Monomials, binomials and Riemann–Roch

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Abstract The Riemann–Roch theorem on a graph G is related to Alexander duality in combinatorial commutative algebra. We study the lattice ideal given by chip firing on G and the initial ideal whose standard monomials are the G -parking functions. When G is a saturated graph, these ideals are generic and the Scarf complex is a minimal free resolution. Otherwise, syzygies are obtained by degeneration. We also develop a self-contained Riemann–Roch theory for Artinian monomial ideals.

Keywords Riemann–Roch theory for graphs · Combinatorial commutative algebra · Chip firing games · Laplacian matrix of a graph · Lattice ideals and their Betti numbers · Alexander duality of monomial ideals · Scarf complex

1 Introduction

We examine the Riemann–Roch theorem on a finite graph G , due to Baker and Norine [3], through the lens of combinatorial commutative algebra. Throughout this paper, G is undirected and connected, has n nodes, and multiple edges are allowed, but we do not allow loops. Its Laplacian is a symmetric $n \times n$ -matrix Λ_G with non-positive integer entries off the diagonal and kernel spanned by $\mathbf{e} = (1, 1, \dots, 1)$. Divisors on G are identified with Laurent monomials $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$. The *chip firing* moves are binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} \geq 0$ and $\mathbf{u} - \mathbf{v}$ is in the lattice spanned by the columns of Λ_G . The lattice ideal I_G spanned by such binomials is here called the *toppling*

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ideal of the graph G . It was introduced by Perkinson, Perlman and Wilmes [11, 15], following an earlier study of the inhomogeneous version of I_G by Cori, Rossin and Salvý [6].

For any fixed node, the toppling ideal I_G has a distinguished initial monomial ideal M_G . This monomial ideal was studied by Postnikov and Shapiro [12], and it is characterized by the property that the standard monomials of M_G are the G -parking functions. We construct free resolutions for both I_G and M_G , and we study their role for Riemann–Roch theory on G . For an illustration, consider the complete graph on four nodes, $G = K_4$. The chip firing moves on K_4 are the integer linear combinations of the columns of

$$A_G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}. \quad (1)$$

The toppling ideal is the lattice ideal in $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, x_2, x_3, x_4]$ that represents $\text{image}_{\mathbb{Z}}(A_G)$:

$$I_G = \langle \underline{x_1^3} - x_2 x_3 x_4, \underline{x_2^3} - x_1 x_3 x_4, \underline{x_3^3} - x_1 x_2 x_4, \underline{x_1 x_2 x_3} - x_4^3, \\ \underline{x_1^2 x_2^2} - x_3^2 x_4^2, \underline{x_1^2 x_3^2} - x_2^2 x_4^2, \underline{x_2^2 x_3^2} - x_1^2 x_4^2 \rangle. \quad (2)$$

This ideal is *generic* in the sense of Peeva and Sturmfels [10], as each of the seven binomials contains all four variables. The minimal free resolution is given by the *Scarf complex*

$$0 \leftarrow \mathbb{K}[\mathbf{x}] \leftarrow \mathbb{K}[\mathbf{x}]^7 \leftarrow \mathbb{K}[\mathbf{x}]^{12} \leftarrow \mathbb{K}[\mathbf{x}]^6 \leftarrow 0. \quad (3)$$

The seven binomials in (2) form a Gröbner basis of I_G , with the underlined monomials generating the initial ideal M_G . That monomial ideal has the irreducible decomposition

$$M_G = \langle x_1, x_2^2, x_3^3 \rangle \cap \langle x_1, x_2^3, x_3^2 \rangle \cap \langle x_1^2, x_2, x_3^3 \rangle \cap \langle x_1^2, x_2^3, x_3 \rangle \cap \langle x_1^3, x_2, x_3^2 \rangle \cap \langle x_1^3, x_2^2, x_3 \rangle.$$

The ideal M_G is the *tree ideal* of [9, §4.3.4]. Its standard monomials are in bijection with the 16 spanning trees. Its Alexander dual is generated by the six socle elements

$$x_2 x_3^2, x_2^2 x_3, x_1 x_3^2, x_1 x_2^2, x_1^2 x_3, x_1^2 x_2. \quad (4)$$

These correspond to the *maximal parking functions* studied in combinatorics; see [5, 12]. We claim that the duality seen in Figs. 4.2 and 4.3 of [9] is the same as that expressed in the Riemann–Roch Theorem for G . This will be made precise in Sections 3 and 4.

The present article is organized as follows. Section 2 is concerned with the case when G is a *saturated graph*, meaning that any two nodes i and j are connected by at least one edge. We show that here I_G is a generic lattice ideal, and we determine its minimal free resolution and its Hilbert series in the finest grading. The Scarf complex of the initial monomial ideal M_G is supported on the barycentric subdivision of the

$(n - 2)$ -simplex [12, §6], and this lifts to the Scarf complex of the lattice ideal I_G by [10, Corollary 5.5].

In Section 3 we revisit the Riemann–Roch formula

$$\text{rank}(D) - \text{rank}(K - D) = \text{degree}(D) - \text{genus} + 1. \quad (5)$$

We prove this formula in an entirely new setting: the role of the curve is played by a monomial ideal, and that of the divisors D and K is played by monomials $\mathbf{x}^{\mathbf{b}}$ and $\mathbf{x}^{\mathbf{K}}$. The identity (5) is shown for monomial ideals that are Artinian, level, and reflection invariant. This includes the parking function ideals M_G derived from saturated graphs G .

In Section 4 we extend our results to the case of graphs G that are not saturated, and we rederive Riemann–Roch for graphs as a corollary. Here M_G is still an initial ideal of I_G , but the choice of term order is more delicate [11, §5]. One choice is the cost function used by Baker and Shokrieh for the integer program in [4, Theorem 4.1]. The Scarf complexes in Section 2 support cellular free resolutions of I_G and M_G , but these resolutions are usually far from minimal. We conclude with several open questions.

This paper demonstrates how Riemann–Roch theory embeds into combinatorial commutative algebra. Our main results are Theorems 2, 13 and 25. These build on earlier work, notably [1] and [12], but they go much further and are new in their current form.

When this collaboration started in the summer of 2011, both authors were unaware of the articles [11, 15] written on similar topics by David Perkinson and his students at Reed College. As our point of departure, we chose to focus on chip firing in the most classical case of undirected graphs, but with the tacit understanding that our ideals and modules generalize to directed graphs, arithmetic graphs, simplicial complexes, matroids, abelian networks, or any of the other extensions seen in the recent chip firing literature (cf. [2]).

2 Saturated graphs

In this section, we assume that the graph G has u_{ij} edges between node i and node j , where u_{ij} is a positive integer, for $i \neq j$. However, we do not allow loops, so that $u_{11} = u_{22} = \cdots = u_{nn} = 0$. Thus, in the language of [12], G is a *saturated graph*. We shall see that, under this hypothesis, the lattice ideal I_G is generic, and an explicit combinatorial description of its minimal free resolution can be given. Throughout this paper we work in the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$ over an arbitrary field $\mathbb{K}$.

We begin by explicitly showing the generators of the lattice ideal I_G in the case $n = 4$.

Example 1 If G is a saturated graph on $[4] = \{1, 2, 3, 4\}$ then I_G is generated by

$$\begin{aligned} & \underline{x_1^{u_{12}+u_{13}+u_{14}} - x_2^{u_{12}} x_3^{u_{13}} x_4^{u_{14}}}, \quad \underline{x_2^{u_{12}+u_{23}+u_{24}} - x_1^{u_{12}} x_3^{u_{23}} x_4^{u_{24}}}, \\ & \underline{x_3^{u_{13}+u_{23}+u_{34}} - x_1^{u_{13}} x_2^{u_{23}} x_4^{u_{34}}}, \quad \underline{x_4^{u_{14}+u_{24}+u_{34}} - x_1^{u_{14}} x_2^{u_{24}} x_3^{u_{34}}}, \end{aligned}$$

$$\begin{aligned} & \underline{x_1^{u_{13}+u_{14}} x_2^{u_{23}+u_{24}}} - x_3^{u_{13}+u_{23}} x_4^{u_{14}+u_{24}}, \quad \underline{x_1^{u_{12}+u_{14}} x_3^{u_{23}+u_{34}}} - x_2^{u_{12}+u_{23}} x_4^{u_{14}+u_{34}}, \\ & \underline{x_2^{u_{12}+u_{24}} x_3^{u_{13}+u_{34}}} - x_1^{u_{12}+u_{13}} x_4^{u_{24}+u_{34}}. \end{aligned}$$

Here the u_{ij} are arbitrary positive integers. These binomials form a Gröbner basis. The initial ideal M_G is generated by the underlined monomials. The minimal free resolution of I_G has the form (3). The same holds for M_G , as was shown in [12, Corollary 6.9]. The minimal resolution of M_G is given by the Scarf complex, which is depicted in Fig. 1.

We now state our main result in this section. For disjoint subsets I and J of $[n]$ we set

$$\mathbf{x}^{I \rightarrow J} := \prod_{i \in I} x_i^{\sum_{k \in J} u_{ik}}.$$

A *split* of the set $[n] = \{1, 2, \dots, n\}$ is an unordered pair (I, J) of non-empty disjoint subsets I and J whose union equals $[n]$. The number of splits equals $2^{n-1} - 1$. With each split (I, J) we associate the following binomial which is well-defined up to sign:

$$\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}. \quad (6)$$

These are precisely the seven binomials in Example 1, one for each split (I, J) .

Let $\text{Cyc}_{n,k}$ denote the set of cyclically ordered partitions of the set $[n]$ into k blocks. Each element of $\text{Cyc}_{n,k}$ has the form $(I_1, I_2, \dots, I_k)$, where $I_1 \cup I_2 \cup \dots \cup I_k = [n]$ is a partition. We regard the $(I_1, I_2, \dots, I_k)$ as formal symbols, subject to the identifications

$$(I_1, I_2, \dots, I_{k-1}, I_k) = (I_2, I_3, \dots, I_k, I_1) = \dots = (I_k, I_1, \dots, I_{k-2}, I_{k-1}).$$

We write $\mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,k}}$ for the free $\mathbb{K}[\mathbf{x}]$ -module generated by these symbols. The rank of this free module equals the number of cyclically ordered partitions, namely

$$|\text{Cyc}_{n,k}| = (k-1)! \cdot S_{n,k}, \quad (7)$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e., the number of partitions of the set $[n]$ into k blocks. Let $\mathcal{CYC}_G$ denote the following complex of free $\mathbb{K}[\mathbf{x}]$ -modules:

$$0 \leftarrow \mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,1}} \leftarrow \mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,2}} \leftarrow \mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,3}} \leftarrow \dots \leftarrow \mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,n}} \leftarrow 0, \quad (8)$$

where the boundary map from $\mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,r}}$ to $\mathbb{K}[\mathbf{x}]^{\text{Cyc}_{n,r-1}}$ is given by the formula

$$\begin{aligned} (I_1, I_2, I_3, \dots, I_r) &\mapsto \sum_{s=1}^{r-1} (-1)^{s-1} \mathbf{x}^{I_s \rightarrow I_{s+1}} (I_1, \dots, I_{s-1}, I_s \cup I_{s+1}, I_{s+2}, \dots, I_r) \\ &\quad - \mathbf{x}^{I_r \rightarrow I_1} \cdot (I_2, I_3, \dots, I_{r-1}, I_1 \cup I_r). \end{aligned} \quad (9)$$

In this formula it is assumed that $n \in I_r$, so as to ensure that all signs are consistent.

Fig. 1 The barycentric subdivision of the $(n-2)$ -simplex supports minimal free resolutions for toppling ideals of saturated graphs with n nodes and their initial ideals. Shown here is the Scarf complex (3) for $n=4$, with seven vertices, 12 edges and six triangles

Theorem 2 Let G be a saturated graph. The toppling ideal I_G is a generic lattice ideal. It is minimally generated by the $2^{n-1}-1$ binomials (6), these form a reverse lexicographic Gröbner basis, the complex $\mathcal{CYC}_G$ coincides with the Scarf complex, and this complex minimally resolves $\mathbb{K}[\mathbf{x}]/I_G$.

Proof We begin by noting that $\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}$ actually lies in the ideal I_G . To see this, let e_I denote the incidence vector in $\{0, 1\}^n$ that represents the subset I of $[n]$. The i th coordinate of the vector $\Lambda_G \cdot e_I$ is equal to $\sum_{k \in J} u_{ik}$ if $i \in I$, and it is $-\sum_{k \in I} u_{ik}$ if $i \in J$. Hence $\Lambda_G \cdot e_I$ is represented algebraically by $\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}$, which is hence in I_G .

Fix any reverse lexicographic term order on $\mathbb{K}[\mathbf{x}]$ that has x_n as the smallest variable, and let $\text{in}(I_G)$ denote the initial monomial ideal of I_G . Since I_G is a lattice ideal, x_n is a non-zero divisor and it does not divide any of the generators of $\text{in}(I_G)$. We may thus regard $\text{in}(I_G)$ as an Artinian ideal in $\mathbb{K}[\mathbf{x}_{\setminus n}] = \mathbb{K}[x_1, \dots, x_{n-1}]$. The index of the Laplacian lattice image $_{\mathbb{Z}}(\Lambda_G)$ in its saturation $\{u \in \mathbb{Z}^n : u_1 + \dots + u_n = 0\}$ equals the number T_G of spanning trees of G . Hence $\text{in}(I_G)$ has T_G standard monomials in $\mathbb{K}[\mathbf{x}_{\setminus n}]$.

Let M_G denote the ideal generated by the initial monomials of the binomials in (6):

$$M_G = \langle \mathbf{x}^{I \rightarrow [n] \setminus I} : I \text{ non-empty subset of } [n-1] \rangle. \quad (10)$$

By construction, the inclusion $M_G \subseteq \text{in}(I_G)$ holds. The monomial ideal M_G was studied in [12] and shown to have precisely T_G standard monomials. Indeed, the standard monomials of M_G are in bijection with the n -reduced divisors. It is known in the chip firing literature (cf. [3, 5, 6]) that their number equals the number T_G of spanning trees. Hence M_G and $\text{in}(I_G)$ are Artinian of the same colength in $\mathbb{K}[\mathbf{x}_{\setminus n}]$, so they must be equal:

$$M_G = \text{in}(I_G).$$

Therefore the binomials (6) form a Gröbner basis, and hence a generating set, of I_G .

The ideal I_G is a generic lattice ideal, in the sense of [10], because all n variables $x_1, \dots, x_n$ occur in the binomial (6). Here we are using the fact that G is saturated. By [10, Theorem 4.2], the Scarf complex is the (essentially unique) minimal free resolution of I_G .

It remains to be seen that the Scarf complex is equal to $\mathcal{C}\mathcal{Y}\mathcal{C}_G$. Postnikov and Shapiro [12, Corollary 6.9] showed that the Scarf complex of the initial ideal M_G is supported on the barycentric subdivision of the $(n-2)$ -simplex, as shown in Fig. 1. The Scarf resolution has the format (8), but with $\mathbb{K}[\mathbf{x}]$ replaced by $\mathbb{K}[\mathbf{x}_n]$. Here, we label the cells in that barycentric subdivision with ordered partitions $(I_1, I_2, \dots, I_r)$ satisfying $n \in I_r$. The boundary maps in the Scarf resolution are then given by (9), namely, by the sum ranging from $s = 1$ to $s = r - 1$, but without the additional term $-\mathbf{x}^{I_r \rightarrow I_1} \cdot (I_2, I_3, \dots, I_{r-1}, I_1 \cup I_r)$.

We pass from the Scarf resolution of M_G to that of I_G by the combinatorial rule in [10, Theorem 5.4]. This adds precisely one term to the boundary of each Scarf simplex of M_G . In our case, that additional term is precisely the one above, and we get (9). $\square$

Example 3 Returning to Example 1, with the seven binomials in that order, here are the matrices over $\mathbb{K}[x_1, x_2, x_3, x_4]$ that represent the first and second syzygies in $\mathcal{C}\mathcal{Y}\mathcal{C}_4$:

$$\begin{pmatrix} (1,2,3,4) & (2,1,3,4) & (1,3,2,4) & (3,1,2,4) & (2,3,1,4) & (3,2,1,4) & (1,2,3,4) & \dots & (12,3,4) \\ (1,2,3,4) & -\mathbf{x}^{2 \rightarrow 3,4} & -\mathbf{x}^{3,4 \rightarrow 2} & -\mathbf{x}^{3 \rightarrow 2,4} & -\mathbf{x}^{2,4 \rightarrow 3} & 0 & 0 & -\mathbf{x}^{2,3 \rightarrow 4} & \dots & 0 \\ (2,1,3,4) & -\mathbf{x}^{3,4 \rightarrow 1} & -\mathbf{x}^{1 \rightarrow 3,4} & 0 & 0 & -\mathbf{x}^{3 \rightarrow 1,4} & -\mathbf{x}^{1,4 \rightarrow 3} & 0 & \dots & 0 \\ (3,1,2,4) & 0 & 0 & -\mathbf{x}^{2,4 \rightarrow 1} & -\mathbf{x}^{1 \rightarrow 2,4} & -\mathbf{x}^{1,4 \rightarrow 2} & -\mathbf{x}^{2 \rightarrow 1,4} & 0 & \dots & -\mathbf{x}^{4 \rightarrow 1,2} \\ (12,3,4) & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{x}^{1 \rightarrow 2,3} & \dots & \mathbf{x}^{1,2 \rightarrow 3} \\ (12,3,4) & \mathbf{x}^{1 \rightarrow 2} & \mathbf{x}^{2 \rightarrow 1} & 0 & 0 & 0 & 0 & 0 & \dots & -\mathbf{x}^{3 \rightarrow 4} \\ (13,2,4) & 0 & 0 & \mathbf{x}^{1 \rightarrow 3} & \mathbf{x}^{3 \rightarrow 1} & 0 & 0 & 0 & \dots & 0 \\ (23,1,4) & 0 & 0 & 0 & 0 & \mathbf{x}^{2 \rightarrow 3} & \mathbf{x}^{3 \rightarrow 2} & -\mathbf{x}^{4 \rightarrow 1} & \dots & 0 \end{pmatrix},$$

$$\begin{pmatrix} (1,2,3,4) & (1,3,2,4) & (2,1,3,4) & (2,3,1,4) & (3,1,2,4) & (3,2,1,4) \\ (1,2,3,4) & \mathbf{x}^{3 \rightarrow 4} & 0 & 0 & -\mathbf{x}^{4 \rightarrow 3} & 0 \\ (2,1,3,4) & 0 & 0 & \mathbf{x}^{3 \rightarrow 4} & 0 & -\mathbf{x}^{4 \rightarrow 3} \\ (1,3,2,4) & 0 & \mathbf{x}^{2 \rightarrow 4} & -\mathbf{x}^{4 \rightarrow 2} & 0 & 0 \\ (3,1,2,4) & 0 & 0 & 0 & -\mathbf{x}^{4 \rightarrow 2} & \mathbf{x}^{2 \rightarrow 4} \\ (2,3,1,4) & -\mathbf{x}^{4 \rightarrow 1} & 0 & 0 & \mathbf{x}^{1 \rightarrow 4} & 0 \\ (3,2,1,4) & 0 & -\mathbf{x}^{4 \rightarrow 1} & 0 & 0 & \mathbf{x}^{1 \rightarrow 4} \\ (1,2,3,4) & -\mathbf{x}^{2 \rightarrow 3} & -\mathbf{x}^{3 \rightarrow 2} & 0 & 0 & 0 \\ (2,3,1,4) & 0 & 0 & 0 & \mathbf{x}^{2 \rightarrow 3} & \mathbf{x}^{3 \rightarrow 2} \\ (2,1,3,4) & 0 & 0 & -\mathbf{x}^{1 \rightarrow 3} & -\mathbf{x}^{3 \rightarrow 1} & 0 \\ (1,3,2,4) & 0 & \mathbf{x}^{1 \rightarrow 3} & 0 & 0 & \mathbf{x}^{3 \rightarrow 1} \\ (3,1,2,4) & 0 & 0 & 0 & 0 & -\mathbf{x}^{1 \rightarrow 2} \\ (12,3,4) & \mathbf{x}^{1 \rightarrow 2} & 0 & \mathbf{x}^{2 \rightarrow 1} & 0 & 0 \end{pmatrix}.$$

Note that the seven binomial generators of I_G appear as the 2×2 -minors of the 3×2 -matrices seen in the six pairs of columns within the 7×12 -matrix of first syzygies. The syzygies of the ideal M_G generated by the underlined monomials in Example 1 are found by replacing with 0 all monomials that have the symbol “4” to the left of the arrow.

One immediate application of our minimal free resolution is a formula for the Hilbert series of the ring $\mathbb{K}[\mathbf{x}]/I_G$ in its natural grading by the group $\text{Div}(G) = \mathbb{Z}^n / \text{image}_{\mathbb{Z}}(\Delta_G)$. As is customary in chip firing theory [2–4, 11], we consider the decomposition

$$\text{Div}(G) = \mathbb{Z} \oplus \text{Div}_0(G),$$

where $\mathbb{Z}$ records the degree of a divisor on G , and $\text{Div}_0(G)$ is the finite subgroup of divisors of degree 0. The order of $\text{Div}_0(G)$ is the number of spanning trees of G . Let t and q denote the generators of the group algebra $\mathbb{Z}[\text{Div}(G)]$ corresponding to this decomposition. The Hilbert series of $\mathbb{K}[\mathbf{x}]/I_G$ equals $1/(1-t)$ times the Hilbert series of $\mathbb{K}[\mathbf{x}_n]/M_G$, where $M_G = \text{in}(I_G)$ is the initial ideal in (10). The latter series equals

$$\sum_{\mathbf{u}} t^{|\mathbf{u}|} q^{\text{div}(\mathbf{u})}.$$

This finite sum is over all elements $\mathbf{u} \in \mathbb{N}^{n-1}$ that represent parking functions on G with respect to the last node n , and $\text{div}(\mathbf{u})$ denotes the class of the reduced divisor of degree 0 given by the vector $(\mathbf{u}, -\sum u_i)$. See [11, Theorem 6.14] for a nice formula, due to Merino [8], which expresses this sum with $q = 1$ in terms of the Tutte polynomial of G .

We fix the natural epimorphism from the semigroup algebra of $\mathbb{N}^{n-1}$ to that of $\text{Div}(G)$:

$$\psi : \mathbb{Z}[\mathbf{x}_n] \rightarrow \mathbb{Z}[\text{Div}(G)], \quad \mathbf{x}^{\mathbf{u}} \mapsto t^{|\mathbf{u}|} q^{\text{div}(\mathbf{u})}.$$

With this notation, our minimal free resolution in Theorem 2 implies the following result:

Corollary 4 *The Hilbert series of $K[\mathbf{x}]/I_G$ in the grading by the group $\text{Div}(G)$ equals*

$$\begin{aligned} & \frac{1}{(1-t)} \sum_{\mathbf{u}} t^{|\mathbf{u}|} q^{\text{div}(\mathbf{u})} \\ &= \frac{1 - \sum_{k=1}^n (-1)^k \sum_{(I_1, I_2, \dots, I_k) \in \text{Cyc}_{n,k}} \psi(\mathbf{x}^{I_1 \rightarrow I_2} \mathbf{x}^{I_2 \rightarrow I_3} \dots \mathbf{x}^{I_{k-1} \rightarrow I_k})}{(1-t)(1-\psi(x_1))(1-\psi(x_2)) \dots (1-\psi(x_{n-1}))}. \end{aligned}$$

Proof It suffices to note that the $\mathbb{Z}^{n-1}$ -degree of the basis element $(I_1, I_2, \dots, I_k)$ of the free $\mathbb{K}[\mathbf{x}_n]$ -module in the k th step of the resolution of M_G is the exponent vector of $\mathbf{x}^{I_1 \rightarrow I_2} \mathbf{x}^{I_2 \rightarrow I_3} \dots \mathbf{x}^{I_{k-1} \rightarrow I_k}$. This monomial does not contain x_n because $n \in I_k$. $\square$

We close this section with a combinatorial recipe for the socle monomials modulo M_G . These are the monomials $\mathbf{x}^{\mathbf{u}}$ that are not in M_G but $\mathbf{x}^{\mathbf{u}} x_i \in M_G$ for $i = 1, \dots, n-1$. Each permutation of $[n-1]$ corresponds to a flag $\mathcal{T}$ of subsets $\emptyset \subset T_1 \subset T_2 \subset T_3 \subset \dots \subset T_{n-1}$. The flag is complete, meaning that each inclusion is strict and each $T_i \setminus T_{i-1}$ is a singleton. Let $\overline{T}_i$ denote the set complement of T_i with respect to $[n]$. For instance, $\overline{T}_{n-1} = \{n\}$.

Corollary 5 *The socle monomials of $\mathbb{K}[\mathbf{x}_{\setminus n}]/M_G$ are precisely the $(n-1)!$ monomials*

$$\mathbf{s}_T = \text{lcm}(\mathbf{x}^{T_1 \rightarrow \bar{T}_1}, \mathbf{x}^{T_2 \rightarrow \bar{T}_2}, \dots, \mathbf{x}^{T_{n-1} \rightarrow \bar{T}_{n-1}}) / (x_1 x_2 \cdots x_{n-1}), \quad (11)$$

where T runs over all complete flags of subsets of $[n-1]$.

Proof The Scarf complex of M_G is a minimal free resolution and it is supported on the barycentric subdivision of the $(n-2)$ -simplex, by [12, Corollary 6.9] and our discussion above. Each facet in that barycentric subdivision corresponds to a complete flag T . The vertices of that facet are labeled by $\mathbf{x}^{T_1 \rightarrow \bar{T}_1}, \mathbf{x}^{T_2 \rightarrow \bar{T}_2}, \dots, \mathbf{x}^{T_{n-1} \rightarrow \bar{T}_{n-1}}$ in the Scarf complex, and the monomial label of the facet is their least common multiple. Facets of the Scarf complex are in bijection with the irreducible components of M_G and also with the socle monomials modulo M_G . By [9, Corollary 6.20], each socle monomial is multiplied by the product of all variables to give the monomial label of the corresponding facet. $\square$

Remark 6 Our results hold verbatim for all generic sublattices of finite index in the root lattice $A_n = \{u \in \mathbb{Z}^n : \sum_{i=1}^n u_i = 0\}$, so we recover the Voronoi theory of [1, 2]. We posit that our commutative algebra derivation of their Voronoi theory is a natural and useful one, and that it opens up new and unexpected connections. For instance, Gröbner bases of lattice ideals are fundamental for integer programming [14]. One original source for that application is Herbert Scarf's seminal work on neighborhood systems in economics. A key example that motivated Scarf was the *Leontief system* [13, §2A]. It turns out that the lattices representing Leontief systems are precisely our generic lattices here. The Gröbner basis property stated in Theorem 2 is in fact equivalent to [13, Theorem 2.2].

3 A Riemann–Roch theorem for monomial ideals

In this section we fix an arbitrary Artinian monomial ideal M in a polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_m]$. We focus on Alexander duality [9, §5], and we establish the Riemann–Roch formula (5) in this new context. Towards the end of this section, and in the next section, we will recover the Riemann–Roch formula for graphs from the Riemann–Roch formula for monomial ideals. To begin with, we need to gather the ingredients, that is, we need to redefine the notions of divisor, genus, rank and degree.

The role of *divisors* on the monomial ideal M is played by Laurent monomials $\mathbf{x}^{\mathbf{b}}$.

Definition 7 (Rank of a monomial) The *rank* of a monomial $\mathbf{x}^{\mathbf{b}}$ with respect to M is one less than the minimum degree of any monomial $\mathbf{x}^{\mathbf{a}}$ that satisfies $\mathbf{x}^{\mathbf{b}} \in \langle \mathbf{x}^{\mathbf{a}} \rangle \setminus \mathbf{x}^{\mathbf{a}} M$.

This definition is restricted to honest monomials $\mathbf{x}^{\mathbf{b}}$, where $\mathbf{b} \geq 0$. Just before the statement of Theorem 13, we shall extend the definition of rank to all Laurent monomials.

The rank measures how deeply a monomial $\mathbf{x}^{\mathbf{b}}$ sits inside the ideal M . We have $\text{rank}(\mathbf{x}^{\mathbf{b}}) \geq 0$ if $\mathbf{x}^{\mathbf{b}} \in M$ and $\text{rank}(\mathbf{x}^{\mathbf{b}}) = -1$ otherwise. Rank zero monomials form

the *border* of M . Let $\text{MonSoc}(M) = \{\mathbf{x}^{\mathbf{c}} \notin M \mid x_i \mathbf{x}^{\mathbf{c}} \in M \forall i\}$ denote the set of *socle monomials* of $\mathbb{K}[\mathbf{x}]/M$. See Fig. 2 for a picture of a monomial ideal. The ideal generators are the large black circles, and monomials in M are labeled by their rank. The socle elements are the black squares, and other standard monomials are white squares.

Definition 8 (Reflection invariance) A monomial ideal M is *reflection invariant* if there exists a *canonical monomial* $\mathbf{x}^{\mathbf{K}}$ such that the map $\phi : \mathbf{x}^{\mathbf{c}} \mapsto \mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{c}}$ defines an involution of the set $\text{MonSoc}(M)$. This requires that every socle monomial divides $\mathbf{x}^{\mathbf{K}}$.

Using notation as in [9, §5], we note that our Artinian monomial ideal M is reflection invariant with canonical monomial $\mathbf{x}^{\mathbf{K}}$ if and only if the following identity holds:

$$M^{[\mathbf{K}+\mathbf{e}]} = \langle \text{MonSoc}(M) \rangle. \quad (12)$$

Here $\mathbf{e} = (1, 1, \dots, 1)$ and $M^{[\mathbf{K}+\mathbf{e}]}$ is the *Alexander dual* of M with respect to $\mathbf{K} + \mathbf{e}$.

Definition 9 (Genus) The ideal M is *level* if all socle monomials have the same degree. If this holds then one plus that degree is called the *genus* of M , denoted $g = \text{genus}(M)$.

Example 10 Let M be the ideal generated by the seven underlined monomials in (2). Then M is level of genus $g = 4$, because all six socle monomials in (4) are cubics, and M is reflection invariant. The canonical monomial $\mathbf{x}^{\mathbf{K}} = x_1^2 x_2^2 x_3^2$ has degree $2g - 2 = 6$. But the rank of $\mathbf{x}^{\mathbf{K}}$ is equal to $g - 2 = 2$, as can be seen from the following lemma.

For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbf{Z}^m$, we abbreviate $\text{degree}^+(\mathbf{u}) = \sum_{i: u_i > 0} u_i$.

Lemma 11 Let M be an Artinian monomial ideal. Then every monomial $\mathbf{x}^{\mathbf{b}}$ satisfies

$$\text{rank}(\mathbf{x}^{\mathbf{b}}) = \min_{\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(M)} \text{degree}^+(\mathbf{b} - \mathbf{c}) - 1. \quad (13)$$

Proof The condition $\mathbf{x}^{\mathbf{b}} \in \langle \mathbf{x}^{\mathbf{a}} \rangle \setminus \mathbf{x}^{\mathbf{a}} M$ in Definition 7 is equivalent to $\mathbf{x}^{\mathbf{b}-\mathbf{a}} \in \mathbb{K}[\mathbf{x}] \setminus M$. Maximizing the degree of $\mathbf{x}^{\mathbf{c}} = \mathbf{x}^{\mathbf{b}-\mathbf{a}}$ subject to this condition is equivalent to minimizing the degree of $\mathbf{x}^{\mathbf{a}}$. But, since M is Artinian, the maximal degree among its finitely many standard monomials is attained by one of the socle monomials $\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(M)$. $\square$

Remark 12 Formula (13) resembles the formula in [3, Lemma 2.2] for the rank of a divisor on a finite graph. We shall exploit this resemblance at the end of this section.

Fig. 2 The monomial ideal $M = \langle x^9, x^6y^4, x^5y^7, x^2y^8, y^{11} \rangle$ is Riemann–Roch of genus 12, with canonical monomial x^9y^{13} and $\text{Soc}(M) = \{x^8y^3, x^5y^6, x^4y^7, xy^{10}\}$. Generators and socle monomials highlighted in dark. Monomials in M are labeled with their rank. The square boxes correspond to the standard monomials of $\mathbb{K}[x]/M$. The dotted lines mark the boundary of the staircase region of $M^{[\mathbf{K}+\mathbf{e}]}$. Note that the identity (12) holds

The formula in (13) can be rewritten to be reminiscent of S-pairs for Gröbner bases:

$$\text{rank}(\mathbf{x}^{\mathbf{b}}) = \min_{\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(M)} \text{degree} \left(\frac{\text{lcm}(\mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}})}{\mathbf{x}^{\mathbf{c}}} \right) - 1. \quad (14)$$

We now define the *rank* of an arbitrary Laurent monomial $\mathbf{x}^{\mathbf{b}}$ by the formula (13). This is consistent with Definition 7, and it is the natural extension to monomials some of whose exponents are negative. The following main result gave this section its title:

Theorem 13 *Let M be a monomial ideal that is Artinian, level, and reflection invariant. Then M satisfies the Riemann–Roch formula, i.e., every Laurent monomial $\mathbf{x}^{\mathbf{b}}$ satisfies*

$$\text{rank}(\mathbf{x}^{\mathbf{b}}) - \text{rank}(\mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{b}}) = \text{degree}(\mathbf{x}^{\mathbf{b}}) - \text{genus}(M) + 1. \quad (15)$$

Proof We denote $\mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{c}}$ by $\bar{\mathbf{c}}$. Using the formula for rank shown in Lemma 11, the left hand side of (15) equals

$$\min_{\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(M)} \text{degree}^+(\mathbf{b} - \mathbf{c}) - \min_{\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(M)} \text{degree}^+(\bar{\mathbf{c}} - \mathbf{b}). \quad (16)$$

For any socle monomial $\mathbf{x}^c$ we have $\text{degree}^+(\mathbf{b} - \mathbf{c}) - \text{degree}^+(\mathbf{c} - \mathbf{b}) = \text{degree}(\mathbf{x}^b) - \text{degree}(\mathbf{x}^c) = \text{degree}(\mathbf{x}^b) - (\text{genus}(M) - 1)$, and hence

$$\text{degree}^+(\mathbf{b} - \mathbf{c}) = \text{degree}^+(\mathbf{c} - \mathbf{b}) + \text{degree}(\mathbf{x}^b) - (\text{genus}(M) - 1). \quad (17)$$

Taking the minimum of $\text{degree}^+(\mathbf{b} - \mathbf{c})$ over $\mathbf{x}^c \in \text{MonSoc}(M)$, (17) implies

$$\begin{aligned} & \min_{\mathbf{x}^c \in \text{MonSoc}(M)} \text{degree}^+(\mathbf{b} - \mathbf{c}) \\ &= \min_{\mathbf{x}^c \in \text{MonSoc}(M)} (\text{degree}^+(\mathbf{c} - \mathbf{b}) + \text{degree}(\mathbf{x}^b) - (\text{genus}(M) - 1)) \\ &= \left(\min_{\mathbf{x}^c \in \text{MonSoc}(M)} \text{degree}^+(\mathbf{c} - \mathbf{b}) \right) + \text{degree}(\mathbf{x}^b) - (\text{genus}(M) - 1). \end{aligned} \quad (18)$$

Since the map ϕ is an involution, we can replace $\bar{\mathbf{c}}$ by $\mathbf{c}$ in the second row of (16). It then follows from (18) that (16) is equal to $\text{degree}(\mathbf{x}^b) - \text{genus}(M) + 1$, as desired. $\square$

Remark 14 Theorem 13 can be extended to monomial ideals M that are Artinian and reflection invariant but not necessarily level. Such M arise as initial ideals from directed regular (indegree = outdegree) graphs. Following [1, §2.3], we define $\text{genus}_{\min}(M)$ as one minus the minimum degree of a socle monomial of M , and $\text{genus}_{\max}(M)$ as one minus the maximum degree of a socle monomial of M . Using a technique similar to that in the proof of Theorem 13, we derive the following Riemann–Roch inequalities:

$$\begin{aligned} \text{genus}_{\min}(M) - 1 &\leq \text{degree}(\mathbf{x}^b) - \text{rank}(\mathbf{x}^b) + \text{rank}(\mathbf{x}^K/\mathbf{x}^b) \\ &\leq \text{genus}_{\max}(M) - 1. \end{aligned} \quad (19)$$

Of course, the above inequality generalizes the Riemann–Roch formula (15): if the ideal M is also level then $\text{genus}_{\max}(M) = \text{genus}_{\min}(M) = \text{genus}(M)$, and the Riemann–Roch formula (15) immediately follows from the inequalities (19).

We say that a monomial ideal M is *Riemann–Roch* if it is Artinian, level, and reflection invariant. See Fig. 2 for an example in two variables. In what follows we assume that M is a Riemann–Roch monomial ideal. The next corollaries are formal consequences of the Riemann–Roch formula, as is the case for algebraic curves and graphs.

Corollary 15 *If $\mathbf{x}^b$ is a multiple of $\mathbf{x}^K$, then $\text{rank}(\mathbf{x}^b) = \text{degree}(\mathbf{x}^b) - \text{genus}(M)$.*

Proof If $\mathbf{x}^K$ divides $\mathbf{x}^b$, then $\text{rank}(\mathbf{x}^K/\mathbf{x}^b) = -1$. Plugging this equation into the Riemann–Roch formula gives the assertion. $\square$

Note that, by definition, the degree of the canonical monomial $\mathbf{x}^K$ equals twice the socle degree. We record the following general facts about the canonical monomial $\mathbf{x}^K$.

Corollary 16 *The canonical monomial of a Riemann–Roch monomial ideal M satisfies*

$$\text{degree}(\mathbf{x}^{\mathbf{K}}) = 2 \cdot \text{genus}(M) - 2 \quad \text{and} \quad \text{rank}(\mathbf{x}^{\mathbf{K}}) = \text{genus}(M) - 2.$$

Experts will note that the rank is off by one when compared to the canonical divisor of an algebraic curve or metric graph. This discrepancy will be addressed in Remark 23 below. We now prepare for an analogue of Clifford’s theorem on special divisors.

Lemma 17 *The rank is superadditive for monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ of non-negative rank:*

$$\text{rank}(\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}) \geq \text{rank}(\mathbf{x}^{\mathbf{a}}) + \text{rank}(\mathbf{x}^{\mathbf{b}}). \quad (20)$$

Proof Consider an arbitrary monomial $\mathbf{x}^{\mathbf{c}}$ of degree at most $\text{rank}(\mathbf{x}^{\mathbf{a}}) + \text{rank}(\mathbf{x}^{\mathbf{b}})$ such that $\mathbf{x}^{\mathbf{c}}$ divides $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}$. The following formulas define monomials $\mathbf{x}^{\mathbf{c}'}$ and $\mathbf{x}^{\mathbf{c}''}$ such that $\mathbf{x}^{\mathbf{c}'}$ divides $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{c}''}$ divides $\mathbf{x}^{\mathbf{b}}$:

$$c'_i = \begin{cases} a_i & \text{if } c_i \geq a_i, \\ c_i & \text{otherwise,} \end{cases} \quad \text{and} \quad c''_i = \begin{cases} c_i - a_i & \text{if } c_i \geq a_i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\mathbf{x}^{\mathbf{c}'} \cdot \mathbf{x}^{\mathbf{c}''} = \mathbf{x}^{\mathbf{c}}$. This implies that either $\text{degree}(\mathbf{x}^{\mathbf{c}'}) \leq \text{rank}(\mathbf{x}^{\mathbf{a}})$ or $\text{degree}(\mathbf{x}^{\mathbf{c}'}) \leq \text{rank}(\mathbf{x}^{\mathbf{b}})$. In other words, either $\mathbf{x}^{\mathbf{a}}/\mathbf{x}^{\mathbf{c}'}$ or $\mathbf{x}^{\mathbf{b}}/\mathbf{x}^{\mathbf{c}''}$ is in M , and hence, their product $\frac{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}}{\mathbf{x}^{\mathbf{c}}}$ is also in M . From this we infer the inequality (20) as follows: Since $\mathbf{x}^{\mathbf{c}}$ is an arbitrary monomial of degree less than or equal to $\text{rank}(\mathbf{x}^{\mathbf{a}}) + \text{rank}(\mathbf{x}^{\mathbf{b}})$, and $\mathbf{x}^{\mathbf{c}}$ divides $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}$, we know that any monomial that “defines” the rank of $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}$ (i.e., a monomial $\mathbf{x}^{\mathbf{d}}$ of minimum degree such that $\mathbf{x}^{\mathbf{d}}$ divides $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}$ and $\frac{\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}}{\mathbf{x}^{\mathbf{d}}} \notin M$) has degree strictly greater than $\text{rank}(\mathbf{x}^{\mathbf{a}}) + \text{rank}(\mathbf{x}^{\mathbf{b}})$. Hence, $\text{rank}(\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}}) \geq \text{rank}(\mathbf{x}^{\mathbf{a}}) + \text{rank}(\mathbf{x}^{\mathbf{b}})$. $\square$

Corollary 18 (Clifford’s Theorem) *Let $\mathbf{x}^{\mathbf{b}}$ be a monomial dividing $\mathbf{x}^{\mathbf{K}}$ such that both $\text{rank}(\mathbf{x}^{\mathbf{b}})$ and $\text{rank}(\mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{b}})$ are non-negative. Then $\text{rank}(\mathbf{x}^{\mathbf{b}}) \leq (\text{degree}(\mathbf{x}^{\mathbf{b}}) - 1)/2$.*

Proof Lemma 17 and Corollary 16 imply

$$\text{rank}(\mathbf{x}^{\mathbf{b}}) + \text{rank}(\mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{b}}) \leq \text{rank}(\mathbf{x}^{\mathbf{K}}) = \text{genus}(M) - 2.$$

From the Riemann–Roch formula we have

$$\text{rank}(\mathbf{x}^{\mathbf{b}}) - \text{rank}(\mathbf{x}^{\mathbf{K}}/\mathbf{x}^{\mathbf{b}}) = \text{degree}(\mathbf{x}^{\mathbf{b}}) - (\text{genus}(M) - 1).$$

The desired conclusion follows by adding these two identities and dividing by 2. $\square$

The construction of all Riemann–Roch monomial ideals of genus g works as follows. We first fix a monomial $\mathbf{x}^{\mathbf{K}}$ with degree $2g - 2$. Next we choose a set $\mathcal{M}$ of

monomials of degree $g - 1$ that divide $\mathbf{x}^{\mathbf{K}}$. Then there exists a unique Artinian monomial ideal M whose socle is spanned by the monomials in $\mathcal{M}$ and their complements relative to $\mathbf{x}^{\mathbf{K}}$:

$$\text{MonSoc} = \mathcal{M} \cup \{ \mathbf{x}^{\mathbf{K}} / \mathbf{x}^{\mathbf{b}} : \mathbf{x}^{\mathbf{b}} \in \mathcal{M} \}. \quad (21)$$

Namely, the ideal M is the intersection of the irreducible ideals $\langle x_1^{c_1+1}, \dots, x_m^{c_m+1} \rangle$ where $\mathbf{x}^{\mathbf{c}}$ runs over the set MonSoc . Then M is Artinian, level, and reflection invariant.

We shall now make the connection to the Riemann–Roch theorem for graphs. As in Section 2, we let G denote a saturated graph on n nodes, with $u_{ij} > 0$ edges between nodes i and j , and M_G the initial monomial ideal in $\mathbb{K}[x_1, \dots, x_{n-1}]$ of the toppling ideal I_G with respect to a reverse lexicographic term order having x_n as smallest variable.

Theorem 19 *Let G be a saturated graph with n vertices, e edges, and node i having degree d_i . Then the monomial ideal M_G is Riemann–Roch with canonical monomial*

$$\mathbf{x}^{\mathbf{K}} = \prod_{i=1}^{n-1} x_i^{d_i+u_{in}-2} \quad \text{and} \quad \text{genus}(M_G) = e - n + 2. \quad (22)$$

Proof The monomial ideal M_G is Artinian, and it is level because all the socle monomials $s_{\mathcal{T}}$ in Corollary 5 have the same degree $e - n + 1$. This quantity is the cyclotomic number (or genus) of the graph G , which, by [5], coincides with the common degree of all maximal parking functions. There is a natural involution ϕ on the set of $(n - 1)!$ maximal flags $\mathcal{T}$ of subsets in $[n - 1]$. It takes a flag $\mathcal{T} : T_1 \subset T_2 \subset \dots \subset T_{n-2} \subset T_{n-1} \subset T_n$ to the reverse flag $\phi(\mathcal{T}) : T_{n-1} \setminus T_{n-2} \subset T_{n-1} \setminus T_{n-3} \subset \dots \subset T_{n-1} \setminus T_1 \subset T_{n-1}$. Using the identification between flags and socle monomials in Corollary 5, we have

$$s_{\phi(\mathcal{T})} = \mathbf{x}^{\mathbf{K}} / s_{\mathcal{T}}, \quad (23)$$

where $\mathbf{x}^{\mathbf{K}}$ is the monomial defined in (22). Hence, M_G is also reflection invariant. $\square$

Remark 20 Not every Riemann–Roch monomial ideal arises as an initial monomial ideal M_G for a connected graph G . To see this, note that the number of socle monomials of M_G is at most $m!$ where m is the number of variables of M_G . On the other hand, Riemann–Roch monomial ideals can in general have more than $m!$ socle monomials. Furthermore, the initial monomial ideal M_G for a connected graph G is not necessarily Riemann–Roch. To see this, consider the four-cycle C_4 . Its initial ideal is $M_{C_4} = \langle x_1, x_2, x_3 \rangle^2$, with socle monomials x_1, x_2 and x_3 , and M_{C_4} is not reflection invariant.

In the remainder of this section we show how the familiar Riemann–Roch theorem for graphs is derived from Theorem 19. While the proof still assumes that G is saturated, that hypothesis will be removed in the next section. The following algebraic

definitions are valid for any undirected connected graph G on $[n]$. Here G need not be saturated.

The Laurent polynomial ring $\mathbb{K}[\mathbf{x}^{\pm 1}] = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a module over the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$. The Laplacian lattice image $_{\mathbb{Z}}(\Lambda_G)$ is a sublattice of rank $n - 1$ in $\mathbb{Z}^n$, and we write L_G for the corresponding *lattice module*, as in [9, Definition 9.11]. Thus, L_G is the $\mathbb{K}[\mathbf{x}]$ -submodule of $\mathbb{K}[\mathbf{x}^{\pm 1}]$ generated by all Laurent monomials $\mathbf{x}^{\mathbf{w}}$ that have degree zero in the grading by the group $\text{Div}(G)$. If G is saturated then L_G is generic and the Scarf complex is a minimal free resolution by [9, Theorem 9.24]. That Scarf complex is precisely the *Delaunay triangulation* in [1], and our point here is to redevelop the Amini–Manjunath approach in the language of commutative algebra.

Consider the set of all Laurent monomials $\mathbf{x}^{\mathbf{c}}$ that are in the socle of the module L_G :

$$\text{MonSoc}(L_G) = \{\mathbf{x}^{\mathbf{c}} \notin L_G \mid x_i \mathbf{x}^{\mathbf{c}} \in L_G \forall i\}.$$

This socle is a set of Laurent monomials on which the lattice L_G acts with finitely many orbits, so the computation of $\text{MonSoc}(L_G)$ is a finite algorithmic problem, as in [9, §9.3]. The problem's solution is given by the socle monomials (11) of our monomial ideal M_G .

Lemma 21 *The socle monomials of the lattice module L_G are precisely of the form $s_{\mathcal{T}} \cdot \mathbf{x}^{\mathbf{w}}/x_n$, where $s_{\mathcal{T}} \in \text{MonSoc}(M_G)$ and $\mathbf{x}^{\mathbf{w}}$ runs over the minimal generators of L_G .*

Proof Since G is assumed to be saturated throughout this section, the lattice module L_G is generic in the sense of [9, Definition 9.23], with M_G being the reverse lexicographic initial ideal of the corresponding lattice ideal I_G . We claim that the stated characterization of the socle is valid for any generic lattice module that is Artinian. Indeed, by the proof of [10, Theorem 5.2], the $\mathbb{Z}^n$ -degrees of the n th syzygies of L_G are the vectors $\mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ runs over the $\mathbb{Z}^{n-1}$ -degrees of the $(n - 1)$ -st syzygies of M_G and $\mathbf{w}$ is any vector in the lattice. The socle degrees of M_G are the vectors $\mathbf{u} - e_1 - \dots - e_{n-1}$ in $\mathbb{Z}^{n-1}$, and the socle degrees of L_G are the vectors $\mathbf{u} + \mathbf{w} - e_1 - \dots - e_{n-1} - e_n$ in $\mathbb{Z}^n$. $\square$

We now identify Laurent monomials $\mathbf{x}^{\mathbf{u}}$ with divisors on the graph G . The i th coordinate u_i of the exponent vector $\mathbf{u}$ is the multiplicity of node i in the divisor $\mathbf{x}^{\mathbf{u}}$. The *degree* of the divisor $\mathbf{x}^{\mathbf{u}}$ is its total degree as a monomial, $\text{degree}(\mathbf{x}^{\mathbf{u}}) = u_1 + \dots + u_n$. The *rank* of the divisor $\mathbf{x}^{\mathbf{u}}$ is defined by the same formula (13) as in Lemma 11:

$$\text{rank}(\mathbf{x}^{\mathbf{u}}) = \min_{\mathbf{x}^{\mathbf{c}} \in \text{MonSoc}(L_G)} \text{degree}^+(\mathbf{u} - \mathbf{c}) - 1. \quad (24)$$

Thus, $\text{rank}(\mathbf{x}^{\mathbf{u}}) \geq 0$ if and only if $\mathbf{x}^{\mathbf{u}}$ lies in L_G . Our definition of rank of the divisor $\mathbf{x}^{\mathbf{u}}$ coincides with the rank of $\mathbf{u}$ as in [3]. To see this, use Lemma 2.7 in [3] and note that the exponents of the socle Laurent monomials of L_G are the elements of the set $\mathcal{N}$ in [3].

We finally define the *canonical divisor* of G to be the monomial

$$\mathbf{x}^{\mathbf{k}} = x_1^{d_1-2} x_2^{d_2-2} \cdots x_n^{d_n-2},$$

where $d_i = \sum_{j \neq i} u_{ij}$ is the degree of node i . Finally, we recall that the genus of G is $e - n + 1$, where e is the number of edges. The following is precisely [3, Theorem 1.12]:

Theorem 22 (Baker–Norine) *Riemann–Roch holds for any divisor $\mathbf{x}^{\mathbf{u}}$ on the graph G :*

$$\text{rank}(\mathbf{x}^{\mathbf{u}}) - \text{rank}(\mathbf{x}^{\mathbf{k}}/\mathbf{x}^{\mathbf{u}}) = \text{degree}(\mathbf{x}^{\mathbf{u}}) - \text{genus}(G) + 1. \quad (25)$$

Proof It follows from Lemma 21 that all socle monomials of L_G have degree equal to the genus of G minus one. The lattice module L_G is also reflection invariant, in the sense that $\mathbf{x}^{\mathbf{u}} \in \text{MonSoc}(L_G)$ implies $\mathbf{x}^{\mathbf{k}}/\mathbf{x}^{\mathbf{u}} \in \text{MonSoc}(L_G)$. Using the representation in Lemma 21, the resulting involution ϕ on $\text{MonSoc}(L_G)$ can be written as follows:

$$\phi((s_T/x_n) \cdot \mathbf{x}^{\mathbf{w}}) = (s_{\phi(T)}/x_n) \cdot \mathbf{x}^{-\mathbf{w}} \cdot \left(\frac{x_n^{d_n}}{\prod_{i=1}^{n-1} x_i^{u_{in}}} \right), \quad (26)$$

where $\phi(T)$ denotes the reverse flag as in (23). Note that the image of ϕ is in $\text{MonSoc}(L_G)$, since $x_n^{d_n} / \prod_{i=1}^{n-1} x_i^{u_{in}}$ is in L_G . The proof of Theorem 22 is now entirely analogous to that of Theorem 13. In other words, our argument for the validity of the Riemann–Roch formula for reflection-invariant Artinian level monomial ideals generalizes in a straightforward manner to reflection-invariant Artinian level lattice modules. $\square$

Remark 23 The rank of the canonical monomial of M_G equals the rank of the canonical divisor of the graph G , but the degree of the former is two more than that of the latter.

4 Non-saturated graphs

We turn to graphs G that are not necessarily saturated. The binomials in (6), their syzygies in (9), and the ideals I_G and M_G are still well-defined. However, the minimality in Theorem 2 is no longer true, and the choice of term order is more subtle, as the next example shows.

Example 24 Let G be the edge graph of a triangular prism, labeled so that I_G equals

$$[(a^3 - bcd, b^3 - ace, c^3 - abf, d^3 - aef, e^3 - bdf, f^3 - cde) : \langle abcdef \rangle^\infty].$$

This toppling ideal has 22 minimal generators, and its free resolution has Betti numbers (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions:

$$M_G = \text{in}(I_G) = \langle a^3, abc, ace^2, b^3, b^2e^2, c^3, cde, d^3, d^2e^2, e^3a^2b^2, a^2c^2, a^2d^2, \\ abde, b^2c^2, bcd^2, a^2be^2, a^2de^2, ab^2d^2, ac^2d^2, b^2d^2e, bc^2e^2 \rangle.$$

Here the reverse lexicographic term order with $a > b > c > d > e > f$ was used. However, if we take $e > d > c > b > a > f$ then the reverse lexicographic Gröbner basis requires two more binomials. Now, the initial monomial ideal has Betti numbers $(1, 24, 98, 153, 104, 26)$.

To explain the phenomenon in this example, we fix a spanning tree T of the graph G that is rooted at the node n , and we order the unknowns according to a linear extension of T . Thus, we fix an ordering of $[n]$ such that $i > j$ if the node i is a descendant of the node j in T . A term order on $\mathbb{K}[\mathbf{x}]$ is a *spanning tree order* if it is a reverse lexicographic term order whose variable ordering is compatible with some spanning tree rooted at n . One spanning tree order is the toppling order considered in [6, Theorem 10]. See also [11, §5] for a discussion of Gröbner bases of toppling ideals in the inhomogeneous case.

Theorem 25 *The toppling ideal I_G is generated by the binomials $\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}$ where (I, J) runs over splits of $[n]$ such that the subgraphs of G induced on I and J are connected. For any spanning tree order, these binomials form a Gröbner basis of I_G with initial monomial ideal M_G . The complexes constructed in (8) are cellular free resolutions.*

Proof The first paragraph in the proof of Theorem 2 is valid in the non-saturated case. It shows that the binomials $\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}$ lie in I_G . For the spanning tree term order, the leading monomials are $\mathbf{x}^{I \rightarrow J}$, where $n \in J \setminus I$, and hence the initial ideal $\text{in}(I_G)$ contains the monomial ideal M_G of (10). Again, both ideals are Artinian of the same colength in $\mathbb{K}[\mathbf{x}_n]$, and hence they are equal. This establishes the Gröbner basis property. The argument in the proof of [6, Theorem 14] shows that the property that the subgraphs of G induced on I and J are connected characterizes a minimal Gröbner basis of I_G and the minimal generators of M_G . In particular, these binomials $\mathbf{x}^{I \rightarrow J} - \mathbf{x}^{J \rightarrow I}$ generate I_G .

Our last assertion states that (8) with differentials (9) gives a free resolution of I_G , and dropping the last term in (9) gives a free resolution of M_G . This claim is proved by deformation to generic monomial modules, as explained in [9, §6.2]. To be precise, in our situation we replace G by a nearby saturated graph G_ϵ with fractional edge numbers $u_{ij}(\epsilon)$ between any pair of nodes. The monomial ideal M_{G_ϵ} is generic and degenerates to M_G . The lattice ideals I_G and I_{G_ϵ} are represented by the corresponding lattice modules L_G and L_{G_ϵ} . These are submodules of the Laurent polynomial ring as in [9, Definition 9.11]. The lattice module L_{G_ϵ} is generic and degenerates to L_G . According to [9, Theorem 6.24], the Scarf complex of M_{G_ϵ} with labels from G gives a free resolution of M_G . Likewise, the Scarf complex of the generic lattice module L_{G_ϵ} with labels from G gives a free resolution of L_G . The resulting minimal free resolution of I_{G_ϵ} degenerates to a (typically non-minimal) resolution of I_G , using [9, Corollary 9.18]. These free resolutions of M_G and I_G are cellular because they are given by labeled simplicial complexes. $\square$

Here is an example that illustrates the degeneration used in the proof above.

Example 26 Let $n = 4$ and G be the 4-cycle $1-2-3-4-1$. For $\delta, \epsilon \in \mathbb{N}$ consider the graph $G_{\delta, \epsilon}$ that has δ edges for every edge in G and ϵ edges for every non-edge of G . Then $G_{\delta, \epsilon}$ is saturated for $\delta, \epsilon > 0$. The Scarf complex in Example 3 gives the minimal free resolution of $M_{G_{\delta, \epsilon}} = \text{in}(I_{G_{\delta, \epsilon}})$ and this lifts to the minimal free resolution of $I_{G_{\delta, \epsilon}}$. By Theorem 19, the monomial ideal $M_{G_{\delta, \epsilon}}$ is Riemann–Roch, its genus is $4\delta + 2\epsilon - 2$, and its canonical monomial equals $\mathbf{x}^{\mathbf{K}} = x_1^{3\delta+\epsilon-2} x_2^{2\delta+2\epsilon-2} x_3^{3\delta+\epsilon-2}$. The involution $\mathbf{x}^{\mathbf{b}} \mapsto \mathbf{x}^{\mathbf{K}-\mathbf{b}}$ on its six socle monomials is given by swapping the two rows below:

$$\begin{aligned} \text{MonSoc}(M_{G_{\delta, \epsilon}}) = \{ & x_1^{\delta-1} x_2^{\delta+\epsilon-1} x_3^{2\delta+\epsilon-1}, x_1^{\delta-1} x_2^{2\delta+\epsilon-1} x_3^{\delta+\epsilon-1}, x_1^{\delta+\epsilon-1} x_2^{2\delta+\epsilon-1} x_3^{\delta-1} \\ & x_1^{2\delta+\epsilon-1} x_2^{\delta+\epsilon-1} x_3^{\delta-1}, x_1^{2\delta+\epsilon-1} x_2^{\epsilon-1} x_3^{2\delta-1}, x_1^{2\delta-1} x_2^{\epsilon-1} x_3^{2\delta+\epsilon-1} \}. \end{aligned} \quad (27)$$

Setting $\delta = 1$ and $\epsilon = 0$, we get the parking function monomial ideal of the 4-cycle

$$M_{G_{1,0}} = M_G = \langle x_1, x_2, x_3 \rangle^2 = \langle x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2 \rangle.$$

Here, $\text{MonSoc}(M_G) = \{x_1, x_2, x_3\}$. This ideal is not reflection invariant and hence not Riemann–Roch. The cellular resolution of M_G induced from $M_{G_{\delta, \epsilon}}$ is not minimal.

We now take a closer look at the combinatorial structure of our resolutions. Let $\text{Bary}(G)$ denote the first barycentric subdivision of the $(n-2)$ -simplex, whose $2^{n-1} - 1$ vertices, namely the non-empty subsets I of $[n-1]$, are labeled by the corresponding monomials $\mathbf{x}^{I \rightarrow [n] \setminus I}$. Thus, $\text{Bary}(G)$ is the cellular free resolution of $M_G = \text{in}(I_G)$ referred to in Theorem 25. Each simplex in $\text{Bary}(G)$ is labeled by the least common multiple of the monomials that label its vertices. For any $\mathbf{c} \in \mathbb{N}^{n-1}$ we write $\text{Bary}(G)_{\prec \mathbf{c}}$ for the subcomplex consisting of all simplices in $\text{Bary}(G)$ whose labels properly divide $\mathbf{x}^{\mathbf{c}}$.

Corollary 27 *The number of minimal i th syzygies of the monomial ideal M_G in degree $\mathbf{c}$ is equal to the rank of the reduced homology group $\tilde{H}_{i-1}(\text{Bary}(G)_{\prec \mathbf{c}}; \mathbb{K})$.*

Proof This follows immediately from Theorem 25 and [9, Theorem 4.7]. $\square$

We next state the analogous result for the lattice module L_G , that is, the $\mathbb{K}[\mathbf{x}]$ -module generated by all Laurent monomials whose exponent vector lies in the Laplacian lattice image $_{\mathbb{Z}}(\Lambda_G)$. We identify this lattice with $\mathbb{Z}^n / \mathbb{Z}\mathbf{e}$ by writing its elements as $\Lambda_G \cdot \mathbf{v}$ where each $\mathbf{v} \in \mathbb{Z}^n$ is unique modulo $\mathbb{Z}\mathbf{e} = \ker(\Lambda_G)$. The *tropical metric* on $\mathbb{Z}^n / \mathbb{Z}\mathbf{e}$ is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \max \{ |u_i + v_j - u_j - v_i| : 1 \leq i < j \leq n \}.$$

We write $\text{Apt}(G)$ for the corresponding *flag simplicial complex*. Thus, $\text{Apt}(G)$ is the simplicial complex whose simplices are subsets S of $\mathbb{Z}^n / \mathbb{Z}\mathbf{e}$ such that $\text{dist}(\mathbf{u}, \mathbf{v}) \leq 1$ for $\mathbf{u}, \mathbf{v} \in S$. The notation “Apt” refers to the fact that this infinite simplicial complex

is the standard *apartment* in the affine building of Lie type A_{n-1} . It is well-known that $\text{Apt}(G)$ is pure of dimension $n - 1$ and that it triangulates the $(n - 1)$ -dimensional affine space $\mathbb{R}^n / \mathbb{R}\mathbf{e}$. For more on buildings and their connection to tropical geometry, see [7].

The apartment $\text{Apt}(G)$ is precisely the same as the Delaunay triangulation constructed in [1], and it also coincides with the Scarf complex of G_ϵ that we used to give a cellular resolution of L_G . The number of i -faces of $\text{Apt}(G)$ modulo the lattice action is given by (7). Each vertex $\mathbf{v}$ of $\text{Apt}(G)$ is labeled by the corresponding Laurent monomial $\mathbf{x}^{A_G \mathbf{v}}$, and each face is labeled by the least common multiple of its vertex labels. Thus, each face of $\text{Apt}(G)$ is labeled by a Laurent monomial of degree ≥ 0 . We write $\text{Apt}(G)_{\prec \mathbf{c}}$ for the subcomplex of all simplices whose label properly divides $\mathbf{x}^{\mathbf{c}}$.

Corollary 28 *The number of minimal $(i + 1)$ -st syzygies of the lattice module L_G in degree $\mathbf{c}$ is the rank of the reduced homology $\tilde{H}_i(\text{Apt}(G)_{\prec \mathbf{c}}; \mathbb{K})$. The sum of these ranks over all $\mathbf{c}$ modulo $\text{image}_{\mathbb{Z}}(\Lambda_G)$ counts the minimal i th syzygies of the toppling ideal I_G .*

We conjecture that the ranks of the homology groups in the two corollaries coincide.

Conjecture 29 *The Betti numbers of the toppling ideal I_G do not increase when passing to the initial ideal M_G . More precisely, for all $i \geq 0$ and all $\mathbf{c} \in \mathbb{N}^{n-1}$, we have*

$$\tilde{H}_{i-1}(\text{Bary}(G)_{\prec \mathbf{c}}; \mathbb{K}) \simeq \tilde{H}_i(\text{Apt}(G)_{\prec \mathbf{c}}; \mathbb{K}). \quad (28)$$

This conjecture has been verified for many graphs using the software `Macaulay2`. We note that the two simplicial complexes appearing in (28) are different from the complex Δ_D used in Hochster's formula for the Betti numbers of a lattice ideal [11, Theorem 7.4].

Example 30 The simplicial complexes $\text{Apt}(G)_{\prec \mathbf{c}}$ can be large even for small graphs. Let G be the graph on four nodes, labeled a, b, c, d , with Laplacian matrix

$$\begin{aligned} \Lambda_G &= \begin{pmatrix} u_{12}+u_{13}+u_{14} & -u_{12} & -u_{13} & -u_{14} \\ -u_{12} & u_{12}+u_{23}+u_{24} & -u_{23} & -u_{24} \\ -u_{13} & -u_{23} & u_{13}+u_{23}+u_{34} & -u_{34} \\ -u_{14} & -u_{24} & -u_{34} & u_{14}+u_{24}+u_{34} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & -3 & 3 \end{pmatrix}. \end{aligned}$$

Both the toppling ideal and the ideal of parking functions are complete intersections:

$$I_G = \langle a^2 - b^2, b - c, c^3 - d^3 \rangle \quad \text{and} \quad M_G = \langle a^2, b, c^3 \rangle.$$

The monomial ideal M_G has one minimal first syzygy in degree $\mathbf{c} = (2, 0, 3, 0)$. The simplicial complex $\text{Bary}(G)_{\prec \mathbf{c}}$ consists of two isolated nodes a^2 and c^3 . The simplicial complex $\text{Apt}(G)_{\prec \mathbf{c}}$ is two-dimensional but it has the homology of a circle. It consists of 12 triangles, 28 edges and 16 vertices, labeled by the following generators of L_G :

$$1, \frac{c}{b}, \frac{c^2}{b^2}, \frac{c^3}{b^3}, \frac{c^2}{a^2}, \frac{c^3}{a^2b}, \frac{c^3}{d^3}, \frac{a^2c}{d^3}, \frac{a^2}{c^2}, \frac{a^2c^2}{bd^3}, \frac{a^2}{bc}, \frac{a^2c^3}{b^2d^3}, \frac{a^2}{b^2}, \frac{a^2c}{b^3}, \frac{a^2c^2}{b^4}, \frac{a^2c^3}{b^5}.$$

The lattice module L_G has one second syzygy in this degree, translating into a first syzygy of I_G . It is represented in $\text{Apt}(G)_{\prec \mathbf{c}}$ by the 4-cycle $1, a^2/b^2, a^2c^3/b^2d^3, c^3/d^3$.

At present, no explicit minimal free resolution of M_G is known. Finding such a resolution was stated as an open problem by Postnikov and Shapiro in [12, §6]. We do not even know whether the Betti numbers depend on the characteristic of the field $\mathbb{K}$.

An explicit formula for the Betti numbers of the toppling ideal I_G was conjectured by Wilmes in [15]. See also [11, §7.4]. Wilmes' formula is combinatorial, and it has been verified for all graphs with $n \leq 6$ nodes. At present we do not know how to relate Wilmes' conjecture to the ranks of the homology groups in Corollaries 27 and 28.

It is known, thanks to [15, Theorem 3.10], that Conjecture 29 is true for the maximal syzygies, with index $i = n - 2$. We have the following combinatorial characterization:

Corollary 31 *The maximal syzygies of the parking function ideal M_G , or of the toppling ideal I_G , are in bijection with the acyclic orientations of G with node n as unique sink.*

See [11, Theorem 7.6] for an alternative but equivalent formulation of this result.

Proof The monomial ideal M_G is Artinian, so its maximal syzygies correspond to the socle elements. These are the maximal parking functions, and, by [5, Theorem 4.1], they correspond to acyclic orientations of G with node n as unique sink. Since all maximal syzygies of $M_G = \text{in}(I_G)$ lift to maximal syzygies of I_G , the same result holds for I_G . $\square$

We now derive the Riemann–Roch theorem for non-saturated graphs G . Let M_G be the initial ideal with respect to a spanning tree order on the variables with x_n as the least. By Corollary 31, we know that the socle monomials of the Laurent monomial module L_G are $s \cdot x_n^{-1} \cdot \mathbf{x}^{\mathbf{w}}$ where s runs over all socle monomials of M_G and $\mathbf{x}^{\mathbf{w}}$ runs over minimal generators of L_G . Unlike in the saturated case, the monomial ideal M_G is generally not reflection invariant. But the Laurent monomial module L_G is always reflection invariant. To see this, we use Lemma 3.2 of [3] to deduce that $s_{\mathcal{T}}/x_n$, defined in (11), is not contained in L_G for any complete flag $\mathcal{T}$ of $[n]$. This implies that $s_{\mathcal{T}}/x_n$ is a socle element of L_G , since every Laurent monomial of degree greater than $\text{degree}(s_{\mathcal{T}}/x_n) = \text{genus}(G) - 1$ is in L_G . We now immediately verify that L_G is reflection invariant with the involution on $\text{MonSoc}(L_G)$ that takes

$s_{\mathcal{T}} \mathbf{x}^{\mathbf{w}} / x_n$ to $s_{\phi(\mathcal{T})} \mathbf{x}^{-\mathbf{w}} x_n^{d_n-1} / \prod_{i=1}^{n-1} x_i^{u_{in}}$, where $\phi(\mathcal{T})$ is the reverse flag of $\mathcal{T}$ exactly as in the proof of Theorem 22. The canonical monomial is

$$\mathbf{x}^{\mathbf{k}} = x_1^{d_1-2} x_2^{d_2-2} \cdots x_n^{d_n-2},$$

where $d_i = \sum_{j \neq i} u_{ij}$ is the degree of node i . Hence, L_G satisfies the Riemann–Roch formula, in its monomial formulation (15), with $M = L_G$ and $\text{genus}(M) = \text{genus}(G)$.

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