesting that $\langle J_{\mu} J_v, J_{\lambda} \rangle$ is symmetric in μ and v , but the symmetry is not at all clear from this form of the formula.
2. By the same method, one can easily generalize this theorem to Macdonald case using the Pieri formula.

Corollary 4.10 *For a marked rectangular partition $\lambda = (r^{s-1}, r - n)$, the matrix coefficient $\langle J_{\mu} J_v, J_{\lambda} \rangle \neq 0$ if and only if $((r^s) -' v) - \mu$ is a horizontal n -strip. In this case $\langle J_{\mu} J_v, J_{\lambda} \rangle$ is a product of non-zero linear polynomials in $\mathbb{Z}_{\geq 0}[\alpha]$, Explicitly, denoting $\overline{v} = (r^s) -' v$, we have*

$$\begin{aligned} \langle J_{\mu} J_v, J_{\lambda} \rangle &= \frac{(r - n)!}{r(r - 1) \cdots (n + 1) \prod_{i=0}^{n-1} [s + i\alpha]} \\ &\quad \cdot \frac{h_*(\overline{v}_u) h^*(\overline{v}_b)}{h_*(\mu_u) h^*(\mu_b)} n! \alpha^n \langle J_{\overline{v}} J_v, J_{r^s} \rangle, \end{aligned} \quad (22)$$

where μ_b and $\overline{v}_b$ (resp., μ_u and $\overline{v}_u$) are the sets of based (resp., un-based) boxes of μ and $\overline{v}$, respectively, for the pair $\mu \subset \overline{v}$.

Proof First, we have

$$\begin{aligned} J_n^* J_{r^s} &= \frac{r\alpha \cdots (n + 1)\alpha}{(r - n)\alpha \cdots 1\alpha} [s + (n - 1)\alpha] \cdots [s + 0\alpha] J_{\lambda}, \\ J_{\lambda} &= \frac{(r - n)!}{r \cdots (n + 1) \prod_{i=0}^{n-1} (s + i\alpha)} J_n^* J_{r^s}, \\ \langle J_{\mu} J_v, J_{\lambda} \rangle &= \frac{(r - n)!}{r(r - 1) \cdots (n + 1) \prod_{i=0}^{n-1} [s + i\alpha]} \langle J_n J_{\mu}, J_v^* J_{r^s} \rangle. \end{aligned}$$

By Theorem 4.7, $\langle J_{\mu} J_v, J_{\lambda} \rangle$ does not vanish if and only if $((r^s) -' v) - \mu$ is a horizontal n -strip. Then the coefficient of $J_{\overline{v}}$ in $J_n J_{\mu}$ is $\frac{h_*(\overline{v}_u) h^*(\overline{v}_b)}{h_*(\mu_u) h^*(\mu_b)} n! \alpha^n$ by Pieri formula of Jack functions [24]. Combining this process with Theorem 4.7 gives the corollary. $\square$

Remark 4.11 This proposition implies that Conjecture 4.1 is true if λ is a rectangle with part of the last row removed. We can also use this method to treat the case of

rectangular partitions with part of the last column removed, and prove that Conjecture 4.1 is true if $\lambda = ((k+1)^s k^t)$ with $k, s, t \in \mathbb{Z}_{\geq 0}$.

We can generalize Theorem 4.7 to the following situation, which is also useful when we consider the $\nu = (2, 1)$ case of Stanley's conjecture.

Proposition 4.12 *Assume that $\langle J_\nu^* J_\lambda, J_\mu \rangle \neq 0$, and $\lambda - \mu$ is an upside-down ω of the t th corner of λ with $|\omega| = |\nu|$, then $\omega = \nu$ and $\langle J_\lambda, J_\mu J_\nu \rangle$ is the product of linear polynomials of α with non-negative integer coefficients.*

Proof As in the proof of Theorem 4.7, we have $\omega \geq \nu$. Note that $\langle J_\lambda, J_\mu J_\nu \rangle = \langle J_\nu^* J_\lambda, J_\mu \rangle \neq 0$ is equivalent to $\langle J_{\lambda'}^*, J_{\mu'} J_{\nu'} \rangle \neq 0$ by Corollary 3.4 in [24], and that $\lambda' - \mu'$ is an upside-down ω' of a corner of λ' with $|\omega'| = |\nu'|$, thus we also have $\omega' \geq \nu'$. Subsequently $\omega = \nu$.

The proof for the other statement is the same as that of Theorem 4.7. One also writes $J_\nu^* = \sum_{\omega \geq \nu} c_\omega q_\omega^*$, and only $q_\nu^* = q_{\nu_1}^* q_{\nu_2}^* \cdots$ contributes to $\langle J_\nu^* J_\lambda, J_\mu \rangle \neq 0$. When $q_{\nu_1}^*$ acts on J_λ , there is only one way to move ν_1 boxes from λ that counts, i.e., to move ν_1 right-most boxes in the last row of the t th corner of λ to get J_{λ_1} with coefficients of the form $f_1(\alpha)/g_1(\alpha)$, where f_1, g_1 are products of linear polynomials of α with non-negative integer coefficients. Then consider $q_{\nu_2}^* J_{\lambda_1}$. There is also one way to move ν_2 right-most boxes in the second last row of the t th corner of α . At last, $\langle J_\nu^* J_\lambda, J_\mu \rangle$ is of the form $A(\alpha)/B(\alpha)$, where both A and B are products of linear polynomials of α with non-negative integer coefficients. But it is also an integral polynomial of α by Remark 4.2, thus all the linear polynomials in the denominator $B(\lambda)$ should be canceled, and the statement is then proved. $\square$

4.2 $\nu = (2, 1)$ case

The following proposition confirms Stanley's conjecture for the case of $\nu = (n, 1)$ subject to a simpler technical condition, but without the condition $C_{\mu\nu}^\lambda = 1$.

Proposition 4.13 *For $\nu = (n, 1)$, $\langle J_\mu J_\nu, J_\lambda \rangle$ is a product of linear polynomials of α with non-negative integer coefficients if the boxes of $\lambda - \mu$ lie on two rows of λ . In particular, for $\nu = (2, 1)$, $\langle J_\mu J_\nu, J_\lambda \rangle \in \mathbb{Z}_{\geq 0}[\alpha]$.*

Proof We first consider the $(n, 1)$ -case. Assume that among the $n+1$ boxes of $\lambda - \mu$, there are $n-a$ boxes in the same row of one corner, and $a+1$ boxes in the same row of a lower corner. We mark the two right-most boxes on the two rows by A and B . We also denote by t (resp., s) the number of rows (resp., columns) between the rows (resp., columns) with removed boxes. See Fig. 3(a).

We can assume that $n-a, a+1 \geq 1$. In fact, if one of them is zero, then $\lambda - \mu$ lies on one row of λ , and $\lambda - \mu$ should be an upside-down $(n+1)$ of a corner of λ . This is a special case of the Proposition 4.12. Note that we have $J_{(n,1)} = (n-1)! \alpha^n [(1+n\alpha)q_n q_1 - (n+1)q_{n+1}]$, and $\langle J_\mu J_\nu, J_\lambda \rangle = \langle J_\mu, J_\nu^* J_\lambda \rangle$. Using Pieri formula, we compute the coefficient c of J_μ in $q_{n+1}^* J_\lambda$. For $q_n^* q_1^* J_\lambda$, we compute the coefficient a_1 of $J_{\lambda-A}$ in $q_1^* J_\lambda$ and the coefficient a_2 of J_μ in $q_n^* J_{\lambda-A}$. We also

Fig. 3 Corners of Young diagram λ

compute the coefficient b_1 of $J_{\lambda-B}$ in $q_1^* J_\lambda$ and the coefficient b_2 of J_μ in $q_n^* J_{\lambda-B}$. Then $\langle J_\mu J_{(n,1)}, J_\lambda \rangle = (n-1)! \alpha^n [(1+n\alpha)(a_1 a_2 + b_1 b_2 - (n+1)c)] \langle J_\mu, J_\mu \rangle$ is found to be a product of linear polynomials of α with non-negative integer coefficients.

Now we consider the case of $(2, 1)$. The idea is same as in the $(n, 1)$ -case. Among the three boxes of $\lambda - \mu$, if any two of them are in the same column of λ , then by Pieri formula only $q_2^* q_1^*$ contributes to J_μ in $J_{(2,1)}^* J_\lambda = \alpha^2 [(1+2\alpha)q_2^* q_1^* - 3q_3^*] J_\lambda$, and it can easily be seen that the proposition is true. If any two of them are in a same column of λ , it is also true by the $(n, 1)$ -case. Thus we only need to prove the case when the three boxes of $\lambda - \mu$ are in three different corners of λ , see Fig. 3(b). We mark the three boxes of $\lambda - \mu$ by A_1, A_2, A_3 , d (resp., c) is the number of rows (resp., columns) between boxes A_1 and A_2 . Similarly, we define a, b as shown in Fig. 3(b). A row (resp., column) of μ is called a critical row (resp., column) if it lies on a row (resp., column) of λ which intersects $\lambda - \mu$. Now

$$\langle J_\mu J_{(2,1)}, J_\lambda \rangle = \alpha^2 h_\lambda^* (R - R \cap C) h_\lambda^* (C - R \cap C) h_\mu^* (\mu - R) h_\mu^* (\mu - C) f(\alpha),$$

where R (resp., C) is the union of the critical rows (resp., columns) of μ , and $f(\alpha) = \sum_{i=0}^7 c_i \alpha^i$, with $c_i \in \mathbb{Z}_{\geq 0}[a, b, c, d]$, thus the positivity follows. $\square$

4.3 Filtration of rectangular partitions

Using Theorem 4.7, we can derive a Jack function of any shape starting from Jack functions of rectangular shapes, which were already realized in [3] using the vertex operator technique.

Definition 4.14 Define maps $\mathfrak{R}, \mathfrak{C}$ on $\mathcal{P}$ by $\mathfrak{R}(\lambda) = (l(\lambda')^{l(\lambda)})$, $\mathfrak{C}(\lambda) = \mathfrak{R}(\lambda) -' \lambda$. Define $R_i(\lambda) = \mathfrak{R}(\mathfrak{C}^{i-1}(\lambda))$ ($i \geq 1$, $\mathfrak{C}^0(\lambda) = \lambda$). Set $s = n_c(\lambda)$ to be the number of corners of λ , $R_\lambda = (R_1, \dots, R_s)$ is called the rectangular filtration of λ .

Example 4.15 For $\lambda = (6, 3, 3, 2, 1)$, note that $l(\lambda') = \lambda_1$, so $R_1(\lambda) = \mathfrak{R}(\lambda) = (6^5)$. $\mathfrak{C}^1(\lambda) = \mathfrak{C}(\lambda) = (6^5) -' (6, 3, 3, 2, 1) = (5, 4, 3, 3)$, $R_2(\lambda) = \mathfrak{R}(\mathfrak{C}^1(\lambda)) = (5^4)$. Similarly, it can be found that $\mathfrak{C}^2(\lambda) = \mathfrak{C}(\mathfrak{C}^1(\lambda)) = (2, 2, 1)$, $R_3(\lambda) = (2^3)$, $\mathfrak{C}^3(\lambda) = (1)$, and $R_4(\lambda) = (1^1)$. Thus the filtration of λ is $R_\lambda = ((6^5), (5^4), (2^3), (1^1))$. Figure 4

Fig. 4 Rectangular filtration

illustrates the pictorial construction of the rectangular filtration. λ is the partition above the bold line in picture 1, the whole rectangle in picture 1 (with diagonal a_0a_1) is R_1 . $R_1 - \lambda$ is the subset under the bold line in picture 1, it is an upside-down $\mathfrak{C}^1(\lambda) = (5, 4, 3, 3)$ of (the corner of) R_1 . The whole rectangle 2 (with diagonal a_1a_2) is R_2 , it is also the sub-rectangle with diagonal a_1a_2 in picture 1. Similarly, the whole rectangle in picture i with diagonal $a_{i-1}a_i$ is R_i . In fact, all rectangles R_i are marked in picture 1 with diagonal $a_{i-1}a_i$.

Lemma 4.16 *For a non-zero partition λ , $\mathfrak{R}(\lambda)$ is the minimum rectangular partition containing λ with respect to the partial order $\subset$, and $n_c(\mathfrak{C}(\lambda)) = n_c(\lambda) - 1$.*

Proof If a rectangular partition $\nu = (a^b)$ contains λ , then we should have $a \geq \lambda_1$ and $b \geq l(\lambda)$, which means $\mathfrak{R}(\lambda) \subset \nu$. This proves the minimality of $\mathfrak{R}(\lambda)$. To prove the statement about the number of corners, we write $\lambda = (a_1^{n_1} \cdots a_s^{n_s})$, with $a_1 > a_2 > \cdots > a_s > 0$ and $m_{a_i}(\lambda) = n_i > 0$. By definition of the compliment partition, we have

$$\begin{aligned} \mathfrak{C}(\lambda) &= (a_1^{n_1 + \cdots + n_s}) -' (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s}) \\ &= ((a_1 - a_s)^{n_s} (a_1 - a_{s-1})^{n_{s-1}} \cdots (a_1 - a_2)^{n_2}). \end{aligned} \quad (23)$$

Thus $n_c(\mathfrak{C}(\lambda^1)) = s - 1$, and the lemma follows because $n_c(\lambda) = s$. $\square$

We need the following lemma in the proof of the realization of Jack functions of general shapes.

Lemma 4.17 *If $R_\lambda = (R_1, \dots, R_s)$ is the rectangular filtration of λ , $s \geq 2$, then $(R_2, R_3, \dots, R_s)$ is a rectangular filtration of $\nu = \mathfrak{C}(\lambda)$.*

Proof By Lemma 4.16, $n_c(\nu) = s - 1$. Thus we can assume that $R_\nu = (R_1^\nu, \dots, R_{s-1}^\nu)$ is the rectangular filtration of ν . By definition, we have $R_i^\nu = \mathfrak{R}(\mathfrak{C}^{i-1}(\nu)) = \mathfrak{R}(\mathfrak{C}^{i-1}(\mathfrak{C}(\lambda))) = \mathfrak{R}(\mathfrak{C}^i(\lambda)) = R_{i+1}$. $\square$

We would like to give a direct description of the rectangular filtration of a partition. For $\vec{n} = (n_1, \dots, n_s) \in \mathbb{Z}^s$, define $\vec{n}^* = (n_1^*, \dots, n_s^*) \in \mathbb{Z}^s$ by setting $n_{2i+1}^* = \sum_{i < j < s-i+1} n_j$, $n_{2i}^* = \sum_{i < j \leq s-i+1} n_j$. For a partition $\lambda = (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s})$,

where $m_{a_i}(\lambda) = n_i > 0$, we call the vector $\vec{n} = (n_1, n_2, \dots, n_s)$ the multiplicity type of λ .

From the Young diagram of λ , we can easily prove the following lemma.

Lemma 4.18 For a partition $\lambda = (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s})$ with $m_{a_i}(\lambda) = n_i > 0$, the multiplicity type of λ' is $\vec{p} = (p_1, p_2, \dots, p_s)$ where $p_i = a_{s+1-i} - a_{s+2-i}$ with $a_{s+1} = 0$.

Lemma 4.19 For $\vec{n} = (n_1, \dots, n_s) \in \mathbb{Z}^s$ ($s \geq 2$), set $\vec{m} = (n_s, n_{s-1}, \dots, n_2)$. Then we have $\vec{m}^* = (n_s^*, n_{s-1}^*, \dots, n_2^*)$.

Proof Set $\vec{m} = (m_1, m_2, \dots, m_{s-1})$. Then $m_j = n_{s+1-j}$ and

$$m_{2i}^* = \sum_{i < j \leq (s-1)-i+1} m_j = \sum_{i < j \leq s-i} n_{s+1-j} = \sum_{i+1 \leq k \leq s+1-i} n_k = n_{2i+1}^*.$$

Similarly, it can be computed that $m_{2i-1}^* = n_{2i}^*$. $\square$

Proposition 4.20 For a partition λ , assume that the multiplicity types of λ and λ' are $\vec{n} = (n_1, \dots, n_s)$, $\vec{p} = (p_1, p_2, \dots, p_s)$, respectively. Then the filtration $R_\lambda = (R_1, R_2, \dots, R_s)$ is given by $R_i = (p_i^{*n_i^*})$.

Proof Set $\lambda = (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s})$, with $m_{a_i}(\lambda) = n_i > 0$. We use induction on the number of the corners $n_c(\lambda) = s$. The claim is obviously true if $n_c(\lambda) = 1$. If it is true when $n_c(\lambda) = s-1$, let us prove the case of $n_c(\lambda) = s$. Note that $p_1^* = p_1 + \cdots + p_s = \sum_{1 \leq i \leq s} (a_{s+1-i} - a_{s+2-i}) = a_1$ ($a_{s+1} = 0$), and $n_1^* = n_1 + \cdots + n_s = l(\lambda)$, so we have $R_1 = (p_1^{*n_1^*})$.

We set $\nu = \mathfrak{C}(\lambda)$. Then $\nu = ((a_1 - a_s)^{n_s} (a_1 - a_{s-1})^{n_{s-1}} \cdots (a_1 - a_2)^{n_2})$. The multiplicity types of ν and ν' are $\vec{m} = (n_s, n_{s-1}, \dots, n_2)$ and $\vec{q} = (p_s, p_{s-1}, \dots, p_2)$, respectively. By the inductive assumption and Lemma 4.19, we have $R_\mu = ((q_1^{*m_1^*}), \dots, (q_{s-1}^{*m_{s-1}^*})) = ((p_2^{*n_2^*}), \dots, (p_s^{*n_s^*}))$. Applying Lemma 4.17, we have $R_i = (p_i^{*n_i^*})$ for $i \geq 2$. $\square$

Recall that Jack functions of rectangular shapes were realized by vertex operators in [3], thus the following theorem gives a vertex operator realization of Jack functions of general shapes.

Theorem 4.21 For a partition λ , let $R_\lambda = (R_1, R_2, \dots, R_s)$ be the rectangular filtration of λ and denote the rectangular vertex operator representation of R_i by $f_i = Q_{R_i}$. Then there is a $c'_\lambda(\alpha) \in \mathbb{Q}(\alpha) \setminus \{0\}$ such that

$$Q_\lambda = c'_\lambda(\alpha) (\cdots ((f_s^* \cdot f_{s-1})^* \cdot f_{s-2})^* \cdots \cdot f_2)^* \cdot f_1. \quad (24)$$

Proof For polynomials F_1, F_2 in Λ_F , we write $F_1 \doteq F_2$ if $F_1 = cF_2$ for some $c \in F - \{0\}$. We use induction on $s = n_c(\lambda)$. If $s = 1$, it is trivial. Assuming (24) holds for the case of $s-1$ ($s \geq 2$),

$$Q_{\mathfrak{C}(\lambda)} \doteq (\cdots ((f_s^* \cdot f_{s-1})^* \cdot f_{s-2})^* \cdots \cdot f_3)^* \cdot f_2.$$

This is due to the fact that $(R_2, \dots, R_s)$ is the rectangular filtration of $\mathfrak{C}(\lambda)$ by Lemma 4.17. Note that by Theorem 4.7, we have $Q_\nu^* \cdot Q_R \doteq Q_{R-\nu}$ if R is a rectangular partition and $\nu \subset R$. Thus we have

$$\begin{aligned} Q_\lambda &\doteq Q_{\mathfrak{R}(\lambda)-\mathfrak{C}(\lambda)} \doteq Q_{\mathfrak{C}(\lambda)}^* \cdot Q_{\mathfrak{R}(\lambda)} \\ &\doteq (\cdots ((f_s^* \cdot f_{s-1})^* \cdot f_{s-2})^* \cdots \cdot f_2)^* \cdot f_1. \end{aligned} \quad (25)$$

□

Example 4.22 Let us look at the λ in Example 4.15. By this theorem, $((Q_1^* Q_{(2^3)})^* Q_{(5^4)})^* Q_{(6^5)}$ is equal to Q_λ up to a non-zero scalar multiple. We can use our vertex operator formula to find the $Q_{(k^s)}$.

Remark 4.23

1. The coefficient $c'_\lambda(\alpha)$ can be explicitly evaluated just as we have done in the proof of Theorem 4.7.
2. The argument of Theorem 4.21 also works for Macdonald symmetric functions, thus we would have a vertex operator realization for Macdonald functions of general shapes if we had that for rectangular shapes.

Note that for the $f_i = Q_{R_i}$ in Theorem 4.21 we have expression $Q_{R_i}(\alpha) = \sum_{\mu^i \vdash |R_i|} g_{R_i, \mu^i}(\alpha) p_{\mu^i}$ by Corollary 3.4. Subsequently, we have a combinatorial formula for Jack functions of general shapes.

Corollary 4.24 *For a partition λ , let $R_\lambda = (R_1, R_2, \dots, R_s)$ be the rectangular filtration of λ . Then we have*

$$Q_\lambda = c'_\lambda(\alpha) \sum_{\underline{\mu}} \prod_{i=1}^s g_{R_i, \mu^i}(\alpha) ((\cdots (p_{\mu^s}^* \cdot p_{\mu^{s-1}})^* \cdots p_{\mu^3})^* \cdot p_{\mu^2})^* \cdot p_{\mu^1}, \quad (26)$$

where $c'_\lambda(\alpha)$ is a non-vanishing rational function of α , $g_{R_i, \xi^i}(\alpha)$ is defined by Definition 3.3, and the sum runs over sequences of partitions $\underline{\mu} = (\mu^1, \dots, \mu^s)$ with $|\mu^i| = |R_i|$.

Lassalle [16, 17] had conjectured about the form of the coefficients of p_μ in Jack functions. It would be very interesting to find the relationship between our results and his conjecture.

Note that $p_\mu^* \cdot p_\nu \neq 0$ only if $\mu \subset' \nu$, thus many summands on the right hand side of (26) vanish. The non-vanishing terms can be iteratively computed by the following identity

$$p_\mu^* \cdot p_\nu = z_\mu \alpha^{l(\mu)} \binom{m(\nu)}{m(\mu)} p_{\nu \setminus \mu}, \quad (27)$$

which is checked by the computation: $(p_i^{m_i})^* \cdot p_i^{n_i} = (i\alpha)^{m_i} m_i! \binom{n_i}{m_i} p_i^{n_i - m_i}$.

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