Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix

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Abstract It is well known that a nilpotent $n \times n$ matrix B is determined up to conjugacy by a partition of n formed by the sizes of the Jordan blocks of B . We call this partition the Jordan type of B . We obtain partial results on the following problem: for any partition P of n describe the type $Q(P)$ of a generic nilpotent matrix commuting with a given nilpotent matrix of type P . A conjectural description for $Q(P)$ was given by P. Oblak and restated by L. Khatami. In this paper we prove “half” of this conjecture by showing that this conjectural type is less than or equal to $Q(P)$ in the dominance order on partitions.

Keywords Jordan type · Nilpotent matrix · Commutator · Partition

1 Introduction

It is well known that the nilpotent commutator $\mathcal{N}_B$ of a Jordan block matrix B whose eigenvalues are in a base field k , is a direct sum of the nilpotent commutators corresponding to the generalized eigenspaces of B [10, p. 338]. The particular eigenvalue in each plays no further role, so henceforth we assume that B is nilpotent.

We fix an n -dimensional vector space V over an infinite field k , and an $n \times n$ nilpotent Jordan block matrix $B = J_P \in \text{Mat}_n(k)$ having t Jordan blocks of sizes p_i given by the partition $P \vdash n$, $P = (p_1, \dots, p_t)$, $p_1 \geq p_2 \geq \dots \geq p_t$. Consider the centralizer $\mathcal{C}_B \subset \text{Mat}_n(k) \cong \text{End}_k(V)$, which is the set of $n \times n$ matrices with entries in k that commute with B , and the subvariety $\mathcal{N}_B$ comprised of those matrices in $\mathcal{C}_B$ that are

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nilpotent. Each element A of $\mathcal{N}_B$ is in the conjugacy class of a Jordan block matrix J_{P_A} of partition $P_A \vdash n$. We term the partition P_A the *Jordan type* of A . It is well known that $\mathcal{N}_B$ is an irreducible algebraic variety [2, Lemma 2.3], [3, Lemma 1.5]. Thus, there is a unique Jordan type $Q(P) = P_A$ associated to a generic matrix $A \in \mathcal{N}_B$ —for A in a suitable Zariski dense open subset of $\mathcal{N}_B$. And $Q(P)$ is greater in the dominance order (1.4) than any other Jordan type occurring for elements of $\mathcal{N}_B$. Of course, a generic $A \in \mathcal{N}_B$ is usually not itself a Jordan block matrix.

Question 1.1 What is $Q(P)$? Determine $Q(P)$ algorithmically from P .

When P is *almost rectangular*—the maximum part of P minus the smallest part is at most one—then it is easy to see that $Q(P) = (n)$, a single block. R. Basili showed that $Q(P)$ has r_P parts, where r_P is the minimum number of almost rectangular subpartitions $P_1, \dots, P_r$ needed for a decomposition $P = P_1 \cup \dots \cup P_r$, where by $P_1 \cup P_2$ we mean the partition whose parts are the concatenation of those of P_1 and P_2 ([2, Proposition 2.4], see also [5, Theorem 2.17]).

Attached to the partition P is a maximal subalgebra $\mathcal{U}_B \subset \mathcal{N}_B$ (Sect. 2.1). A key combinatorial object attached to the partition P and defined from $\mathcal{U}_B$ is the poset $\mathcal{D}_P$, which has n elements corresponding to a certain basis $B = \{b_1, \dots, b_n\}$ of V . We regard these as being arranged in t rows: each row corresponds to a part p_i of P : the i th row is comprised of the basis of a B -invariant subspace of V isomorphic to $k[B]/B^{p_i}$, $i \in \{1, \dots, t\}$ (see Definition 2.3 below). This poset was defined by P. Oblak as the digraph associated to the maximal subalgebra $\mathcal{U}_B$ of $\mathcal{N}_B$: for $b, b' \in B$ set $b \leq b'$ in $\mathcal{D}_P$ if $A_{b,b'} \neq 0$ for A generic in $\mathcal{U}_B$, when A is expressed in the basis B [5, 18].¹ Furthermore, any matrix $A \in \mathcal{N}_B$ is conjugate by a matrix in the centralizer $\mathcal{C}_B$ to one in $\mathcal{U}_B$, so we may restrict to $\mathcal{U}_B$ in determining $Q(P)$ [2, 5, 29].

P. Oblak [18] for $\text{char } k = 0$ and subsequently the first author and R. Basili for k algebraically closed (unpublished) determined the *index*—largest part— $i(Q(P))$ of $Q(P)$ in terms of the poset $\mathcal{D}_P$. Let $P = (\dots, i^{n_i}, \dots)$ where i has multiplicity n_i . A U -chain of $\mathcal{D}_P$ is a maximal chain whose vertices are comprised of those in the rows of $\mathcal{D}_P$ corresponding to the parts of an almost rectangular (AR) subpartition $P' \subset P$, union two *hooks*—one from the source and the other to the sink of $\mathcal{D}_P$ (see Definition 2.7). We associate to an AR subpartition $P' = (a^{n_a}, (a-1)^{n_{a-1}})$ of P the invariant $ob(P')$ which is the length—number of vertices—of the unique U -chain U_a containing P' :

$$ob(P') = |U_a| = an_a + (a-1)n_{a-1} + \sum_{c>a} 2n_c. \quad (1.1)$$

Theorem 1.2 [18] *Let $\text{char } k = 0$.² The index $i(Q(P))$ is the length of the longest chain in $\mathcal{D}_P$, and is also the length of the longest U -chain in $\mathcal{D}_P$.*

¹The poset $\mathcal{D}_P$ is used implicitly by P. Oblak—the possible edges in her $(\mathcal{N}_B, A)$ graphs determine the comparable elements of $\mathcal{D}_P$. The algebra $\mathcal{U}_B$ was used in Sect. 4 of [18] and formally defined and studied in [5]. Our graphs are drawn as the transpose of those in [18], and are rotated 90 degrees.

²See [5, Theorem 3.3] for the case k algebraically closed. Corollary 3.10 shows that Theorem 1.2 implies its analogue for k infinite.

1.1 A new lower bound for $Q(P)$

C. Greene, E.R. Gansner, and S. Poljak associate to any finite poset $\mathcal{D}$ a partition $\lambda(\mathcal{D})$ defined from its chains, as follows [7, 9, 11, 21]. Let $a_{\mathcal{D}}$ be the minimum number of chains needed to cover $\mathcal{D}$. Set $c_0(\mathcal{D}) = 0$, and for every $i \in \{1, \dots, a_{\mathcal{D}}\}$ set

$$c_i = c_i(\mathcal{D}) = \max\{\#\text{ vertices of } \mathcal{D} \text{ covered by } i \text{ chains}\}, \quad (1.2)$$

$$\lambda_i(\mathcal{D}) = c_i - c_{i-1}. \quad (1.3)$$

One can construct similarly to $\lambda(\mathcal{D}_P)$ a possibly different partition $\lambda_U(\mathcal{D}_P)$ using s - U -chains in place of arbitrary chains (Definitions 2.10, 2.14). P. Oblak had conjectured that $Q(P)$ could be obtained by a recursive process, first picking a maximum-length chain C_1 in $\mathcal{D}_P$, then a maximum length chain C_2 in a new, smaller poset $\mathcal{D}_{P'}$ where the partition $P' = P - C_1$ is defined through removing C_1 from $\mathcal{D}_P$ and counting the vertices left in each row (warning: $\mathcal{D}_{P'}$ does not have the induced partial order from $\mathcal{D}_P$). And so on for r_P steps. Then $Q(P)$ is conjecturally the set of lengths of the chains [4]. The second author has shown that any such Oblak process $\mathcal{O}$ yields a partition $\text{Ob}_{\mathcal{O}}(P) = (|C_1|, |C_2|, \dots, |C_{r_P}|)$ satisfying $\text{Ob}_{\mathcal{O}}(P) = \lambda_U(\mathcal{D}_P)$ [14, §2]. Thus, the Oblak conjecture for $Q(P)$ is equivalent to a positive answer to

Question 1.3 Is $Q(P) = \lambda_U(\mathcal{D}_P)$?

Recall the *dominance* or *orbit closure* order on the set of partitions of n [10]. Let $P = (p_1, \dots, p_t)$ with $p_1 \geq \dots \geq p_t$ and $P' = (p'_1, \dots, p'_{t'})$ with $p'_1 \geq \dots \geq p'_{t'}$ be partitions of n . Then

$$P \geq P' \Leftrightarrow \forall i, \sum_{k=1}^i p_k \geq \sum_{k=1}^i p'_k. \quad (1.4)$$

Our main result is

Theorem 3.9 *Let k be an infinite field, then*

$$Q(P) \geq \lambda_U(\mathcal{D}_P). \quad (1.5)$$

To prove this, we first work over a polynomial ring R over k and define in (2.18) a nilpotent matrix $A_R \in \text{Mat}_R(n) \cong \text{End}_R(V \otimes R)$ which commutes with B . We then show that $P_{A_R} \geq \lambda_U(\mathcal{D}_P)$, when we consider A_R as an element of $\text{Mat}_F(n)$, with F the quotient field of R (Corollary 3.8).

To prove that $P_{A_R} \geq \lambda_U(\mathcal{D}_P)$, we show in Theorem 3.7 that for every $s \in \{1, \dots, r_P\}$ there exist $v_1, \dots, v_s \in V$ such that $\dim_F \langle F[A_R] \circ \{v_1, \dots, v_s\} \rangle$, the dimension of the vector space span of the ring $F[A_R]$ acting on the set $\{v_1, \dots, v_r\}$, is at least the sum $c_U^s(\mathcal{D}_P)$ of the first s parts of $\lambda_U(\mathcal{D}_P)$. Indeed, together with a well-known property of nilpotent matrices (Lemma 3.1), this establishes the desired inequality.

In turn, the proof of Theorem 3.7 boils down to showing that for a maximal s - U -chain $\mathfrak{A}$, a certain F linear map $\pi_{\mathfrak{A}} = \pi_{\mathfrak{A}}(U, A_R)$ defined between a subspace of

Fig. 1 $P = (4, 2, 1)$,
 $Q(P) = (5, 2)$

a	b	c	d
e	f		
g			

$F[x] \otimes_k V \cong F[x] \otimes_F V_F$, and a subspace of $V \otimes F$, both of dimension $c_U^s(\mathcal{D}_P)$, is an isomorphism. The domain of $\pi_{\mathfrak{A}}$ is a subspace $\mathcal{T}_{\mathfrak{A}}$ of $F[x] \otimes_F \langle v_1, \dots, v_s \rangle$ where the v_i are the initial vertices of the s component chains of $U_{\mathfrak{A}}$, and its co-domain is the span of all the vertices covered by $U_{\mathfrak{A}}$: it has matrix $M_{\mathfrak{A}}$ in a suitable basis for each (Definition 2.23). We show, and this is the heart of the matter, that $\det(M_{\mathfrak{A}}) \neq 0$ by an analysis of the sets of chains from the initial vertices v_i to all the vertices covered by $U_{\mathfrak{A}}$. A final step in the proof of (1.5) is to specialize to k (Theorem 3.9).

We next state some further results and questions concerning $Q(P)$. In Sect. 2 we define the poset $\mathcal{D}_P$, the multi- U -chains, the homomorphism $\pi_{\mathfrak{A}}$ and show some properties we will need. In Sect. 3 we show Theorem 3.9. We first give a simple example where P is not AR to illustrate naively the problem of determining $Q(P)$.

Example 1.4 Let $P = (4, 2, 1)$. Since $P = (4) \cup (2, 1)$ is a minimal decomposition into almost rectangular subpartitions, we have $r_P = 2$, and we shall see that $Q(P) = (5, 2)$. Here the basis $B = \{a, b, c, d, e, f, g\}$ with $Ba = b$, $Bb = c$, $Bc = d$, $Bd = 0$, $Be = f$, $Bf = 0$, and $Bg = 0$. Since A and B commute, $A \in \mathcal{U}_B$ is determined by its action on the B -cyclic vectors $\{a, e, g\}$ of V . To obtain a general enough A so that $P_A = (5, 2) = Q(P)$ we may take (Fig. 1)

$$A \cdot a = b + e, \quad A \cdot e = c + g, \quad A \cdot g = f. \quad (1.6)$$

In Example 3.14 we apply the proof method of this paper to $P = (4, 2, 1)$: the endomorphism A above is obtained by substituting 1 for each of the variables of $\mathbb{R}$ in the matrix $A_{\mathbb{R}}$ of (3.9).

In Examples 2.15 and 2.16 below we determine $\lambda_U(P)$ for $P = (4, 2, 2, 1)$ and $P = (5, 4, 3, 3, 2, 1)$. By Corollary 3.12, $Q(P) = \lambda_U(P)$ for these P , since $r_P \leq 3$. For a discussion of different Oblak processes and the relation to U -chains see [14].

1.2 Some open questions

Recall that the incidence algebra $\mathfrak{I}(\mathcal{D})$ of the n -element poset $\mathcal{D}$ is the algebra of $n \times n$ matrices M satisfying $M_{uv} \in k$ if $u \leq v$, and $M_{uv} = 0$ if $u \not\leq v$.

The nilpotent matrices $\mathcal{N}(\mathcal{D})$ in $\mathfrak{I}(\mathcal{D})$ are those such that $\forall u \ m_{uu} = 0$. Suppose that $\mathcal{D}$ is acyclic, as is true for the posets $\mathcal{D}_P$ we consider. Then these nilpotent matrices have entries $m_{uv} \in k$ that are arbitrary for intervals $[u, v]$ with $u < v$. Then, evidently, $\mathcal{N}(\mathcal{D})$ is an irreducible variety. We have

Theorem ([9, 23], see also [7, Theorem 6.1]) (A) Let $\mathcal{D}$ be a finite poset and suppose $\text{char } k = 0$. A generic nilpotent matrix $M \in \mathcal{N}(\mathcal{D})$ has Jordan type $P_M = \lambda(\mathcal{D})$.

(B) ([24], see also [7, Proof of Theorem 6.1])³ Let k be an infinite field, let $\mathcal{D}$ be an acyclic poset, and $M \in \mathcal{N}(\mathcal{D})$. Then

$$\lambda(\mathcal{D}) \geq P_M. \quad (1.7)$$

The commutator subset $\mathcal{C}_B \cap \mathfrak{I}(\mathcal{D}_P) \subset \mathfrak{I}(\mathcal{D}_P)$ of the incidence algebra of $\mathcal{D}_P$ consists of those $A \in \mathfrak{I}(\mathcal{D}_P)$ whose entries A_{uv} satisfy, for $u, v \in B$

$$u \leq v \text{ and } Bu, Bv \neq 0 \Rightarrow A_{B \cdot u, B \cdot v} = A_{uv}. \quad (1.8)$$

This is a Toeplitz condition on the blocks of A (see [2, Lemma 2.2], [29]).

Since $\mathcal{D}_P$ is acyclic, the nilpotent matrices $\mathcal{N}(\mathcal{D}_P) \subset \mathfrak{I}(\mathcal{D}_P)$ form an irreducible family satisfying $\mathcal{N}(\mathcal{D}_P) \supset \mathcal{U}_B = \mathcal{C}_B \cap \mathcal{N}(\mathcal{D}_P)$. By (1.7) we have $\lambda(\mathcal{D}_P) \geq Q(P)$.

Question 1.5 Is $Q(P) = \lambda(\mathcal{D}_P)$?

We can also ask the seemingly purely combinatorial question

Question 1.6 Is $\lambda(\mathcal{D}_P) = \lambda_U(\mathcal{D}_P)$?

In view of Theorem 3.9 and (1.7) a positive answer to Question 1.6 would also imply “yes” to Questions 1.3 and 1.5. In this direction P. Oblak in Theorem 1.2 showed that the index—largest part—of $\lambda(\mathcal{D}_P)$ and of $\lambda_U(\mathcal{D}_P)$ are the same, and the second author has shown that the minimum parts of $\lambda(\mathcal{D}_P)$ and $\lambda_U(\mathcal{D}_P)$ are the same [15]. Together with Theorem 3.9 their results imply “Yes” to Question 1.6 and Question 1.3 when $r_P \leq 3$ (Corollary 3.12).

Even if the Questions above were answered, it could still be a nontrivial combinatorial problem to identify compactly which partitions P satisfy $Q(P) = Q$ for a given partition Q . This is discussed in [19].

1.2.1 What else is known about $Q(P)$?

T. Košir and P. Oblak showed that the Artinian algebra $\mathcal{A} = k[A, B]$ is Gorenstein for general enough $A \in \mathcal{U}_B$ [16, Corollary 5]. The present authors with R. Basili gave sufficient conditions on $A \in \mathcal{U}_B$ for the algebra $\mathcal{A}$ to be Gorenstein [5, Theorem 2.20]. When $\text{char } k = 0$ or $\text{char } k > n$ the partition $Q(P)$ is dual to the Hilbert function $H(k[A, B])$, viewed as a partition of n , for generic $A \in \mathcal{N}_B$ [3, Theorem 2.23]. From these last results it follows that when $\text{char } k = 0$ or $\text{char } k > n$, $Q(P)$ has parts differing pairwise by at least two, and that $Q(Q(P)) = Q(P)$ ([16, Theorem 6], see also [5, Sect. 2.5]).

A. Premet, G. McNinch and D.I. Panyushev studied pairs of commuting nilpotent matrices in the broader context of Lie algebras [17, 20, 22]. V. Baranovsky, R. Basili and others have related the study of commuting nilpotent matrices to the punctual

³This is stated in slightly different language for $k = \mathbb{C}$ in [24, Theorem 5.16ii], however, the proof there of (1.7) does not depend on characteristic, nor require A to have generic entries nor be “free” in the language of [24]. Likewise, the proof of Theorem 6.1 in [7], although stated for $k = \mathbb{R}$, shows (1.7) for k infinite.

Hilbert scheme of a plane [1–3, 22]. R. Guralnick and A. Sethuranam, K. Šivic and others have studied commuting pairs and triples of matrices: see [12, 25–27] and the references given there.

2 The algebra $\mathcal{U}_B$ and the poset $\mathcal{D}_P$

2.1 The algebra $\mathcal{U}_B$

We now define a maximal subalgebra $\mathcal{U}_B$ of $\mathcal{N}_B$. Fix an integer n and let $P \vdash n$ be the partition $P = (p_1, p_2, \dots, p_t)$, $p_1 \geq p_2 \geq \dots \geq p_t$ or, in second notation $P = (p_1^{n_{p_1}}, \dots, i^{n_i}, \dots, 1^{n_1})$ —where n_i possibly zero—is the multiplicity of the part i of P . Here $n = \sum p_i = \sum i \cdot n_i$. Denote by $S_P = \{i \mid n_i \neq 0\}$ the set of integers occurring as parts of P . For each subpartition P' of P we denote by $\iota(P') \subset S_P$ the set of integers occurring in P' . We have $V = \bigoplus_{i \in S_P} V_i$, where V_i has a decomposition

$$V_i = \bigoplus_{k=1}^{n_i} V_{i,k} \quad (2.1)$$

into cyclic B -modules $V_{i,k}$, each of length i . The subspace $V_{i,k}$ has a cyclic vector $(1, i, k)$ and basis

$$\{(u, i, k) = B^{u-1}(1, i, k), 1 \leq u \leq i\}. \quad (2.2)$$

So $V_{i,k} \cong k[x]/x^i$ as a $k[x]$ -module through the action of B .

Definition 2.1 We denote by B the basis of V that is the union of the bases for $V_{i,k}$ defined in (2.1), (2.2). For each $i \in S_P$ denote by $W_i \subset V_i$ the n_i -dimensional subspace of V spanned by the level- i cyclic vectors,

$$W_i = \langle \{(1, i, k), 1 \leq k \leq n_i\} \rangle, \quad (2.3)$$

with basis ordered by “ k ”. Let $W = \bigoplus_{i \in S_P} W_i$.

We have $W \cong V/\text{Im}(B)$, where $\text{Im}(B)$ denotes the image $B(V)$. Denote by κ_i the natural projection: $V \rightarrow W \rightarrow W_i$. Let $\mathcal{M}_B = \prod_{i \in S_P} \text{End}_k(W_i)$ and define

$$\varphi_i : \mathcal{C}_B \rightarrow \text{End}_k W_i \cong \text{Mat}_{n_i}(k) : \varphi_i(A) = \kappa_i(A|_{W_i}) \quad (2.4)$$

and

$$\varphi = \prod \varphi_i : \mathcal{C}_B \rightarrow \mathcal{M}_B. \quad (2.5)$$

It is well known that φ is, up to an automorphism of $\mathcal{M}_B$, the canonical projection from $\mathcal{C}_B$ to its semisimple quotient, with kernel the Jacobson radical $\mathfrak{J}_B \subset \mathcal{C}_B$ (see [2, Lemma 2.3], [5, Theorem 2.3], [13, Theorem 6]). Denote by $\mathcal{N}$ the product $\mathcal{N} = \prod_{i \in S_P} N(W_i)$ where $N(W_i) \subset \text{End}_k(W_i)$ are the nilpotent elements; and by $\mathcal{U} = \prod_{i \in S_P} U_T(W_i) \subset \mathcal{M}_B$ the products of the subalgebras of strictly upper triangular

elements $U_T(W_i) \subset \text{End}_k(W_i)$, in the ordered basis (2.3). Since the Jacobson radical $\mathfrak{J}_B$ is already comprised of nilpotent elements of $\mathcal{C}_B$, it follows from (2.5) that the nilpotent commutator $\mathcal{N}_B$ satisfies

$$\mathcal{N}_B = \varphi^{-1}(\mathcal{N}). \quad (2.6)$$

We define

$$\mathcal{U}_B = \varphi^{-1}(\mathcal{U}). \quad (2.7)$$

For $v \in V$ we denote by $\langle v, (u, i, k) \rangle$ the coefficient of v on (u, i, k) , when v is written in the basis B of Definition 2.1.

Lemma 2.2 ([2, Lemma 2.3], [5, Theorem 2.3B]) *Let $C \in \mathcal{N}_B$. Then $C \in \mathcal{U}_B$ iff C satisfies the following condition for all $i \in S_P$:*

$$\langle C((1, i, k)), (1, i, k') \rangle = 0 \quad \text{whenever } 1 \leq k' \leq k \leq n_i. \quad (2.8)$$

Also, $\mathcal{U}_B$ is a maximal nilpotent subalgebra of $\mathcal{C}_B$, and is isomorphic as a variety to an affine space.

Proof The condition (2.8) is equivalent to the strict upper triangularity of $\varphi_i(C)$. That $\mathcal{U}_B$ is a maximal nilpotent subalgebra of $\mathcal{N}_B$ follows from (2.7), and the fact that each $U_T(W_i)$ is a maximal nilpotent subalgebra of $\text{End}_k(W_i)$. It is straightforward to write coordinates for $\mathcal{U}_B$ as an affine space, using (1.8), and the B -action from (2.1), (2.2). $\square$

2.2 The poset $\mathcal{D}_P$

We stated earlier that $\mathcal{D}_P$ is the poset (or digraph) associated to the algebra $\mathcal{U}_B \subset \mathcal{C}_B$. That is, the elements (or vertices) of $\mathcal{D}_P$ correspond 1-1 to the basis elements of V from (2.1), (2.2); two elements b, b' in $\mathcal{D}_P$ satisfy

$$b < b' \Leftrightarrow \exists A \in \mathcal{U}_B \quad \text{such that} \quad \langle A(b), b' \rangle \neq 0. \quad (2.9)$$

We now give a second definition of $\mathcal{D}_P$ by specifying its diagram $\text{Diag}(\mathcal{D}_P)$, comprised of the pairs $b < b'$ in $\mathcal{D}_P$ such that b' covers b (there are no vertices $x, b < x < b'$; we will also say b precedes b'). We determine these pairs by their corresponding elementary maps in $\mathcal{U}_B$ (see below). For the equivalence of the two definitions, see [5, Theorem 2.5, (2.18), Remark 2.10]. For $i \in S_P$ we denote by $i^+ = \{s \mid s \in S_P, s > i\}$ and $i^- = \max\{s \mid s \in S_P, s < i\}$ the next largest and next smaller elements of S_P , respectively, when they exist.

Definition 2.3 [5, Def. 2.9]. (Maps and poset $\mathcal{D}_P$ associated to P)

- (a) Vertices of $\mathcal{D}_P$. For each pair (u, i) with $i \in S_P$ and $1 \leq u \leq i$, there are n_i vertices $\{(u, i, k), 1 \leq k \leq n_i\}$. We visualize these as a vertical column parallel to the z -axis in 3-space where $(u, i, 1)$ is the bottom vertex and (u, i, n_i) is the top vertex of the column.

Fig. 2 $\text{Diag}(\mathcal{D}_P)$ and maps for $P = (4, 2, 2, 1)$

- (b) Elementary maps of $\text{End}_k(V)$. The maps defined below are zero on those basis elements of V from (2.1) and (2.2) not specifically listed.
- (i) for $i \in S_P \setminus p_t$, $\beta_i = \beta_{i,i^-}$ maps the vertex (u, i, n_i) to $(u, i^-, 1)$, whenever $1 \leq u \leq i^-$.
  - (ii) for $i \in S_P \setminus p_t$, $\alpha_i = \alpha_{i^-,i}$ maps (u, i^-, n_{i^-}) to $(u + i - i^-, i, 1)$, whenever $1 \leq u \leq i^-$.
  - (iii) $e_{i,k}$ maps the vertex (u, i, k) to $(u, i, k + 1)$, $1 \leq u \leq i$, $1 \leq k < n_i$.
  - (iv) When $i \in S_P$ is isolated (when neither $i - 1 \in S_P$ nor $i + 1 \in S_P$), the map w_i sends (u, i, n_i) to $(u + 1, i, 1)$ whenever $1 \leq u < i$.
- (c) There is an edge $v \rightarrow v'$ in the diagram $\text{Diag}(\mathcal{D}_P)$ iff $\exists$ an elementary map γ such that $\gamma(v) = v'$.

Example 2.4 $\mathcal{D}_P$ for $P = (4, 2, 2, 1)$. There are four rows, three levels $i = 4, 2, 1$. See Fig. 2.

By giving maps corresponding to the edges of the diagram of $\mathcal{D}_P$ we have in effect defined a large quiver Ω_P with identities ([5, Definition 2.9]). The B -action on $\mathcal{D}_P$ maps vertices one step to the right: $B(u, i, k) = (u + 1, i, k)$ if $u < i$ and is 0 when $u = i$ (see (2.2)). Evidently, the elementary maps commute with the action of B . The B -orbit of an edge $v \rightarrow v'$ of the diagram is the set of edges $B^s v \rightarrow B^s v'$, $s = 1, 2, \dots$ for which $B^s \cdot v' \neq 0$. Definition 2.3b above assigns a unique map to each maximal B -orbit of $\text{Diag}(\mathcal{D}_P)$. Thus, there is a 1-1 correspondence between the maximal B -orbits of edges of $\text{Diag}(\mathcal{D}_P)$ and the set of elementary maps. Also, if (u, i, k) precedes (u', i', k') then either

$$B((u, i, k)) \text{ precedes } B((u', i', k')), \quad \text{or} \quad B((u', i', k')) = 0.$$

Likewise $v \leq v'$ and $Bv' \neq 0$ imply $Bv \leq Bv'$. It follows from (2.7) and Definition 2.3 that $\mathcal{U}_B \subset \mathfrak{I}(\mathcal{D}_P)$ is the subalgebra of the incidence algebra $\mathfrak{I}(\mathcal{D}_P)$ over k comprised of its nilpotent elements satisfying $[A, B] = 0$, or, equivalently, (1.8).

The i -level of $\mathcal{D}_P$ is the set of all vertices with second entry i : equivalently, the vertices of $\{\bigcup_{1 \leq k \leq n_i} V_{i,k}\}$. We denote by $\varrho(u, i, k) = \varrho(u, i) = 2u - i - 1$ the integer giving the relative position of a vertex with respect to the vertical center of symmetry of $\mathcal{D}_P$, determined by the involution τ of $\mathcal{D}_P$ (see [5] and (2.14) below).

Lemma 2.5 [5, Theorem 2.13] *Let $(u, i, k) \leq (u', i', k')$ in $\mathcal{D}_P$. Then*

$$u \leq u' \quad (2.10)$$

$$i - u \geq i' - u' \quad (2.11)$$

$$\varrho(u, i) + |i' - i| \leq \varrho(u', i'). \quad (2.12)$$

Proof (2.10) is immediate from (2.9) and $[A, B] = 0$. We have

$$B^{i+1-u} \cdot A((u, i, k)) = A \cdot B^{i+1-u}((u, i, k)) = 0,$$

implying (2.11). (2.12) follows from (2.10) and (2.11). $\square$

We may write (u, i) for $(u, i, 1) \in \mathcal{D}_P$ when the multiplicity $n_i = 1$.

Corollary 2.6 *Let p be a chain from $v = (u, i, k)$ to $v' = (u', i', k')$ in $\mathcal{D}_P$. Then the total number $m(p, b)$ of β_b and w_b edges in p satisfies*

$$m(p, b) \leq u' - u - \min\{|i - b|, |i' - b|\} \quad (2.13)$$

if $b < \min(i, i')$ or $b > \max(i, i')$; and it satisfies $m(p, b) \leq u' - u$ otherwise.

Proof Assume $b < \min(i, i')$, and that p is saturated. Then β_b, α_b edges are paired in p . Thus, if the chain first hits level b at $(u_1, b, 1)$ and last at (u_2, b, n_b) we have $m(p, b) \leq u_2 - u_1$; from (2.10), (2.11) we have $u \leq u_1, u_2 + i' - b \leq u'$ so $m(p, b) \leq u' - u - |i' - b|$. Likewise for $b > \min(i, i')$ we have $m(p, b) \leq u' - u - |i - b|$. $\square$

See [5, Proposition 2.14] for a generalization specifying $\varrho(v') - \varrho(v)$ for p .

2.3 The U -chains of $\mathcal{D}_P$

For $S \subset S_P$ we denote by $\iota^{-1}(S)$ the subpartition of P comprised of all parts of P having lengths in S . An s -chain of a poset $\mathcal{D}$ is a union of s chains of $\mathcal{D}$. The length of a chain is its number of vertices. The concept of U -chains of $\mathcal{D}_P$ is essentially due to P. Oblak (“ B_k paths” in [18], see also [5, §3]).

Definition 2.7 A simple U -chain $U_a \subset \mathcal{D}_P$ is comprised of the following vertices, and edges in $\mathcal{D}_P$ between adjacent vertices:

- (i) all the vertices at levels $a, a - 1$ of $\mathcal{D}_P$.

(ii) two *hooks* above the a -level:

- the first is comprised of all vertices $(1, i, k) \mid i > a, 1 \leq k \leq n_i$;
- the second is comprised of all vertices $(i, i, k) \mid i > a, 1 \leq k \leq n_i$.

Note 2.8 The simple chain U_a in $\mathcal{D}_P$ is comprised of the $\mathcal{D}_P$ levels $a, a-1$ corresponding to an almost rectangular subpartition $P' = \iota^{-1}\{a, a-1\}$ of P , union the two hooks, one on the left from the source $(1, p_1, 1)$ of $\mathcal{D}_P$ down to P' and the other symmetrically located on the right from P' up to the sink (p_1, p_1, n_{p_1}) of $\mathcal{D}_P$. The length $|U_a|$ satisfies Eq. (1.1). When a is isolated in S_P the simple U -chain U_a in $\mathcal{D}_P$ is comprised of the chain at level a of $\mathcal{D}_P$ union the two hooks.

The diagram of $\mathcal{D}_P$ is comprised of the covering edges of $\mathcal{D}_P$ (Definition 2.3). We need an augmented diagram, whose role will become apparent after we define s - U -chains, in Lemma 2.13.

Definition 2.9 A maximal consecutive subsequence (MCS) of S_P is one not properly contained in a larger consecutive subsequence. We denote by S_P'' the subset of S_P comprised of minimum elements of all MCS having odd cardinality. The *augmented diagram* $\text{Diag}^{\text{aug}}(\mathcal{D}_P) \supset \text{Diag}(\mathcal{D}_P)$ is the diagram $\text{Diag}(\mathcal{D}_P)$ supplemented by new edges $(u, \ell, n_\ell) \mapsto (u+1, \ell, 1)$ for each pair (u, ℓ) such that $1 \leq u < \ell$ and $\ell \in S_P''$, ℓ not isolated.

An isolated $\ell \in S_P$ is the minimum of an MCS of length one in S_P : the corresponding edges are already in $\text{Diag}(\mathcal{D}_P)$.

Recall from [5, Definition 2.15] the order reversing involution

$$\tau : \mathcal{D}_P \rightarrow \mathcal{D}_P, \quad \tau(u, i, k) = (i+1-u, i, n_i+1-k). \quad (2.14)$$

A U -chain U_a is evidently mapped to itself by τ , the left hand hook mapping to the right hand hook.

Let C, C' be two disjoint τ -symmetric chains of $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$ that are maximal with respect to the properties of being disjoint and symmetric. We say that C' is *inside* C if for each row $(1, i, k) \leq (2, i, k), \dots \leq (i, i, k)$ of $\mathcal{D}_P$ (so i, k are fixed), all vertices of U' in the row lie between the outside two vertices of U in that row. A *shelling* of a τ -symmetric subset $\mathcal{D}$ of the vertices of $\mathcal{D}_P$ is a sequence of s disjoint τ -symmetric chains $C_1, \dots, C_s$ of $\mathcal{D}_P$ whose union is $\mathcal{D}$ and such that C_{i+1} is inside C_i for $i = 1, \dots, s-1$.

We now define s - U -chains of $\mathcal{D}_P$.

Definition 2.10 (A) Let $\mathfrak{A} = (a_1, a_2, \dots, a_s)$ be an s -tuple of positive integers satisfying $a_i \in S_P$ and $a_i \geq a_{i+1} + 2$ for $1 \leq i < s$. We define $\{\mathfrak{A}\} = \{a_1, a_1-1, a_2, a_2-1, \dots, a_s, a_s-1\}$.

- (a) We denote by $\{U_{\mathfrak{A}}\}$ the subset of vertices of $\mathcal{D}_P$ comprised of
- (i) all vertices in the levels of $\mathcal{D}_P$ given by $\iota^{-1}(\{\mathfrak{A}\} \cap S_P)$;
  - (ii) for each level $\ell > a_s, \ell \in S_P \setminus (\{\mathfrak{A}\} \cap S_P)$, all vertices

- (a) (u, ℓ, k) with $u \leq \#\{a_i < \ell\}$, $1 \leq k \leq n_\ell$ (at the left of the ℓ level);
 (u, ℓ, k) , with $u \geq \ell + 1 - \#\{a_i < \ell\}$, $1 \leq k \leq n_\ell$ (at the right of the ℓ level).
- (b) We define the s - U -chain $U_{\mathfrak{A}}$ as the unique shelling of $\{U_{\mathfrak{A}}\}$ by a set of s disjoint τ -symmetric chains of $\mathcal{D}_P$. The first and outside chain in the shelling is the simple U -chain $U_{\mathfrak{A},1} = U_{a_s}$ of $\mathcal{D}_P$. The γ th component chain $U_{\mathfrak{A},\gamma}$ —counting from the outside of $U_{\mathfrak{A}}$ —has vertices

$$\begin{aligned} \{U_{\mathfrak{A},\gamma}\} \\ &= \{(u, \ell, k) \mid \ell \in \{a_{s+1-\gamma}, a_{s+1-\gamma} - 1\}, \\ &\quad \gamma - 1 \leq u \leq \ell + 2 - \gamma, 1 \leq k \leq n_\ell\} \\ &\cup \{(\gamma, \ell, k), (\ell + 1 - \gamma, \ell, k) \text{ with } \ell > a_{s+1-\gamma}, \ell \in S_P \setminus \{\mathfrak{A}\} \cap S_P, \\ &\quad 1 \leq k \leq n_\ell\}. \end{aligned} \quad (2.15)$$

(B) We denote by $|U_{\mathfrak{A}}|$ and $|U_{\mathfrak{A},\gamma}|$ the lengths of the s -chain, and of the γ th component chain, respectively. We denote by $v_{\mathfrak{A},\gamma,j}$, $1 \leq j \leq |U_{\mathfrak{A},\gamma}|$ the j th vertex of the chain $U_{\mathfrak{A},\gamma}$: so its initial vertex is $v_\gamma = v_{\mathfrak{A},\gamma,1}$. Given $U_{\mathfrak{A}}$ and $v_{\mathfrak{A},\gamma,j}$ we will term the portion of $U_{\mathfrak{A},\gamma}$ from v_γ to $v_{\mathfrak{A},\gamma,j}$ the *standard* chain from v_γ to $v_{\mathfrak{A},\gamma,j}$.

(C) We denote by $\langle U_{\mathfrak{A}} \rangle$ and $\langle U_{\mathfrak{A},\gamma} \rangle$ the $\mathbf{k}$ span of the elements of $\mathbf{B}$ (vertices of $\mathcal{D}_P$) in $U_{\mathfrak{A}}$ and $U_{\mathfrak{A},\gamma}$, respectively, and by $\langle U_{\mathfrak{A}} \rangle_L$ and $\langle U_{\mathfrak{A},\gamma} \rangle_L$ the L spans of the same elements when $L \supset \mathbf{k}$ is a field.

(D) We say that the s - U -chain is *maximal* if it is not a proper subset of another s - U -chain (with the same s).

Remark 2.11 The chain $U_{\mathfrak{A},\gamma}$ is made up of vertices in what is left of the set $\{U_{\mathfrak{A}}\}$, after removal of the previous $\gamma - 1$ chains. It has an almost rectangular part (first subset of Eq. (2.15)); it has as well the two outside hooks of what is left above the $a_{s+1-\gamma}$ level of $\mathcal{D}_P$ (second subset of (2.15)). The chain $U_{\mathfrak{A},\gamma}$ has initial vertex $v_\gamma = v_{\mathfrak{A},\gamma,1} = (\gamma, p_1, 1)$ and terminal vertex $\tau(v_\gamma) = v_{\mathfrak{A},\gamma,|U_{\mathfrak{A},\gamma}|} = (p_1 + 1 - \gamma, p_1, n_1)$.

Definition 2.12 We say that $U_{\mathfrak{A},\gamma}$ has a *singleton level* if $a_{s+1-\gamma} - 1 \notin S_P$: so its almost rectangular portion has only one level.

We will need the following characterization of the levels $\ell \in S_P$ that may occur as singleton levels in a s - U -chain of $\mathcal{D}_P$.

Lemma 2.13 Let $U_{\mathfrak{A}}$ be a maximal s - U -chain. If $U_{\mathfrak{A},\gamma}$ has a singleton level then $a_{s+1-\gamma}$ is the minimum of an odd length MCS of S_P included in $\{\mathfrak{A}\}$. Conversely, let ℓ be the minimum of a length $(2k + 1)$ MCS of S_P : then the $(k + 1)$ - U -chain $U_{\mathfrak{A}}$ where $\mathfrak{A} = (\ell + 2k, \ell + 2k - 2, \dots, \ell)$ has the singleton level ℓ .

Definition 2.14 We define the partition $\lambda_U(\mathcal{D}_P)$ from the s - U -chains of $\mathcal{D}_P$. For $1 \leq i \leq r_P$ the i th part of $\lambda_U(\mathcal{D}_P)$ is

$$\begin{aligned} (\lambda_U(\mathcal{D}_P))_i &= u_i(\mathcal{D}_P) - u_{i-1}(\mathcal{D}_P) \quad \text{where } u_0(\mathcal{D}_P) = 0 \text{ and for } i > 0 \\ u_i(\mathcal{D}_P) &= \max\{|U_{\mathfrak{A}}| \text{ such that } \mathfrak{A} \text{ is an } i\text{-}U\text{-chain in } \mathcal{D}_P\}. \end{aligned} \quad (2.16)$$

Although the component chains of the s - U -chains are disjoint, this is otherwise analogous to the definition of $\lambda(\mathcal{D})$ in (1.3) from the sets of all chains of $\mathcal{D}_P$.

Example 2.15 The poset $\mathcal{D}_P$ for $P = (4, 2, 2, 1)$ has $t = 4$ rows, $\#S_P = 3$ levels of which $\ell = 4 \in S_P''$ is isolated. The source is $(1, 4)$ the sink is $(4, 4)$. The two simple U -chains of $\mathcal{D}_P$ are (see Fig. 2)

$$(1, 4) \leq (2, 4) \leq (3, 4) \leq (4, 4), \quad \text{and}$$

$$(1, 4) \leq (1, 2, 1) \leq (1, 2, 2) \leq (1, 1) \leq (2, 2, 1) \leq (2, 2, 2) \leq (4, 4).$$

The 2- U -chain $U_{\mathfrak{U}}$, $\mathfrak{U} = (4, 2)$ has a singleton level $\ell = 4$. Thus we have $\lambda_U(P) = (7, 2)$, the first difference of $(u_0 = 0, u_1 = 7, u_2 = 9)$.

Example 2.16 For $P = (5, 4, 3, 3, 2, 1)$ the simple U -chains are U_5, U_4, U_3, U_2 of lengths 9, 12, 12, 11, respectively, according to (1.1). The 2- U -chain $U_{(4,2)}$ (Fig. 3) has length 17 and is comprised of the outer chain

$$U_{(4,2),1} = (1, 5), (1, 4), (1, 3, 1), (1, 3, 2), (1, 2), (1, 1), (2, 2), (3, 3, 1), \\ (3, 3, 2), (4, 4), (5, 5)$$

and the inner chain

$$U_{(4,2),2} = (2, 5), (2, 4), (2, 3, 1), (2, 3, 2), (3, 4), (4, 5).$$

Fig. 3 2- U -chain $U_{4,2}$ for $P = (5, 4, 3, 3, 2, 1)$

The other maximal 2- U -chains are $U_{5,3}$ and $U_{5,2}$ of lengths 17 and 16, respectively. The unique 3- U -chain $U_{(5,3,1)}$ has a singleton level $\ell = 1$; it has the shelling

$$\begin{aligned} U_{(5,3,1),1} &= (1, 5), (1, 4), (1, 3, 1), (1, 3, 2), (1, 2), (1, 1), (2, 2), (3, 3, 1), (3, 3, 2), \\ &\quad (4, 4), (5, 5), \\ U_{(5,3,1),2} &= (2, 5), (2, 4), (2, 3, 1), (2, 3, 2), (3, 4), (4, 5), \quad \text{and} \\ U_{(5,3,1),3} &= (3, 5). \end{aligned}$$

Thus, $v_{(5,3,1),2,3} = (2, 3, 1)$, the third vertex of $U_{(5,3,1),2}$; and $v_{(5,3,1),2,6} = (4, 5)$.

The partition $\lambda_U(P) = (12, 5, 1)$, the first differences of $(u_0 = 0, u_1 = 12, u_2 = 17, u_3 = 18)$. Note that neither of the maximum-length simple U -chains U_4, U_3 is the first component $U_{(4,2),1}$ or $U_{(5,3),1}$ of a maximum-length 2- U -chain!

2.4 The homomorphism A_R

We first define a polynomial ring R over k , most of whose variables correspond 1-1 to the maximal B -orbits of edges in the diagram of $\mathcal{D}_P$; then we will define a certain sparse matrix $A_R \in \mathcal{U}_{B,R} = \mathcal{U}_B \otimes_k R$. We let

$$R = k[s_i, t_i, t_{j,k}, z_\ell \mid i \in S_P \setminus p_t, j \in S_P, 1 \leq k \leq n_i, \ell \in S_P'']. \quad (2.17)$$

Let F be the quotient field of R , $V_R = V \otimes_k R$, $V_F = V \otimes_k F$. We identify $\text{End}_k V$ with $\text{Mat}_n k$, $\text{End}_R V_R$ with $\text{Mat}_n R$ and $\text{End}_F V_F$ with $\text{Mat}_n F$ in the basis B .

Definition 2.17 We define the *simply adequate* matrix $A_R \in \text{End}_R V_R = \text{Mat}_n R$ as

$$A_R = \sum_{i \in S_P \setminus p_t} (s_i \beta_i + t_i \alpha_i) + \sum' t_{i,k} e_{i,k} + \sum_{\ell \in S_P''} z_\ell w_\ell \quad (2.18)$$

where $\sum'$ is the sum over couples (i, k) with $1 \leq k < n_i, i \in S_P$. Here β_i, α_i and e_{ik} are the elementary endomorphisms of V given in Definition 2.3b; and w_ℓ is the endomorphism of V taking (u, ℓ, n_ℓ) to $(u + 1, \ell, 1)$ for $1 \leq u < \ell$, which is elementary only when the MCS containing $\ell \in S_P''$ is a singleton.

Equivalently, we have the following description of the entries of the matrix A_R :

$$(A_R)_{v,v'} = \begin{cases} \text{the variable of } R \text{ determined by the map } v \rightarrow v' \text{ when } v \text{ precedes } v'; \\ \text{the variable } z_i \text{ when } v = (u, \ell, n_\ell) \text{ and } v' = (u + 1, \ell, 1) \text{ and } \ell \in S_P''; \\ 0 \quad \text{otherwise.} \end{cases} \quad (2.19)$$

In particular the variables z_ℓ of the simply adequate A_R of (2.18) correspond 1-1 to the singleton levels in maximal s - U -chains $\mathfrak{A}$ of $\mathcal{D}_P$ (Lemma 2.13).

Definition 2.18 Let L be a field containing k . We call $A \in \mathcal{U}_B \otimes_k L$ *adequate* if there exist $s_i, t_i, t_{i,k}, z_\ell \in L \setminus 0$ for every $i \in S_P \setminus p_t$, every $k \in \{1, \dots, n_i\}$, and every $\ell \in S_P''$,

such that, in the notation of (2.18),

$$A = \sum_{i \in S_P \setminus p_t} (s_i \beta_i + t_i \alpha_i) + \sum' t_{i,k} e_{i,k} + \sum_{\ell \in S_P''} z_\ell w_\ell. \quad (2.20)$$

In [5] we conjectured that if A is adequate, then $P_A = Q(P) = \text{Ob}(P)$. We will show the weaker result that if k is an infinite field and A_R is simply adequate then $P_{A_R} \geq \lambda_U(\mathcal{D}_P)$, where A_R is considered as an element of $\text{Mat}_{\mathbb{F}}(n)$ (Corollary 3.8). We then show that there exists an adequate A over k such that $P_A \geq \lambda_U(\mathcal{D}_P)$ (Theorem 3.9). The need for a hypothesis such as “adequate” is shown by [5, Example 3.17c].

Note 2.19 A chain in $\text{Diag}(\mathcal{D}_P)$ or $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$ from vertex v to vertex v' is *saturated* if it is not a proper subset of another chain from v to v' in $\text{Diag}(\mathcal{D}_P)$ or $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$, respectively. For $u \in \mathbb{N}$ the entry $(A_R^u)_{v,v'}$ of A_R^u is the projection $\langle A_R^u(v), v' \rangle$. It is a sum of terms, most of which are monomials in R corresponding to a saturated chain in $\text{Diag}(\mathcal{D}_P)$ from v to v' . However, we have included extra variables z_ℓ , each corresponding to a map w_ℓ and to the B orbit of an edge $(1, \ell, n_\ell) \rightarrow (2, \ell, 1)$ in $\text{Diag}^{\text{aug}}(\mathcal{D}_P) \subset \mathcal{D}_P$, where $\ell \in S_P''$, ℓ not isolated. Thus, $(A_R^u)_{v,v'}$ includes monomials corresponding to chains from v to v' in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$ (Lemma 2.21). We chose the simply adequate A_R —a relatively sparse matrix—in order to simplify a key step in our proof (see (3.6)ff of Proposition 3.5): A_R has the minimum number of variables that we need for this step. We could have worked directly with a generic $A' = A_{R'} \in \mathcal{U}_{B,R'}$ over a large ring R' : for $v < v'$, $v, v' \in \mathcal{D}_P$ the entry $A'_{v,v'}$ is a variable of R' corresponding to the maximal B -orbit containing the interval $[v, v']$ in $\mathcal{D}_P$. Using the sparse matrix A_R leads to a more precise statement.

2.5 The projection $\pi_{\mathfrak{A}} : \mathcal{T}_{\mathfrak{A}} \rightarrow \mathcal{U}_{\mathfrak{A}}$, and the matrix $M_{\mathfrak{A}}$

We fix $A_R \in \mathcal{U}_{B,R}$ to be the simply adequate matrix of Definition 2.17. Let $U_{\mathfrak{A}}$ be an s - U -chain of $\mathcal{D}_P$. Recall that the initial vertices v_γ of the γ -component chain $U_{\mathfrak{A},\gamma}$ of $U_{\mathfrak{A}}$ satisfy $v_\gamma = v_{\mathfrak{A},\gamma,1} = (\gamma, p_1, 1) \in V$, $1 \leq \gamma \leq s$. The matrix A_R^u has entries $(A_R^u)_{b,b'}$ on each pair $b, b' \in B$ of basis vectors.

Definition 2.20 We associate to a chain p in the augmented diagram $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$ the monomial μ_p obtained by multiplying the variables of R in (2.18) that are the coefficients for the elementary maps of (2.3) and also those variables z_ℓ with $\ell \in S_P''$ corresponding to the edges of p .

Lemma 2.21 For $v < v'$ vertices of $\mathcal{D}_P$, the entry $(A_R^u)_{v,v'}$ of the u th power A_R^u is the sum of degree- u monomials in R ,

$$(A_R^u)_{v,v'} = \sum_p \mu_p \quad (2.21)$$

where the sum is over all chains p of length $u + 1$ from v to v' in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$.

Proof This is a standard result concerning the incidence algebra of a poset. $\square$

Example 2.22 Set $P = (4, 2, 2, 1)$. (See Fig. 2 and Example 2.15.) The chain $p : (1, 4) \rightarrow (2, 4) \rightarrow (3, 4) \rightarrow (4, 4)$ in $\mathcal{D}_P$ contributes the monomial $\mu_p = z_4^3 \in \mathbb{R}$ to the entry $(A_R^3)_{(1,4),(4,4)}$. The chain

$$p' = (1, 4) \rightarrow (1, 2, 1) \rightarrow (1, 2, 2) \rightarrow (1, 1) \rightarrow (2, 2, 1) \rightarrow (2, 2, 2) \rightarrow (4, 4)$$

contributes the monomial $\mu_{p'} = \beta_4 \cdot e_{2,1}^2 \cdot \beta_2 \cdot \alpha_2 \cdot \alpha_4 \in \mathbb{R}$ to the entry $(A_R^6)_{(1,4),(4,4)}$.

Definition 2.23 (Projection $\pi_{\mathfrak{A}}$ from $\mathcal{T}_{\mathfrak{A}}$ to $\mathcal{U}_{\mathfrak{A}}$) (A) Denote by $\mathbb{F}[x]$ the polynomial ring in one variable. Let $U_{\mathfrak{A}}$ be an s - U -chain. For every $u, \gamma \in \mathbb{Z}$ with

$$1 \leq \gamma \leq s \quad \text{and} \quad 0 \leq u \leq |U_{\mathfrak{A},\gamma}| - 1 \quad (2.22)$$

we put

$$T(U_{\mathfrak{A}})_{u,\gamma} = x^u \otimes_{\mathbb{F}} v_{\mathfrak{A},\gamma,1} \in \mathbb{F}[x] \otimes_{\mathbb{F}} V_{\mathbb{R}}. \quad (2.23)$$

We set

$$T(U_{\mathfrak{A}}) = \{T(U_{\mathfrak{A}})_{u,\gamma} \mid 1 \leq \gamma \leq s, 0 \leq u \leq (|U_{\mathfrak{A},\gamma}| - 1)\}. \quad (2.24)$$

Denote by $\mathcal{T}_{\mathfrak{A}} = \langle T(U_{\mathfrak{A}}) \rangle \subset \mathbb{F}[x] \otimes_{\mathbb{F}} V_{\mathbb{F}}$ and $\mathcal{U}_{\mathfrak{A}} = \langle \{U_{\mathfrak{A}}\} \rangle \subset V_{\mathbb{F}}$ the respective $\mathbb{F}$ -linear spans.

(B) There is a natural homomorphism $\omega : \mathbb{F}[x] \otimes_{\mathbb{F}} V_{\mathbb{F}} \rightarrow V_{\mathbb{F}}$

$$\omega(x^s \otimes_{\mathbb{F}} v) = A^s(v), \quad (2.25)$$

and, since $\{U_{\mathfrak{A}}\}$ is a subset of the basis $\mathbf{B}$ for $V_{\mathbb{F}}$, a natural projection ρ from $V_{\mathbb{F}}$ to the subspace $\mathcal{U}_{\mathfrak{A}}$. We denote by $\pi_{\mathfrak{A}} : \mathcal{T}_{\mathfrak{A}} \rightarrow \mathcal{U}_{\mathfrak{A}}$ the composition $\rho \circ \omega$. To define the matrix $M_{\mathfrak{A}}$ of $\pi_{\mathfrak{A}}$ we simply order the set $\{U_{\mathfrak{A}}\}$ by $v_{\mathfrak{A},\gamma,j} < v_{\mathfrak{A},\gamma',j'}$ if $\gamma < \gamma'$ or $\gamma = \gamma'$ and $j < j'$. We similarly order the set $T(U_{\mathfrak{A}})$ by setting $x^u \otimes_{\mathbb{F}} v_{\mathfrak{A},\gamma,1} < x^{u'} \otimes_{\mathbb{F}} v_{\mathfrak{A},\gamma',1}$ if $\gamma < \gamma'$ or $\gamma = \gamma'$ and $u < u'$. We denote by $M_{\mathfrak{A}}$ the $|U_{\mathfrak{A}}| \times |U_{\mathfrak{A}}|$ matrix of $\pi_{\mathfrak{A}}$ with respect to these ordered bases. That is, the entry of $M_{\mathfrak{A}}$ in the $x^u \otimes_{\mathbb{F}} v_{\mathfrak{A},\gamma,1}$ row and the $v_{\mathfrak{A},j,u'}$ column, with $1 \leq \gamma, j \leq s$ and $0 \leq u < |U_{\mathfrak{A},\gamma}|, 1 \leq u' < |U_{\mathfrak{A},j}|$ is

$$\langle A^u(v_{\mathfrak{A},\gamma,1}), v_{\mathfrak{A},j,u'} \rangle. \quad (2.26)$$

(See Fig. 4 and Example 3.14 for $M_{\mathfrak{A}}$ when $P = (4, 2, 1)$ and $\mathfrak{A} = (4, 2)$.)

(C) We define the *standard chain*

$$p_{\mathfrak{A},\gamma,j} : v_{\mathfrak{A},\gamma,1} \rightarrow v_{\mathfrak{A},\gamma,2} \rightarrow \cdots \rightarrow v_{\mathfrak{A},\gamma,j} \quad (2.27)$$

in $U_{\mathfrak{A},\gamma}$ from the initial vertex $v_{\mathfrak{A},\gamma,1}$ to $v_{\mathfrak{A},\gamma,j}$. We denote by $\mu_{\mathfrak{A},\gamma,j} \in \mathbb{R}$ the monomial of degree $j - 1$ in $\mathbb{R}$ arising as in Lemma 2.21, from this standard chain. We denote by $\mu_{\mathfrak{A},\gamma}$ the monomial

$$\mu_{\mathfrak{A},\gamma} = \prod_{1 \leq j \leq |U_{\mathfrak{A},\gamma}|} \mu_{\mathfrak{A},\gamma,j}. \quad (2.28)$$

$$M_{\mathfrak{A}} = \begin{pmatrix} & \mathbf{a} & \mathbf{e} & \mathbf{g} & \mathbf{f} & & \mathbf{d} & & \mathbf{b} & & \mathbf{c} \\ 1 \otimes_{\mathbb{F}} \mathbf{a} & 1 & 0 & \dots & & & & & & & 0 \\ x \otimes_{\mathbb{F}} \mathbf{a} & 0 & s_4 & 0 & 0 & & 0 & & z_4 & & 0 \\ x^2 \otimes_{\mathbb{F}} \mathbf{a} & 0 & 0 & s_2 s_4 & s_4 z_4 & & 0 & & 0 & & t_4 s_4 + z_4^2 \\ x^3 \otimes_{\mathbb{F}} \mathbf{a} & 0 & 0 & 0 & t_2 s_2 s_4 & & t_4 s_4 z_4 + z_4 t_4 s_4 + z_4^3 & & 0 & & 0 \\ x^4 \otimes_{\mathbb{F}} \mathbf{a} & 0 & 0 & 0 & 0 & & t_4 t_2 s_2 s_4 & & 0 & & 0 \\ 1 \otimes_{\mathbb{F}} \mathbf{b} & 0 & & \dots & & & 0 & & 1 & & 0 \\ x \otimes_{\mathbb{F}} \mathbf{b} & 0 & 0 & 0 & s_4 & & 0 & & 0 & & z_4 \end{pmatrix}$$

Fig. 4 Matrix $M_{\mathfrak{A}}$ for $\mathfrak{A} = (4, 2)$, $P = (4, 2, 1)$ (see Example 3.14)

The *distinguished* monomial of $\det M_{\mathfrak{A}}$ for the s - U -chain $\mathfrak{A}$ is the product

$$\mu_{\mathfrak{A}} = \prod_{1 \leq \gamma \leq s} \mu_{\mathfrak{A}, \gamma}. \quad (2.29)$$

Note 2.24 Evidently, the dimensions of the vector spaces $\mathcal{T}_{\mathfrak{A}}$ and $\mathcal{U}_{\mathfrak{A}}$ are the same. The degree of $\mu_{\mathfrak{A}, \gamma}$ satisfies

$$\deg \mu_{\mathfrak{A}, \gamma} = 1 + 2 + \dots + (|U_{\mathfrak{A}, \gamma}| - 1) = (|U_{\mathfrak{A}, \gamma}| - 1)(|U_{\mathfrak{A}, \gamma}|)/2$$

The entry $\langle A^u(v_{\mathfrak{A}, \gamma, 1}), v_{\mathfrak{A}, j, u'} \rangle$ of $M_{\mathfrak{A}}$ is the sum of the monomials μ_p of $\mathbb{R}$ corresponding as in (2.21) to length- $(u + 1)$ chains p from $v_{\mathfrak{A}, \gamma, 1}$ to $v_{\mathfrak{A}, j, u'}$ in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$.

The distinguished monomial $\mu_{\mathfrak{A}}$ occurs in the main diagonal term of $\det M_{\mathfrak{A}}$. For $\mathfrak{A}' \subset \mathfrak{A}$, $M_{\mathfrak{A}'}$ is a principal submatrix of $M_{\mathfrak{A}}$. For example, when $P = (4, 2, 1)$, $\mathfrak{A}' = (4)$, $\mathfrak{A} = (4, 2)$, $M_{\mathfrak{A}'}$ is the leading 5×5 principal submatrix of $M_{\mathfrak{A}}$ (Example 3.14).

3 Lower bound for $Q(P)$

The key steps in the proof of Theorem 3.9 involve an analysis of the sets of chains from the initial vertices of the s - U -chains to all the vertices of $U_{\mathfrak{A}}$. Each such set leads to a factorization of a monomial $v \in \mathbb{R}$ occurring in the expansion of the determinant $\det(M_{\mathfrak{A}})$. Using the sparseness of $A_{\mathbb{R}}$ —that simplifies our work—we show that there is a unique such factorization leading to the monomial $\mu_{\mathfrak{A}}$ of (2.29) (Proposition 3.5 for 2-chains and Theorem 3.7 for s -chains). This shows that the Jordan block partition $P_{A_{\mathbb{R}}}$ dominates $\lambda_U(\mathcal{D}_P)$ (Corollary 3.8).

The following result is well known.

Lemma 3.1 *Let V be an n -dimensional vector space over a field $\mathbb{F}$ and let A be a nilpotent matrix in $\text{Mat}_n(\mathbb{F}) = \text{End}_{\mathbb{F}} V_{\mathbb{F}}$. The Jordan type $Q = P_A = (q_1, \dots, q_r)$, $q_1 \geq \dots \geq q_r$ of A satisfies*

$$\forall i \in \{1, \dots, r\}, \quad \sum_{k=1}^i q_k = \max \{ \dim_{\mathbb{F}} \langle \mathbb{F}[A] \circ \{v_1, \dots, v_i\} \rangle \mid v_1, \dots, v_i \in V \}. \quad (3.1)$$

Proof By the action of A as X , V is a finitely generated torsion $k[X]$ -module, the direct sum of cyclic modules $V = \bigoplus_{k=1}^r F[A] \circ v_i \cong \bigoplus_{k=1}^r F[X]/(X^{q_k})$ whose lengths correspond to the Jordan type of A . This provides a set of cyclic vectors $z_1, \dots, z_r$ satisfying, for each i , $1 \leq i \leq r$, $\dim_F \langle F[A] \circ \{z_1, \dots, z_i\} \rangle = \sum_{k=1}^i q_k$. That $\sum_{k=1}^i q_k$ is the maximum dimension of a subspace generated by i vectors is a consequence of the uniqueness of the Jordan partition. $\square$

We now prepare to show that the monomial $\mu_{\mathfrak{A}}$ occurs only once in the expansion of $\det M_{\mathfrak{A}}$, where $M_{\mathfrak{A}}$ is the $|U_{\mathfrak{A}}| \times |U_{\mathfrak{A}}|$ matrix of Definition 2.23. There is a natural bijection η from the set of rows to the set of columns of $M_{\mathfrak{A}}$:

$$\eta : T(U_{\mathfrak{A}}) \rightarrow \{U_{\mathfrak{A}}\}; \quad \eta(x^u \otimes_F v_{\mathfrak{A}, \gamma, 1}) = v_{\mathfrak{A}, \gamma, u+1}, \quad 0 \leq u < |U_{\mathfrak{A}, \gamma}|, \quad 1 \leq \gamma \leq s. \quad (3.2)$$

Here $\det M_{\mathfrak{A}}$ is the sum of $|U_{\mathfrak{A}}|!$ terms, one for each permutation σ of $\{U_{\mathfrak{A}}\}$. The term corresponding to σ is

$$\text{sgn}(\sigma) \prod_{1 \leq \gamma \leq s, 0 \leq u < |U_{\mathfrak{A}, \gamma}|} \langle A^u(v_{\mathfrak{A}, \gamma, 1}), \sigma(v_{\mathfrak{A}, \gamma, u+1}) \rangle, \quad (3.3)$$

where the sign is that of σ . Indeed, the entry in row (u, i) and column $\sigma \circ \eta(u, i)$ of $M_{\mathfrak{A}}$ is the sum of monomials, one for each chain $c_{\gamma, u}$ of length $u + 1$ from $v_{\mathfrak{A}, \gamma, 1}$ to $\sigma(v_{\mathfrak{A}, \gamma, u+1})$ in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$. Consequently, the term of $\det M_{\mathfrak{A}}$ corresponding to σ is the sum of signed monomials $\text{sgn}(\sigma)v$, one for each array C_f of chains as in (ii) of Definition 3.2.

Definition 3.2 Let $U_{\mathfrak{A}}$ be an s -chain. A *chain factorization* f of a signed monomial $\pm v \in \mathbb{R}$ in the expansion of $\det M_{\mathfrak{A}}$, is a triple $f = (v_f, \sigma_f, C_f)$ where $C_f = \{c_{\gamma, u, f}\}$ is an array of chains, and $v_f = \{v_{\gamma, u, f}\}$ is an array of monomials, comprised of

- (i) A choice of a permutation σ_f of $\{U_{\mathfrak{A}}\}$. This determines the map

$$\sigma_f \circ \eta : T(U_{\mathfrak{A}}) \rightarrow \{U_{\mathfrak{A}}\}.$$

- (ii) C_f : For each pair (γ, u) , $1 \leq \gamma \leq s, 0 \leq u < |U_{\mathfrak{A}, \gamma}|$, the choice of a chain $c_{\gamma, u, f}$ of length $u + 1$ from $v_{\mathfrak{A}, \gamma, 1}$ to $\sigma_f(v_{\mathfrak{A}, \gamma, u+1})$ in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$.
- (iii) v_f : the array of monomials $v_{\gamma, u, f} = \mu_{c_{\gamma, u, f}}$, each the product of variables of $\mathbb{R}$ corresponding to the edges of $c_{\gamma, u, f}$ (Definition 2.20).

When $f = (v_f, \sigma_f, C_f)$ is a chain factorization of $\pm v$, then

$$|v| = \prod_{1 \leq \gamma \leq s, 0 \leq u < |U_{\mathfrak{A}, \gamma}|} v_{\gamma, u, f}, \quad (3.4)$$

and $\text{sgn}(\sigma_f) \cdot v$ is a signed monomial of $\mathbb{R}$ in the expansion of $\det(M_{\mathfrak{A}})$, before any cancellation.

We say that C_f is a complete set of chains for $\mathfrak{A}$, and that $f = (v_f, \sigma_f, C_f)$ *encodes* the chains C_f . We may omit subscripts on v_f, σ_f, C_f when f is clear.

Note 3.3 A complete set C_f of chains for $\mathfrak{A}$ includes one chain to each vertex of $\{U_{\mathfrak{A}}\}$, but the chains may include vertices outside of $\{U_{\mathfrak{A}}\}$: see Example 3.4.

Among the chain factorizations is $g_{\mathfrak{A}} = (v_{\mathfrak{A}}, e, C_{\mathfrak{A}})$ of the distinguished monomial $\mu_{\mathfrak{A}}$, given in Definition 2.23C, where $\sigma = e$, the identity permutation, and $C_{\mathfrak{A}}$ is comprised of the standard chains as in (2.27) from the initial vertex of each component chain of $U_{\mathfrak{A}}$ to every vertex of that chain.

In principle, a monomial term v in the expansion of the determinant $\det(M_{\mathfrak{A}})$ may equal $\mu_{\mathfrak{A}}$ even though the component chains encoded by the factorization of v do not lie in $U_{\mathfrak{A}}$. This is so as there may be occurrences of an α_k or β_k at the same level as an edge of $U_{\mathfrak{A}}$, but coming from an edge not in $U_{\mathfrak{A}}$ —a result of the Toeplitz condition that A_R commutes with B .

Example 3.4 (Standard chains and monomials of $\det M_{\mathfrak{A}}$) Let $P = (5, 4, 3, 3, 2, 1)$ and $\mathfrak{A} = (4, 2)$ (Fig. 3). Then $\mu_{\mathfrak{A}} = \mu_{\mathfrak{A},1} \cdot \mu_{\mathfrak{A},2}$,

$$\begin{aligned}\mu_{\mathfrak{A},1} &= s_5^{10} s_4^9 t_{31}^8 s_3^7 s_2^6 t_2^5 t_3^4 t_{31}^3 t_4^2 t_5, \\ \mu_{\mathfrak{A},2} &= s_5^5 s_4^4 t_{31}^3 t_4^2 t_5.\end{aligned}$$

Recall from Example 2.16 that $v_{\mathfrak{A},1,7} = (2, 2, 1) = (2, 2)$ and $v_{\mathfrak{A},2,5} = (3, 4)$ (see Fig. 3). The standard chains to these vertices are

$$\begin{aligned}(1, 5) &\rightarrow (1, 4) \rightarrow (1, 3, 1) \rightarrow (1, 3, 2) \rightarrow (1, 2) \rightarrow (1, 1) \rightarrow (2, 2), \\ (2, 5) &\rightarrow (2, 4) \rightarrow (2, 3, 1) \rightarrow (2, 3, 2) \rightarrow (3, 4),\end{aligned}$$

respectively, corresponding to factors $\mu_{\mathfrak{A},1,7} = s_5 s_4 t_{31} s_3 s_2 t_2$ and $\mu_{\mathfrak{A},2,5} = s_5 s_4 t_{31} t_4$, respectively of the distinguished monomial $\mu_{\mathfrak{A}}$. Let $\sigma = (v_{\mathfrak{A},1,7}, v_{\mathfrak{A},2,5})$, the transposition taking $(2, 2)$ to $(3, 4)$. There is a unique length 5 chain c from $v_{\mathfrak{A},2,1} = (2, 5)$ to $(2, 2) = \sigma(v_{\mathfrak{A},2,5})$,

$$c = (2, 5) \rightarrow (2, 4) \rightarrow (2, 3, 1) \rightarrow (2, 3, 2) \rightarrow (2, 2),$$

an encoding (factorization) of the monomial $p_c = s_5 s_4 t_{31} s_3$. There are two length 7 chains from $v_{\mathfrak{A},1,1} = (1, 5)$ to $(3, 4) = \sigma(v_{\mathfrak{A},1,7})$, namely,

$$\begin{aligned}c_1 &= (1, 5) \rightarrow (1, 4) \rightarrow (1, 3, 1) \rightarrow (1, 3, 2) \rightarrow (2, 4) \rightarrow (3, 5) \rightarrow (3, 4) \quad \text{and} \\ c_2 &= (1, 5) \rightarrow (1, 4) \rightarrow (2, 5) \rightarrow (2, 4) \rightarrow (2, 3, 1) \rightarrow (2, 3, 2) \rightarrow (3, 4),\end{aligned}$$

each encoding the monomial $p_{c_1} = s_5^2 s_4 t_{31} t_4 t_5$. Let μ denote the monomial

$$\mu = \mu_{\mathfrak{A}} \cdot (\mu_{\mathfrak{A},1,7})^{-1} \cdot (\mu_{\mathfrak{A},2,5})^{-1} \cdot (s_5 s_4 t_{31} s_3) \cdot (s_5^2 s_4 t_{31} t_4 t_5)$$

obtained from $\mu_{\mathfrak{A}}$ by replacing the monomials $\mu_{\mathfrak{A},1,7}$ and $\mu_{\mathfrak{A},2,5}$ for the standard chains in $U_{\mathfrak{A}}$ to $(2, 2)$ and $(3, 4)$, by p_c and p_{c_1} , respectively; let C_1, C_2 , respectively, denote the corresponding complete sets of chains. Then $-\mu$ occurs twice in the σ term of the expansion of $\det(M_{\mathfrak{A}})$. Once for the factorization $f_1 = (v_1, \sigma, C_1)$ and once for the factorization $f_2 = (v_2, \sigma, C_2)$ where for $k = \{1, 2\}$, C_k is the array with

$c_{14} = c$ and $c_{16} = c_k$ and all the other c_{γ, u, f_i} are the standard chains from $v_{\mathfrak{A}, \gamma, 1}$ to $\sigma(v_{\mathfrak{A}, \gamma, u+1})$ of length $u + 1$.

Note that there is a second length 8 chain c' from $v_{\mathfrak{A}, 1, 1} = (1, 5)$ to the vertex $v_{\mathfrak{A}, 1, 8} = (3, 3, 1)$ besides the standard one, namely

$$c' = (1, 5) \rightarrow (1, 4) \rightarrow (1, 3, 1) \rightarrow (1, 3, 2) \rightarrow (2, 4) \rightarrow (3, 5) \rightarrow (3, 4) \rightarrow (3, 3, 1),$$

that contains the vertex $(3, 5)$ not in $\{U_{\mathfrak{A}}\}$. Thus, there are other chain factorizations corresponding to the same transposition σ above that use the chain c' to the vertex $(3, 3, 1)$.

We begin the proof of our main results with the special case of 2- U -chains to illustrate our method. Recall from (2.18) that s_i and t_i are the coefficients of $A_{\mathbb{R}}$ on β_i and α_i , respectively.

Given a chain factorization $f = (v_f, \sigma_f, C_f)$ of the monomial v we may write

$$f = f_1 \cdot f_2 \cdots f_s, \quad v_f = v_{1,f} \cdot v_{2,f} \cdots v_{s,f}, \quad (3.5)$$

where f_{γ} collects all elements of C_f to vertices v in $U_{\mathfrak{A}, \gamma}$. We similarly write $g_{\mathfrak{A}} = g_{\mathfrak{A}, 1} \cdots g_{\mathfrak{A}, s}$ and $\mu_{\mathfrak{A}} = \mu_{\mathfrak{A}, 1} \cdots \mu_{\mathfrak{A}, s}$ or $\mu_{\mathfrak{A}} = \mu_1 \cdots \mu_s$ for short. Here σ_f determines for each vertex $v = v_{\mathfrak{A}, \gamma, u}$ in the γ component chain (a vertex corresponds to a column of $M_{\mathfrak{A}}$), the corresponding row $\eta^{-1} \sigma_f^{-1} v = A_{\mathbb{R}}^{u'}(v_{\mathfrak{A}, \gamma', 1})$ for a suitable power u' and index γ' . The sub-factorization $f_{\gamma'}$ determines a length u' chain in $\text{Diag}^{\text{aug}}(\mathcal{D}_P)$ from $v_{\mathfrak{A}, \gamma', 1}$ to v and a corresponding monomial factor of v_{γ} .

Proposition 3.5 *Let $U_{\mathfrak{A}} = U_{a,b}$ be a maximal 2- U -chain in the augmented diagram of $\mathcal{D}_P$ and suppose that $A_{\mathbb{R}} \in \text{Mat}_n(\mathbb{R}) \cap \mathcal{U}_{\mathbb{B}, \mathbb{R}}$ is simply adequate (Definition 2.17). Let $f = (v_f, \sigma_f, C_f)$ be a $\mathfrak{A}$ -factorization of a monomial $v = \pm \mu_{\mathfrak{A}}$. Then $f = g_{\mathfrak{A}}$, that is, $\sigma_f = \text{id}$ and every $c_{\gamma, u, f}$ is the standard chain from $v_{\mathfrak{A}, \gamma, 1}$ to $v_{\mathfrak{A}, \gamma, u+1}$. The coefficient of $\mu_{\mathfrak{A}}$ in $\det(M_{\mathfrak{A}})$ is 1.*

Proof We will show that $g_{\mathfrak{A}}$ is the unique chain factorization $f = (v_f, \sigma_f, C_f)$ for a monomial v of $\det(M_{\mathfrak{A}})$ such that v has both the minimum possible multiplicity of s_{a-1} (or of s_a if a is a singleton level of S_P), and the maximum possible multiplicity of s_b (or of z_b if b is a singleton level of S_P).

Let the monomial $v = \pm \mu_{\mathfrak{A}}$ in the expansion of $\det M_{\mathfrak{A}}$ have chain factorization $f = (v_f, \sigma_f, C_f)$ as in Definition 3.2. We write $v = v_1 \cdot v_2$ as in (3.5). We need to show that $f = g_{\mathfrak{A}}$. We have $a \geq b + 2$, so the almost rectangular levels of $\mathfrak{A}$ are $(a, a - 1, b, b - 1)$ or, in the special case of a length-3 spread $(a, a - 1, a - 2)$, or also $(a, b, b - 1)$, $(a, a - 1, b)$ when there is a singleton row. The component chains of $U_{\mathfrak{A}}$ are U_b and $U_{\mathfrak{A}, 2}$.

Claim A $\sigma_f(U_b) \subset U_b$ and $s_b, z_b \nmid v_2$.

Proof of claim Assume first that a is not a singleton level of S_P . The multiplicity of s_{a-1} as a factor of $\mu_{\mathfrak{A}}$ and also of $|v| = \mu_{\mathfrak{A}}$ is

$$(bn_b + (b - 1)n_{b-1}) + 2 \cdot \sum_{b < c \leq a-2} n_c + \sum_{a-2 < c} n_c, \quad (3.6)$$

where the first summand comes from chains to vertices in the b and $b - 1$ levels of U_b , the middle from chains to vertices in the left hook and right hook that are below level $a - 1$, and the right summand from chains to vertices in the top of the right hook of U_b . We will show that the sum of all but the last term in (3.6) is a lower bound for the multiplicity of s_{a-1} in any monomial of $\det(M_{\mathfrak{A}})$; and equality in (3.6) will greatly restrict the chain factorization f of $v = \pm\mu_{\mathfrak{A}}$. By (2.10) there is a unique saturated chain between two vertices $v = (0, i, a)$ and $v' = (0, i', a')$ at the extreme left of $\mathcal{D}_P$: there can be no return chain from a vertex $v'' = (u'', i'', a'')$ with $u'' > 0$ to v' . It follows that the chains encoded by f to the left-hook vertices of U_b are the same as those to the same vertices encoded by $g_{\mathfrak{A}}$. Since A_R is simply adequate the chain encoded by f to each vertex of the $b, b - 1$ levels of U_b must have at least one β_{a-1} edge encoded by s_{a-1} (see Note 2.19).

Similarly for each vertex of the right hook of U_b , lying at level $a - 2$ or below. There are $\sum_{c \geq a-1} n_c$ vertices at the top of the right hook of U_b that might be reached by a chain encoded by f_1 lying entirely on or above the $a - 1$ levels. Suppose now that $\kappa \leq \sum_{c \geq a-1} n_c$ chains encoded by f_2 to vertices of $U_{\mathfrak{A},2}$ dip to the $a - 2$ or lower level: each such chain contributes an s_{a-1} factor for v_2 . Then κ chains of f_1 to the top vertices of the right hook of U_b must lie entirely at or above the $a - 1$ level, in order for the power of s_{a-1} dividing v to not exceed the value given by (3.6) for $\mu_{\mathfrak{A}}$.

We now compare the factors s_b^k encoded by f and by $g_{\mathfrak{A}}$. The multiplicity of s_b as a factor of $\mu_{\mathfrak{A}}$ is the same as in μ_1 and is

$$(b(b-1)/2) \cdot (n_b + n_{b-1}) + (b-1) \cdot \sum_{c > b} n_c. \quad (3.7)$$

Here the first summand is from chains to vertices in the $b, b - 1$ levels, and the second counts $b - 1$ occurrences of s_b in each chain to a vertex of the right hook of U_b . We will see that the sum in (3.7) is an upper bound for the multiplicity of s_b as a factor of any monomial of $\det(M_{\mathfrak{A}})$ containing exactly the power of s_{a-1} specified in (3.6), so equality in $v = \pm\mu_{\mathfrak{A}}$ will further restrict f . For each vertex of U_b , the power of s_b encoded by a chain of f_1 to that vertex is no greater than the power of s_b encoded by the standard chain of $g_{\mathfrak{A},1}$ to the same vertex. The highest power of s_b that a chain p encoded by f_2 could contribute to v_2 is s_b^{b-2} , since $U_{\mathfrak{A},2}$ lies to the left of the rightmost column $0 : B$ of $\mathcal{D}_P$: by (2.11) there can be no chain from $(b - 1, b - 1, k)$ to (u, c, k') for $u < c$ when $b < c$.⁴ Thus, when one replaces κ standard chains of $g_{\mathfrak{A},1}$ to top vertices of the right hook of U_b by κ chains whose levels are entirely on or above the $a - 1$ level in f_2 , and makes up the missing s_{a-1}^{κ} power in $v = \pm\mu$ by adding κ chains of f_2 dipping to the $b - 1$ level, one loses at least a total multiplicity of κ for s_b in v in comparison to the multiplicity of s_b in μ given by (3.7). It follows that $\kappa = 0$.

Thus $s_b \nmid f_2$. If any component chain for f_1 began from $(2, a, 1) = v_{\mathfrak{A},2,1}$, it would contribute at least one less power of s_b to v_1 than the power contributed to μ_1 by the standard chain from $(1, a, 1)$ to the corresponding vertex encoded by $g_{\mathfrak{A},1}$, again by (2.10). Furthermore, there is no way to increase the total s_b power by choosing

⁴Corollary 2.6 and (2.13) give a sharper bound on the multiplicity of the edge β_b .

different chains than the standard chains encoded by g_1 to the vertices of U_b . This implies that all chains encoded by f_1 begin from $(1, a, 1)$: equivalently, $\sigma_f(U_b) \subset U_b$.

In the special case that the a level is a singleton we replace s_{a-1} by s_a above. Since $\mathfrak{A}$ is maximal, the case of b being a singleton level occurs only when b is the minimum level of an odd-length MCS of S_P (Lemma 2.13.) That we have included the added variables z_ℓ in (2.18) allows us to carry out the second part of the argument, replacing s_b by z_b and using (3.7) as lower bound for the multiplicity of z_b in v , when b is a singleton level in $\mathfrak{A}$. This completes the proof of Claim A.

Claim B *The restriction of σ_f to U_b is the identity, and $f_1 = g_{\mathfrak{A},1}$.*

The chains encoded by f_1 and $g_{\mathfrak{A},1}$ agree for vertices of the left or right hook of U_b . On the left hook by uniqueness of the chains to vertices of the left hook; and on the right hook by the argument above requiring each such vertex v to contribute s_b^{b-1} to μ_1 —the only way to do so is for the chain to the vertex v to pass through $(b-1, b-1, 1)$: then it is the standard chain to v . Since $s_b \nmid v_2$ by Claim A, in order for the power of s_b given by (3.7) to divide v , the chain encoded by f_1 to each vertex $v = (u, b, k)$ must contribute $u-1$ and that to $v = (u, b-1, k)$ must contribute u to the s_b power of v_1 : the only way to do so is for each such chain to pass through $(1, b, 1)$, and to be the standard chain to v . This completes the proof of Claim B.

Claim C *The restriction of σ_f to $U_{\mathfrak{A},2}$ is the identity, and $f_2 = g_{\mathfrak{A},2}$.*

We have shown that $\sigma_f(U_{\mathfrak{A},2}) \subset U_{\mathfrak{A},2}$. Since $v = \mu$ and $v_1 = \mu_1$ we have $v_2 = \mu_2$; since the factorization of v_1 is that of μ_1 , all chains contributing factors to v_2 must start from $(2, a, 1)$ and lie entirely within the chain $U_{\mathfrak{A},2}$. It follows similarly to the proof of Claim B that σ_f on $U_{\mathfrak{A},2}$ is the identity, and $f_2 = g_{\mathfrak{A},2}$.

This completes the proof of the Proposition. $\square$

Example 3.6 Let P have $n_i > 0$ parts i for $1 \leq i \leq 5$ and consider the 2- U -chain $U_{\mathfrak{A}} = U_{4,2}$ of $\mathcal{D}_P$. By (3.6) the power of s_3 dividing the $\mu_{\mathfrak{A}}$ term of $\det M_{\mathfrak{A}}$ is $(2n_2 + n_1) + (n_5 + n_4 + n_3)$. By (3.7) the power of s_2 dividing the $\mu_{\mathfrak{A}}$ term of $\det M_{\mathfrak{A}}$ is $(n_2 + n_1) + (n_5 + n_4 + n_3)$: this is the maximum power of s_2 possible for terms containing exactly the power of s_3 given by (3.6) and only $u_{\mathfrak{A}}$ attains this maximum. For $P = (5, 4, 3, 3, 2, 1)$ of Example 3.4 and Fig. 3 these powers are s_3^7 and s_2^6 .

Theorem 3.7 *Let $U_{\mathfrak{A}}$ be a maximal s - U -chain in the augmented diagram of $\mathcal{D}_P$ and suppose that $A_{\mathbf{R}} \in \text{Mat}_n(\mathbf{R}) \cap \mathcal{U}_{B,\mathbf{R}}$ is simply adequate. Let $f = (v_f, \sigma_f, C_f)$ be a $\mathfrak{A}$ -factorization of $v = \pm \mu_{\mathfrak{A}}$. Then $f = g_{\mathfrak{A}}$. The coefficient of $\mu_{\mathfrak{A}}$ in $\det(M_{\mathfrak{A}})$ is 1.*

Proof We will show this by induction on s . The case $s = 1$ is essentially Claim B of Proposition 3.5, and the case $s = 2$ is Proposition 3.5. Suppose that the theorem is known for $s-1$ and all partitions P . Let the monomial $v = \pm \mu_{\mathfrak{A}}$ in the expansion of $\det M_{\mathfrak{A}}$ have chain factorization $f = (v_f, \sigma_f, C_f)$ as in Definition 3.2. Let $a = a_1, b = a_s$. We will show first that $\sigma_f(U_b) \subset U_b$, then that the restriction σ_f to U_b is the identity, and $f_1 = g_{\mathfrak{A},1}$. Then induction suffices to complete the result.

Claim A $\sigma_f(U_b) \subset U_b$ and $s_b, z_b \nmid v_2$.

Assume first that $m = a_{s-1}$ is not a singleton of S_P , and consider the variable s_{m-1} . The multiplicity of s_{m-1} in $\mu_{\mathfrak{A}}$ is

$$(bn_b + (b-1)n_{b-1}) + 2 \cdot \sum_{b < c \leq m-2} n_c + \sum_{m-2 < c} n_c. \quad (3.8)$$

We will see that equality in (3.8) for $v_f = \pm \mu_{\mathfrak{A}}$ will greatly restrict f . The chains encoded by f to the left hook vertices of U_b are the same as those encoded by $g_{\mathfrak{A}}$ to the same vertices. By the definition of A_R the chain encoded by f_1 to each vertex of U_b at the $b, b-1$ levels must have at least one β_m edge encoded by an s_m factor. Likewise for the chain encoded by f_1 to any right hook vertex at level $m-2$ or below. However, $\kappa \leq \sum_{c \geq m} n_c$ chains encoded by f might lie entirely at level $m-1$ or above; the missing s_{m-1} powers for these chains must be replaced by those in κ chains encoded by $f_2, \dots, f_s$ to vertices in $U_{\mathfrak{A}} - U_b$.

The multiplicity of s_b as a factor of $\mu_{\mathfrak{A}}$ is given by (3.7). The right hook vertices of U_b each contribute s_b^{b-1} to μ_1 . Any chain to a vertex of one of the top $s-1$ component chains $\mu_{\mathfrak{A},i}, 2 \leq i \leq s$ can contribute at most s_b^{b-2} to v , by (2.11) or (2.13). As before for $s=2$, we conclude that $\kappa=0$, that $s_b \nmid v_2 \cdots v_s$, and that $\sigma_f \mid U_b \subset U_b$. In the special case that the a -level is a singleton we replace s_{a-1} by s_a above; in case U_b is a singleton level, we replace s_b by z_b in the above argument, as in the proof of Claim A of Proposition 3.5. This proves Claim A.

The same argument as in the proof of Proposition 3.5 Claim B now shows that $\sigma_f \mid U_b$ is the identity, and $f_1 = g_{\mathfrak{A},1}$. We have also shown that $\sigma_f(U_{\mathfrak{A}} - U_b) \subset U_{\mathfrak{A}} - U_b$, and that the portion of the factorization of v coming from vertices in $\{U_{\mathfrak{A}} - U_b\}$ involves no edges of $\mathcal{D}_P$ below level $m-1$. Since these chains encoded by f to vertices of $U_{\mathfrak{A}} - U_b$ start and end in $U_{\mathfrak{A}} - U_b$ they don't involve edges in the hooks of U_b .

Now peel the chain $U_b = U_{\mathfrak{A},1}$ from $U_{\mathfrak{A}}$ to form $U_{\mathfrak{A}'}, \mathfrak{A}' = (a_1, \dots, a_{s-1})$, and regard its image $s-1$ chain U' with label $\mathfrak{A}'_{P'} = (a_1-2, \dots, a_{s-1}-2)$ in $\mathcal{D}_{P'}$, where P' is obtained from P by peeling off U_b and omitting parts below U_b :

$$n_i(P') = \begin{cases} n_{i+2}(P) & \text{for } i \geq b-1 \\ 0 & \text{for } i \leq b-2. \end{cases}$$

Since $U_b = U_{\mathfrak{A},1}$ was an outside chain, $\mathcal{D}_{P'} \subset \mathcal{D}_P$. The induction step applied to U' and $\mathcal{D}_{P'}$ now shows that the portion $f_2 \cdot f_3 \cdots f_s$ of the factorization f corresponding to vertices of $U_{\mathfrak{A}} - U_b$ agrees with the factorization $g_2 \cdot g_3 \cdots g_s$ of the corresponding portion of $g = g_{\mathfrak{A}}$. Putting this together with $f_1 = g_{\mathfrak{A},1}$ we conclude that $f = g_{\mathfrak{A}}$. This completes the proof of the induction step and the theorem. $\square$

Corollary 3.8 *Let k be an infinite field, let R be the polynomial ring of (2.18), and suppose that $A_R \in \text{End}_R V_F$ is the simply adequate element of $\mathcal{U}_{B,R}$. Then the Jordan partition P_{A_R} over the quotient field F of R satisfies, $P_{A_R} \geq \lambda_U(\mathcal{D}_P)$.*

Proof Let $A_{\mathbb{R}} \in \mathcal{U}_{B,\mathbb{F}}$ be simply adequate. Let $U_{\mathfrak{A}(s)}$ be a maximum-length s U -chain of $\mathcal{D}_P$, $1 \leq s \leq r_P$. Theorem 3.7 shows that for each s , the projections $\pi_{\mathfrak{A}(s)} : \mathcal{T}_{\mathfrak{A}(s),A} \rightarrow \mathcal{U}_{\mathfrak{A}(s)}$ have the maximum possible rank $|U_{\mathfrak{A}(s)}|$. By Lemma 3.1 the partition $P_{A_{\mathbb{R}}}$ has first s parts summing to at least $|U_{\mathfrak{A}(s)}|$. By Definition 2.14 and (1.4) this implies that over the quotient field $\mathbb{F}$ we have $P_{A_{\mathbb{R}}} \geq \lambda_U(\mathcal{D}_P)$. $\square$

Theorem 3.9 *Let $\mathbb{k}$ be an infinite field. Then $Q(P) \geq \lambda_U(\mathcal{D}_P)$. In particular, there is an adequate $A \in \mathcal{U}_B$ over $\mathbb{k}$ satisfying $P_A \geq \lambda_U(\mathcal{D}_P)$.*

Proof Let $A_{\mathbb{R}} \in \mathcal{U}_{B,\mathbb{R}}$ be simply adequate. Let $U_{\mathfrak{A}(s)}$ be a maximum-length s U -chain of $\mathcal{D}_P$, $1 \leq s \leq r_P$. Theorem 3.7 shows that when $\det(M_{\mathfrak{A}(s)})$ is expanded into a sum of monomials of $\mathbb{R}$ over $\mathbb{k}$ corresponding each to a chain factorization $f = (v_f, \sigma_f, C_f)$, $C_f = \{c_{iu}\}$ as in Definition 3.2, there is a unique term $\mu_{\mathfrak{A}(s)}$. Since $\mathbb{k}$ is an infinite field, we can choose $\theta : \mathbb{F} \rightarrow \mathbb{k}$, that is, substitute for the variables of $\mathbb{R}$, so that each $\theta(\det(M_{\mathfrak{A}(s)})) \neq 0$. Thus, $A = \theta(A_{\mathbb{R}}) \in \text{Mat}_n(\mathbb{k}) \cap \mathcal{U}_B$ satisfies $\text{rank}(\pi_{\mathfrak{A}(s)}) = |U_{\mathfrak{A}(s)}|$ on $\mathcal{T}_{\mathfrak{A}(s),A}$ for each $U_{\mathfrak{A}(s)}$. As in Corollary 3.8, this implies $P_A \geq \lambda_U(\mathcal{D}_P)$. By the irreducibility of $\mathcal{U}_B$, we have $Q(P) \geq \lambda_U(\mathcal{D}_P)$. $\square$

Write $Q(P)_{\mathbb{k}} = P_A$ for a generic $A \in \mathcal{N}_B$, $B = J_P$ over the field $\mathbb{k}$. Write $Q(P)_{\mathbb{R}}$ over the reals $\mathbb{R}$ as $Q(P)_{\mathbb{R}} = (q_1(\mathbb{R}), \dots, q_{r(P)}(\mathbb{R}))$ where $q_1(\mathbb{R}) \geq q_2(\mathbb{R}) \geq \dots$, and write $\lambda_U(\mathcal{D}_P) = (\lambda_{1,U}(P), \lambda_{2,U}(P), \dots)$ where $\lambda_{1,U}(P) \geq \lambda_{2,U}(P) \geq \dots$.

Corollary 3.10 *Let $\mathbb{k}$ be an infinite field. Fix a partition $P \vdash n$ and an integer k , $1 \leq k \leq r_P$. Assume that $\sum_{i=1}^k q_i(\mathbb{R}) = \sum_{i=1}^k \lambda_{i,U}(P)$. Then the analogous equality holds for $Q(P)_{\mathbb{k}}$.*

Proof This follows from $Q(P)_{\mathbb{R}} \geq Q(P)_{\mathbb{k}}$ and Theorem 3.9. $\square$

The second author has shown

Theorem [15] *The minimum part of $\lambda(\mathcal{D}_P)$ is equal to the minimum part of $\lambda_U(\mathcal{D}_P)$.*

This together with Theorem 3.9 and (1.7) shows

Corollary 3.11 [15] *Let $\mathbb{k}$ be an infinite field. The minimum part of $Q(P)$ is equal to the minimum part m_P of $\lambda_U(\mathcal{D}_P)$.*

An explicit formula for m_P in terms of P is given in [15]. Our result also has the corollary of extending P. Oblak's Theorem 1.2 to an infinite field $\mathbb{k}$. These show

Corollary 3.12 ([18] $r_P = 2$, [15] $r_P = 3$) *Let $\mathbb{k}$ be an infinite field. When $r_P \leq 3$, $Q(P) = \lambda_U(\mathcal{D}_P) = \lambda(\mathcal{D}_P)$ and can be explicitly written in terms of P .*

Example 3.13 For $P = (5, 4, 3^3, 2^3, 1^2)$, $r_P = 3$. The maximum-length simple U -chains are U_3 of length $|U_3| = 3(3) + 3(2) + 2(2) = 19$, where the two hooks each have length 2, and also U_2 . The maximum-length 2- U chain is $U_{4,2}$ of length $|U_{4,2}| = 25$. So $Q(P) = (19, 6, 1)$.

Fig. 5 $\text{Diag}(\mathcal{D}_P)$ and variables in $\mathbb{R}$ for $P = (4, 2, 1)$

Example 3.14 Recall from Example 1.4 that for $P = (4, 2, 1)$ we have $r_P = 2$; from this and Oblak's index formula (1.1) we have $Q(P) = (5, 2)$. We use the notation of Example 1.4 and Fig. 1 for the basis $\mathcal{B}$ of V . The simply adequate $A_{\mathbb{R}}$ of (2.18) and Corollary 3.8 with coefficients in $\mathbb{R}$ satisfies

$$A_{\mathbb{R}} \cdot a = z_4 b + s_4 e, \quad A_{\mathbb{R}} \cdot e = t_4 c + s_2 g, \quad A_{\mathbb{R}} \cdot g = t_2 f, \quad (3.9)$$

where s_4, s_2, t_4, t_2, z_4 are the variables of $\mathbb{R}$. Since $A_{\mathbb{R}}$ commutes with B , these determine $A_{\mathbb{R}}$. The matrix $M_{\mathfrak{A}}$ is given in Fig. 4; the entries can be obtained from Fig. 5 by multiplying the variables of $\mathbb{R}$ labelling the edges of the chain corresponding to each entry.

The determinant of $M_{\mathfrak{A}}$ has a unique non-zero term: its unique chain factorization arises from $\mu_{\mathfrak{A}}$:

$$\det M_{\mathfrak{A}} = \mu_{\mathfrak{A}} = 1 \cdot s_4 \cdot s_2 s_4 \cdot t_2 s_2 s_4 \cdot t_4 t_2 s_4 s_2 \cdot 1 \cdot z_4 = s_4^4 s_2^3 t_2^2 t_4 z_4.$$

When, as here, there is a unique maximum length chain from the source a to the sink d of $\mathcal{D}_P$, any matrix A as in (2.18) such that the values of $s_i, t_i, t_{i,k}, z_\ell$ are non-zero in $\mathbb{k}$ satisfies, the maximum part of the Jordan type of A is the index $i(Q(P))$. Here, setting the variables of $\mathbb{R}$ of (3.9) equal to 1 yields the matrix $A \in \mathcal{U}_B$ of (1.6) satisfying $\dim \langle \mathbb{k}[A] \cdot \{a, b\} \rangle = 7$ and $\dim \langle \mathbb{k}[A] \cdot \{a\} \rangle = 5$.

Also the matrix $\text{Mat}_{\mathfrak{A}'}$ for $\mathfrak{A}' = (2)$ is the leading 5×5 minor of $\text{Mat}_{\mathfrak{A}}$, with determinant the monomial $\mu_{\mathfrak{A}',1}$ in $\mathbb{R}$. This shows that here $P_A = Q(P) = (5, 2)$, as stated in Example 1.4 and Corollary 3.12.

Remark 3.15 Even if the questions of Sect. 1 be answered, it still appears subtle to understand, given a stable partition Q , the set of partitions P such that $Q(P) = Q$: see [19] for some results and open problems in this direction.

We have wondered why this problem of understanding the map $P \rightarrow Q(P)$ was not posed much earlier in the literature. Perhaps it was supplanted by another natural problem, to characterize maximal vector spaces of commuting matrices [28]. One colleague speculated that the map was not considered much earlier because of the difficulty of computing large enough examples by hand to see interesting behavior.

Recent work of E. Friedlander, J. Pevtsova, and A. Suslin on modular representations has involved both Jordan types and the variety of commuting nilpotent matrices [8].

Remark 3.16 (The field k) J.R. Britnell and M. Wildon have shown that over the finite field $k(p^r)$ having p^r elements, the Jordan types $P_A = (d+1, d-1)$ and $P_B = (d, d)$ occur for two commuting nilpotent matrices A, B if and only if d is not divisible by $p(p^{2r}-1)/2$ for $p > 2$, and by $2(4^r-1)$ when $p = 2$ [6, Proposition 4.12]. However, when k is infinite, they show that there are always commuting matrices A, B in these two orbits [6, Remark 4.15].

G. McNinch showed in [17, Example 22] (see also [3, Example 2.18]) that the class of a generic linear combination $A + tB$ of certain nilpotent A, B in a tensor product $V = V_d \otimes V_2$ over an infinite field k depends on the characteristic of k : this class is $(d+1, d-1)$ for d invertible, but is (d, d) when d divides $\text{char } k$. It appears to be open whether the set of pairs of Jordan partitions for the similarity classes of two commuting nilpotent matrices depends on $\text{char } k$ when k is an infinite field.

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⁵R. Basili [arXiv:1202.3369](https://arxiv.org/abs/1202.3369), revised in June 2012.

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