Combinatorial properties of the Temperley–Lieb algebra of a Coxeter group

Alfonso Pesiri

Received: 2 November 2011 / Accepted: 19 June 2012 / Published online: 30 June 2012
© Springer Science+Business Media, LLC 2012

Abstract We study two families of polynomials that play the same role in the Temperley–Lieb algebra of a Coxeter group as the Kazhdan–Lusztig and R -polynomials play in the Hecke algebra of the group. Our results include recursions, non-recursive formulas, symmetry properties and expressions for the constant term. We focus mainly on non-branching Coxeter graphs.

Keywords Temperley–Lieb algebra · Hecke algebra · Kazhdan–Lusztig basis · Coxeter group

1 Introduction

The Temperley–Lieb algebra $TL(X)$ is a quotient of the Hecke algebra $\mathcal{H}(X)$ associated to a Coxeter group $W(X)$, X being an arbitrary Coxeter graph. It first appeared in [49], in the context of statistical mechanics (see, e.g., [30]). The case $X = A$ was studied by Jones (see [31]) in connection to knot theory. For an arbitrary Coxeter graph, the Temperley–Lieb algebra was studied by Graham. More precisely, in [20] Graham showed that $TL(X)$ is finite dimensional whenever X is of type A, B, D, E, F, H or I . If $X \neq A$ then $TL(X)$ is usually referred to as the generalized Temperley–Lieb algebra.

The algebra $TL(X)$ has many properties similar to the Hecke algebra $\mathcal{H}(X)$. In particular, it is shown in [24] that $TL(X)$ inherits an involution from $\mathcal{H}(X)$ and that it always has a basis, indexed by the fully commutative elements of $W(X)$, with some

This work is part of the author's doctoral dissertation, written under the direction of Prof. F. Brenti at the University of Rome "Tor Vergata".

A. Pesiri (✉)

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica,
00133 Roma, Italy
e-mail: pesiri@mat.uniroma2.it

remarkable properties, called an IC basis (see [15] and [24]). Thus, one has two families of polynomials, indexed by pairs of fully commutative elements of $W(X)$, which are analogous to the Kazhdan–Lusztig and R -polynomials of $\mathcal{H}(X)$. Although the Kazhdan–Lusztig and R -polynomials have been extensively studied (see, e.g., [2, 4, 5, 7–10, 12, 14, 16, 23, 28, 29, 34, 35, 39, 43]) and $TL(X)$ plays an important role in several areas and has also been extensively studied (see, e.g., [11, 17–21, 25, 31, 32, 37, 41, 42]), these polynomials have not been investigated very much. Our purpose in this work is to begin the study of these polynomials from a combinatorial point of view. More precisely we obtain recursions, non-recursive formulas, symmetry properties and expressions for the constant terms, for these polynomials. To do this, we need to study some auxiliary polynomials (which have no analogue in $\mathcal{H}(X)$, and which in some sense express the relationship between $\mathcal{H}(X)$ and $TL(X)$) which were first defined in [24]. Most of our results hold for non-branching Coxeter graphs, although some hold in full generality. Our results show that there is a close relationship between Kazhdan–Lusztig and R -polynomials and their analogue in $TL(X)$.

The organization of the paper is as follows. In the next section we recall some generalities on the Hecke algebra, Kazhdan–Lusztig polynomials and the Kazhdan–Lusztig basis for $\mathcal{H}(X)$. Moreover, we introduce the Temperley–Lieb algebra and the families of polynomials $\{a_{x,w}\}$ and $\{L_{x,w}\}$ that we study in this work. In Sect. 3 we prove our main result (Theorem 3.7) on D -polynomials, which holds for all finite irreducible and affine non-branching Coxeter graphs X such that $X \neq \tilde{F}_4$, and we obtain some explicit formulas for the D -polynomials in type A . In Sects. 4 and 5 we use Theorem 3.7 to obtain recursions, non-recursive formulas, symmetry properties and expressions for the constant term, for the polynomials $\{a_{x,w}\}$ and $\{L_{x,w}\}$.

2 Preliminaries

In this section we recall some basic facts about Hecke algebras $\mathcal{H}(X)$ and Temperley–Lieb algebras $TL(X)$, X being any Coxeter graph. Let $W(X)$ be the Coxeter group having X as Coxeter graph and $S(X)$ as set of generators. Let $\mathcal{A}$ be the ring of Laurent polynomials $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The Hecke algebra $\mathcal{H}(X)$ associated to $W(X)$ is an $\mathcal{A}$ -algebra with linear basis $\{T_w : w \in W(X)\}$ (see, e.g., [3, §6.1]). For all $w \in W(X)$ and $s \in S(X)$ the multiplication law is determined by

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases} \quad (2.1)$$

where ℓ denotes the usual length function of $W(X)$. We refer to $\{T_w : w \in W(X)\}$ as the T -basis for $\mathcal{H}(X)$.

Let e be the identity element of $W(X)$. One easily checks that $T_s^2 = (q-1)T_s + qT_e$, being T_e the identity element, and so $T_s^{-1} = q^{-1}(T_s - (q-1)T_e)$. It follows that all the elements T_w are invertible, since, if $w = s_1 \cdots s_r$ and $\ell(w) = r$, then $T_w = T_{s_1} \cdots T_{s_r}$. To express T_w^{-1} as a linear combination of elements in the basis, one obtains the so-called R -polynomials. For a proof of the following result we refer to [27, §7.4].

Theorem 2.1 *There is a unique family of polynomials $\{R_{x,w}(q)\}_{x,w \in W(X)} \subseteq \mathbb{Z}[q]$ such that*

$$T_{w^{-1}}^{-1} = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x,$$

and $R_{x,w}(q) = 0$ if $x \not\leq w$, where $\varepsilon_x \stackrel{\text{def}}{=} (-1)^{\ell(x)}$. Furthermore, $R_{x,w}(q) = 1$ if $x = w$.

Define a map $\iota : \mathcal{H} \rightarrow \mathcal{H}$ such that $\iota(T_w) = (T_{w^{-1}})^{-1}$, $\iota(q) = q^{-1}$ and extend by linearity. We refer the reader to [27, §7.7] for the proof of the following result.

Proposition 2.2 *The map ι is a ring homomorphism of order 2 on $\mathcal{H}(X)$.*

In [33], the Kazhdan and Lusztig prove this basic theorem:

Theorem 2.3 *There exists a unique basis $\{C_w : w \in W(X)\}$ for $\mathcal{H}(X)$ such that the following properties hold:*

- (i) $\iota(C_w) = C_w$,
- (ii) $C_w = \varepsilon_w q^{\frac{\ell(w)}{2}} \sum_{x \leq w} \varepsilon_x q^{-\ell(x)} P_{x,w}(q^{-1}) T_x$,

where $\{P_{x,w}(q)\} \subseteq \mathbb{Z}[q]$, $P_{w,w}(q) = 1$ and $\deg(P_{x,w}(q)) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ if $x < w$.

The polynomials $\{P_{x,w}(q)\}_{x,w \in W(X)}$ are the so-called *Kazhdan–Lusztig polynomials* of $W(X)$. In [27, §7.9] it is shown that one can substitute the basis $\{C_w : w \in W(X)\}$ with the equivalent basis $\{C'_w : w \in W(X)\}$, where

$$C'_w = q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x,w}(q) T_x. \quad (2.2)$$

For the rest of this paper we will refer to the latter basis as the *Kazhdan–Lusztig basis* for $\mathcal{H}(X)$.

It is a routine exercise to prove the following properties (see, e.g., [3, §5, Exercises 3 and 7(a)]).

Lemma 2.4 *Let $x, w \in W(X)$ be such that $x \leq w$. If $\ell(w) - \ell(x) \leq 2$ then*

- (i) $R_{x,w}(q) = (q - 1)^{\ell(w) - \ell(x)}$;
- (ii) $P_{x,w}(q) = 1$.

Let $s_i, s_j \in S(X)$ and denote by $\langle s_i, s_j \rangle$ the parabolic subgroup of $W(X)$ generated by s_i and s_j . Following [20], we consider the two-sided ideal $J(X)$ generated by all elements of $\mathcal{H}(X)$ of the form

$$\sum_{w \in \langle s_i, s_j \rangle} T_w,$$

where (s_i, s_j) runs over all pairs of non-commuting generators in $S(X)$ such that the order of $s_i s_j$ is finite.

Definition 2.5 The generalized Temperley–Lieb algebra is $TL(X) \stackrel{\text{def}}{=} \mathcal{H}(X)/J(X)$.

When X is of type A , we refer to $TL(X)$ as the Temperley–Lieb algebra. In order to describe a basis for $TL(X)$, we recall the notion of a *fully commutative element* for $W(X)$ (see [48]).

Definition 2.6 An element $w \in W(X)$ is fully commutative if any reduced expression for w can be obtained from any other by applying Coxeter relations that involve only commuting generators. We let

$$W_c(X) \stackrel{\text{def}}{=} \{w \in W(X) : w \text{ is a fully commutative element}\}.$$

If $X = A_{n-1}$ then $W(X) \simeq S_n$ (see [3, Example 1.2.3]) and $W_c(A_{n-1})$ may be described as the set of elements of $W(A_{n-1})$ all of whose reduced expressions avoid substrings of the form $s_i s_{i \pm 1} s_i$, for all $s_i \in S$ (see [48, Proposition 1.1]). Another description of $W_c(A_{n-1})$ may be given in terms of pattern avoidance: namely, in [1, Theorem 2.1] Billey, Jockusch and Stanley show that $W_c(A_{n-1})$ coincides with the set of permutations avoiding the pattern 321. Moreover $|W_c(A_{n-1})| = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes the n th Catalan number (see [18, Proposition 3] for further details).

Let $t_w = \sigma(T_w)$, where $\sigma : \mathcal{H} \rightarrow \mathcal{H}/J$ is the canonical projection. A proof of the following can be found in [20].

Theorem 2.7 $TL(X)$ admits an $\mathcal{A}$ -basis of the form $\{t_w : w \in W_c(X)\}$.

We call $\{t_w : w \in W_c(X)\}$ the t -basis of $TL(X)$. By (2.1), it satisfies

$$t_w t_s = \begin{cases} t_{ws} & \text{if } \ell(ws) > \ell(w), \\ q t_{ws} + (q-1)t_w & \text{if } \ell(ws) < \ell(w). \end{cases} \quad (2.3)$$

Observe that if $ws \notin W_c(X)$, then t_{ws} can be expressed as linear combination of the t -basis elements by means of the following result (see [24, Lemma 1.5]).

Proposition 2.8 Let $w \in W(X)$. Then there exists a unique family of polynomials $\{D_{x,w}(q)\}_{x \in W_c(X)} \subseteq \mathbb{Z}[q]$ such that

$$t_w = \sum_{\substack{x \in W_c(X) \\ x \leq w}} D_{x,w}(q) t_x,$$

where $D_{w,w}(q) = 1$ if $w \in W_c(X)$. Furthermore, $D_{x,w}(q) = 0$ if $x \not\leq w$.

From the fact that the involution ι fixes the ideal $J(X)$ (see [24, Lemma 1.4]), it follows that ι induces an involution on $TL(X)$, which we still denote by ι , if there is no danger of confusion. More precisely, we have the following result.

Proposition 2.9 The map ι is a ring homomorphism of order 2 such that $\iota(t_w) = (t_{w^{-1}})^{-1}$ and $\iota(q) = q^{-1}$.

To express the image of t_w under ι as a linear combination of elements of the t -basis, one defines a new family of polynomials (see [24, §2]).

Proposition 2.10 *Let $w \in W_c(X)$. Then there exists a unique family of polynomials $\{a_{y,w}(q)\} \subseteq \mathbb{Z}[q]$ such that*

$$(t_{w^{-1}})^{-1} = q^{-\ell(w)} \sum_{\substack{y \in W_c(X) \\ y \leq w}} a_{y,w}(q) t_y,$$

where $a_{w,w}(q) = 1$.

The polynomials $\{a_{x,w}(q)\}$ associated to $TL(X)$ play the same role as the polynomials $\{R_{x,w}(q)\}$ associated to $\mathcal{H}(X)$. They both represent the coordinates of elements of the form $\iota(t_w)$ (respectively, $\iota(T_w)$) with respect to the t -basis (respectively, T -basis).

The generalized Temperley–Lieb algebra admits a basis $\{c_w : w \in W_c(X)\}$ which is analogous to the Kazhdan–Lusztig basis $\{C'_w : w \in W(X)\}$ of $\mathcal{H}(X)$. The following is a restatement of [24, Theorem 3.6].

Theorem 2.11 *There exists a unique basis $\{c_w : w \in W_c\}$ of $TL(X)$ such that*

- (i) $\iota(c_w) = c_w$,
- (ii) $c_w = \sum_{\substack{x \in W_c \\ x \leq w}} q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}) t_x$,

where $\{L_{x,w}(q^{-\frac{1}{2}})\} \subseteq q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$, $L_{x,x}(q^{-\frac{1}{2}}) = 1$, and $L_{x,w}(q^{-\frac{1}{2}}) = 0$ if $x \not\leq w$.

This basis is often called an *IC basis* (see [24, §2]).

Comparing the definition of c_w with that of C'_w , we notice that the polynomials $L_{x,w}(q^{-\frac{1}{2}})$ play the same role as $q^{\frac{\ell(x)-\ell(w)}{2}} P_{x,w}(q)$, where $P_{x,w}(q)$ are the Kazhdan–Lusztig polynomials defined in Theorem 2.3. Since the Kazhdan–Lusztig basis and the IC basis are both ι -invariant and since $\iota(J) = J$, it is natural to ask to what extent $\{\sigma(C'_w) : w \in W(X)\}$ coincides with $\{c_w : w \in W_c(X)\}$.

Definition 2.12 We say that a Coxeter graph X satisfies the projection property if

$$\{\sigma(C'_w) : w \in W_c(X)\} = \{c_w : w \in W_c(X)\}.$$

A sufficient condition for a Coxeter graph to have the projection property is given in [25, Proposition 1.2.3].

Proposition 2.13 *Let $\sigma : \mathcal{H}(X) \rightarrow \mathcal{H}(X)/J(X) = TL(X)$ be the canonical projection. If $\text{Ker}(\sigma)$ is spanned by the basis elements C'_w that it contains, then X satisfies the projection property.*

The kernel of the canonical projection $\sigma : \mathcal{H}(A_n) \rightarrow \mathcal{H}(A_n)/J(A_n)$ is spanned by all elements C'_w such that $w \notin W_c(A_n)$. This stems from the fact that $W_c(A_n)$ is

a union of two-sided Kazhdan–Lusztig cells (see [19, Proposition 3.1.1]). Therefore, type A has the projection property and the same argument holds for types B and $I_2(m)$ (see [26, Theorem 3.1.1] and [22, Proposition 6.14]). This fact was also verified for types H_3 , H_4 , and F_4 , by means of computer calculations (see [26, §3]). The converse of Proposition 2.13 is not true in general. A counterexample is given in [38, Example 2.5], where Losonczy shows that type D_n , with $n \geq 4$, has the projection property, but Proposition 2.13 does not apply. Similar problems arise whenever X is a Coxeter graph that contains a vertex connected to at least three other vertices. Graphs having this property are sometimes called *branching graphs*. Some examples of branching graphs are types D , E_6 , E_7 , E_8 , and in these cases Proposition 2.13 does not apply (see [26, Corollary 3.1.3]). No example of a Coxeter group that fails to satisfy the projection property is known and even in type E it is an open problem (see [26, §2]).

We remark that the hypothesis of Proposition 2.13 is equivalent to asking that $\sigma(C'_w) = 0$, for all elements $w \notin W_c(X)$ (see [26, Theorem 2.2.3]). In particular, one may wonder whether the map $\sigma : \mathcal{H}(X) \rightarrow \mathcal{H}(X)/J(X)$ satisfies

$$\sigma(C'_w) = \begin{cases} c_w & \text{if } w \in W_c(X), \\ 0 & \text{if } w \notin W_c(X). \end{cases} \quad (2.4)$$

The answer is affirmative in types A , B , $I_2(m)$, F_4 , H_3 and H_4 , and negative for types D , E_6 , E_7 and E_8 (for a complete discussion of these results, see [22] and [26]). More generally, if $W(X)$ is a finite irreducible or affine Coxeter group, relation (2.4) holds if and only if $W_c(X)$ is a union of two-sided Kazhdan–Lusztig cells (see [46, Lemma 2.4] and [26, Theorem 2.2.3]). On the other hand, in [45, §3] Shi shows that $W_c(X)$ is a union of two-sided Kazhdan–Lusztig cells if and only if X is non-branching and $X \neq \tilde{F}_4$. We sum up these properties in the following.

Theorem 2.14 *Let X be a finite irreducible or affine Coxeter graph. Then, relation (2.4) holds if and only if X is non-branching and $X \neq \tilde{F}_4$.*

3 The polynomials $D_{x,w}$

In this section we study the D -polynomials defined in Proposition 2.8. More precisely, we obtain a recurrence relation for the polynomials $\{D_{x,w}(q)\}_{x \in W_c(X), w \in W(X)}$, X being an arbitrary Coxeter graph. Then we focus on the Coxeter graphs satisfying equation (2.4) and obtain some results concerning symmetry properties, the value of the constant term, and explicit formulas for the D -polynomials of some families of Bruhat intervals. From now on we will denote by $\ell(x, w)$ the number $\ell(w) - \ell(x)$, for every $x, w \in W(X)$ such that $x \leq w$.

Let X be an arbitrary Coxeter graph. We begin with the following recursion for the D -polynomials.

Theorem 3.1 Let $w \notin W_c(X)$ and $s \in S(X)$ be such that $ws \notin W_c(X)$, with $\ell(ws) < \ell(w)$. Then, for all $x \in W_c(X)$, $x \leq w$, we have

$$D_{x,w} = \widetilde{D_{x,w}} + \sum_{\substack{y \in W_c(X), ys \notin W_c(X) \\ ys > y}} D_{x,ys} D_{y,ws},$$

where

$$\widetilde{D_{x,w}} = \begin{cases} D_{xs,ws} + (q-1)D_{x,ws} & \text{if } xs < x, \\ qD_{xs,ws} & \text{if } x < xs \in W_c(X), \\ 0 & \text{if } x < xs \notin W_c(X). \end{cases}$$

Proof On the one hand, by Proposition 2.8,

$$t_w = \sum_{\substack{x \in W_c(X) \\ x \leq w}} D_{x,w} t_x.$$

On the other hand, letting $v \stackrel{\text{def}}{=} ws$,

$$\begin{aligned} t_w &= t_v t_s = \left(\sum_{\substack{y \in W_c(X) \\ y \leq v}} D_{y,v} t_y \right) t_s \\ &= \sum_{\substack{y \in W_c(X) \\ y \leq v, ys > y}} D_{y,v} t_{ys} + \sum_{\substack{y \in W_c(X) \\ y \leq v, ys < y}} D_{y,v} (q t_{ys} + (q-1) t_y) \\ &= \sum_{\substack{y \in W_c(X), ys \in W_c(X) \\ y \leq v, ys > y}} D_{y,v} t_{ys} + \sum_{\substack{y \in W_c(X), ys \notin W_c(X) \\ y \leq v, ys > y}} D_{y,v} t_{ys} \\ &\quad + \sum_{\substack{y \in W_c(X) \\ y \leq v, ys < y}} D_{y,v} q t_{ys} + \sum_{\substack{y \in W_c(X) \\ y \leq v, ys < y}} (q-1) D_{y,v} t_y \\ &= \sum_{\substack{y \in W_c(X), ys \in W_c(X) \\ y \leq v, ys > y}} D_{y,v} t_{ys} + \sum_{\substack{y \in W_c(X) \\ y \leq v, ys < y}} D_{y,v} q t_{ys} \\ &\quad + \sum_{\substack{y \in W_c(X) \\ ys < y \leq v}} (q-1) D_{y,v} t_y + \sum_{\substack{y \in W_c(X), ys \notin W_c(X) \\ y \leq v, ys > y}} D_{y,v} \left(\sum_{\substack{x \in W_c(X) \\ x \leq ys}} D_{x,ys} t_x \right) \\ &= \sum_{\substack{x \in W_c(X) \\ xs < x \leq w}} D_{xs,ws} t_x + \sum_{\substack{xs \in W_c(X) \\ xs = w, xs > x}} q D_{xs,ws} t_x \\ &\quad + \sum_{\substack{x \in W_c(X) \\ xs < x \leq w}} (q-1) D_{x,ws} t_x + \sum_{x \in W_c(X)} \left(\sum_{\substack{y \in W_c(X), ys \notin W_c(X) \\ y < ys}} D_{x,ys} D_{y,ws} \right) t_x \end{aligned}$$

(note that $xs \in W_c(X)$, $x < xs \Rightarrow x \in W_c(X)$). Extracting the coefficient of t_x we get

$$D_{x,w} = \begin{cases} D_{xs,ws} + (q-1)D_{x,ws} + b(x,w) & \text{if } xs < x, \\ qD_{xs,ws} + b(x,w) & \text{if } x < xs \in W_c(X), \\ b(x,w) & \text{if } x < xs \notin W_c(X), \end{cases}$$

where

$$b(x,w) = \sum_{\substack{y \in W_c(X), ys \notin W_c(X) \\ y < ys}} D_{x,ys} D_{y,ws},$$

as desired. $\square$

It is interesting to note that the recursion in Theorem 3.1 is similar to the one for the parabolic Kazhdan–Lusztig polynomials (see [13]).

The preceding recursion can sometimes be solved explicitly. In the proof of the next result we need the notion of Grassmannian and bi-Grassmannian elements (see, e.g., [36, §3] and [3, §5, Exercise 38]).

Definition 3.2 Let $w \in W(A_{n-1})$ and define $D_R(w) \stackrel{\text{def}}{=} |\{s \in S(X) : ws < w\}|$. The permutation w is called Grassmannian if $|D_R(w)| \leq 1$ and bi-Grassmannian if $|D_R(w)| = |D_R(w^{-1})| = 1$.

As a consequence of [1, Theorem 2.1], if $w \in W(A_{n-1})$ is Grassmannian then $w \in W_c(A_{n-1})$.

In the sequel we will need the following properties of the Bruhat order, which are usually referred to as the *Subword Property* (see, e.g., [3, Theorem 2.2.2]) and the *Lifting Property* (see, e.g., [3, Proposition 2.2.7]).

Theorem 3.3 Subword property Let $s_1 \cdots s_r$ be a reduced expression of $v \in W(X)$ and let $u \in W(X)$. Then $u \leq v$ if and only if there exists a reduced expression $s_{i_1} \cdots s_{i_k}$ of u such that $1 \leq i_1 \leq \cdots \leq i_k \leq r$.

Lemma 3.4 Lifting property Let $x, w \in W$ be such that $x < w$ and suppose $s \in S$ such that $ws < w$ and $xs > x$. Then $x \leq ws$ and $xs \leq w$.

Corollary 3.5 Let X be of type A and let $x_0 \in W_c(X)$ be a bi-Grassmannian element. If x_0 is a maximal element in the Bruhat order of $W_c(X)$, then $D_{x_0,w}(q) = \varepsilon_{x_0} \varepsilon_w$, for all elements $w \geq x_0$.

Proof If $w \in W_c(X)$ then the result is trivial. Therefore, assume $w \notin W_c(X)$. Observe that if $s \in S(X)$ is such that $x_0s > x_0$, then $x_0s \notin W_c(X)$. Moreover, if $x_0s > x_0$ and $y \in W_c(X)$ is such that $ys > x_0$ then $y \geq x_0$ by Lemma 3.4, so $y = x_0$. Hence

$$\{y \in W_c(X) : ys > x_0\} = \{x_0\},$$

for any $s \notin D_R(x_0)$. Choosing s such that $x_0s > x_0$, $ws < w$ (there exists such an s since x_0 is a bi-Grassmannian element, while $w \notin W_c(X)$ is not Grassmannian), the

third case of Theorem 3.1 applies, so

$$D_{x_0, w}(q) = D_{x_0, x_0 s}(q) D_{x_0, w s}(q).$$

We proceed by induction on $\ell(x_0, w)$. Suppose $\ell(x_0, w) = 1$, with $w \notin W_c(A_{n-1})$. By Theorem 3.3 it follows that w admits a reduced expression of the form $x_1 s_i s_{i+1} s_i x_2$, with $x_1, x_2 \in W_c(A_{n-1})$, $s_i, s_{i+1} \in S(A_{n-1})$, and x_0 admits a reduced expression of the form $x_1 \hat{s}_i s_{i+1} s_i x_2$ or $x_1 s_i s_{i+1} \hat{s}_i x_2$, since $x_0 \in W_c(A_{n-1})$. Therefore

$$t_w = t_{x_1} t_{s_i s_{i+1} s_i} t_{x_2} = t_{x_1} (-t_{s_i s_{i+1}} - t_{s_{i+1} s_i} - t_{s_{i+1}} - t_{s_i} - t_e) t_{x_2},$$

and the statement follows by applying (2.3). If $\ell(x_0, w) > 1$, then

$$D_{x_0, w}(q) = D_{x_0, x_0 s}(q) D_{x_0, w s}(q) = -D_{x_0, w s}(q) = -\varepsilon_{x_0} \varepsilon_{w s} = \varepsilon_{x_0} \varepsilon_w.$$

□

From here to the end of this section we will denote by X a Coxeter graph satisfying (2.4). Observe that $D_{x, w}(q) = \delta_{x, w}$ if $x, w \in W_c(X)$.

Lemma 3.6 *For all $y \in W_c(X)$ and $w \notin W_c(X)$, we have*

$$\sum_{y \leq x \leq w} D_{y, x}(q) P_{x, w}(q) = 0.$$

Proof Let $w \in W(X)$. Then, by Proposition 2.8,

$$\begin{aligned} \sigma(C'_w) &= q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x, w}(q) \sigma(T_x) \\ &= q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x, w}(q) \left(\sum_{\substack{y \in W_c(X) \\ y \leq x}} D_{y, x}(q) t_y \right) \\ &= q^{-\frac{\ell(w)}{2}} \sum_{\substack{y \in W_c(X) \\ y \leq w}} \left(\sum_{y \leq x \leq w} D_{y, x}(q) P_{x, w}(q) \right) t_y. \end{aligned}$$

When $w \notin W_c(X)$ we get $\sigma(C'_w) = 0$, so the expression in round brackets must vanish and the statement follows. □

The following is the second main result of this section, and gives a non-recursive formula for the D -polynomials of a Coxeter group $W(X)$.

Theorem 3.7 *Let X be such that (2.4) holds. For all $x \in W_c(X)$ and $w \notin W_c(X)$ such that $x < w$, we have*

$$D_{x, w}(q) = \sum \left((-1)^k \prod_{i=1}^k P_{x_{i-1}, x_i}(q) \right),$$

where the sum is over all the chains $x = x_0 < x_1 < \cdots < x_k = w$ such that $x_i \notin W_c(X)$ if $i > 0$, and $1 \leq k \leq \ell(x, w)$.

Proof We proceed by induction on $\ell(x, w)$.

If $\ell(x, w) = 1$ then, from Lemma 3.6, we get $D_{x,w}(q) = -P_{x,w}(q)$, proving the claim in this case. If $\ell(x, w) > 1$ then, from Lemma 3.6 and our induction hypothesis, we have

$$\begin{aligned}
 D_{x,w}(q) &= -P_{x,w}(q) - \sum_{\substack{t \notin W_c(X) \\ x < t < w}} D_{x,t}(q) P_{t,w}(q) \\
 &= -P_{x,w}(q) - \sum_{\substack{t \notin W_c(X) \\ x < t < w}} P_{t,w}(q) \sum_{k=1}^{\ell(x,t)} \left(\sum_{x=x_0 < \cdots < x_k=t} (-1)^k \prod_{i=1}^k P_{x_{i-1}, x_i}(q) \right) \\
 &= -P_{x,w}(q) + \sum_{\substack{t \notin W_c(X) \\ x < t < w}} \left(\sum_{k=1}^{\ell(x,t)} \left(\sum_{\substack{x=x_0 < \cdots < x_{k+1}=w \\ x_k=t}} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1}, x_i}(q) \right) \right) \\
 &= \sum_{\substack{t \notin W_c(X) \\ x < t < w}} \left(\sum_{k \geq 0} \left(\sum_{\substack{x=x_0 < \cdots < x_{k+1}=w, \\ x_k=t \text{ if } k \neq 0}} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1}, x_i}(q) \right) \right) \\
 &= \sum_{k \geq 0} \left(\sum_{\substack{t \notin W_c(X) \\ x < t < w}} \left(\sum_{\substack{x=x_0 < \cdots < x_{k+1}=w, \\ x_k=t \text{ if } k \neq 0}} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1}, x_i}(q) \right) \right) \\
 &= \sum_{k \geq 0} \left(\sum_{\substack{x=x_0 < \cdots \\ \cdots < x_{k+1}=w}} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1}, x_i}(q) \right),
 \end{aligned}$$

and the result follows. $\square$

Theorem 3.7 shows that the D -polynomials are intimately related to the Kazhdan–Lusztig polynomials, which is not at all obvious from their definition.

Lemma 3.8 *Let $W(X)$ be a finite Coxeter group. Then*

$$x \in W_c(X) \iff x^{-1} \in W_c(X) \iff w_0 x w_0 \in W_c(X),$$

for all $x \in W(X)$.

Proof The maps $x \mapsto w_0 x w_0$ and $x \mapsto x^{-1}$ are Bruhat order automorphisms (see [3, Proposition 2.3.4 and Corollary 2.3.6]). Moreover, both these maps send Coxeter generators to Coxeter generators, since $\ell(x) = \ell(x^{-1}) = \ell(w_0 x w_0)$ (see [3, Corollary 2.3.3]). $\square$

We now derive some consequences of Theorem 3.7. First we obtain some symmetry properties of the polynomials $\{D_{x,w}(q)\}_{x \in W_c(X), w \in W(X)}$.

Corollary 3.9 *Let $x \in W_c(X)$, $w \notin W_c(X)$ and $x < w$. Then we have*

- (i) $D_{x,w}(q) = D_{x^{-1},w^{-1}}(q)$;
- (ii) $D_{x,w}(q) = D_{w_0 x w_0, w_0 w w_0}(q)$.

Proof By Lemma 3.8, $x^{-1} \in W_c(X)$ for every $x \in W_c(X)$. Therefore we get

$$\begin{aligned} D_{x^{-1},w^{-1}}(q) &= \sum_{\substack{x^{-1}=x_0 < x_1 < \dots \\ \dots < x_k = w^{-1}}} \left((-1)^k \prod_{i=1}^k P_{x_{i-1},x_i}(q) \right) \\ &= \sum_{\substack{x=x_0^{-1} < x_1^{-1} < \dots \\ \dots < x_k^{-1} = w}} \left((-1)^k \prod_{i=1}^k P_{x_{i-1},x_i}(q) \right) \\ &= \sum_{\substack{x=x_0^{-1} < x_1^{-1} < \dots \\ \dots < x_k^{-1} = w}} \left((-1)^k \prod_{i=1}^k P_{x_{i-1}^{-1},x_i^{-1}}(q) \right) \\ &= \sum_{\substack{x=y_0 < y_1 < \dots \\ \dots < y_k = w}} \left((-1)^k \prod_{i=1}^k P_{y_{i-1},y_i}(q) \right) \\ &= D_{x,w}(q), \end{aligned}$$

where we have used a well known property of the Kazhdan–Lusztig polynomials (see, e.g., [3, §5, Exercise 12]). The same holds for $D_{w_0 x w_0, w_0 w w_0}(q)$, using Lemma 3.8 and the properties in [3, §5, Exercise 13]. $\square$

Next we compute the constant term of the polynomials $D_{x,w}(q)$.

Corollary 3.10 *For all $x \in W_c(X)$ and $w \notin W_c(X)$ such that $x < w$, we have*

$$D_{x,w}(0) = \sum_{\substack{x=x_0 < \dots \\ \dots < x_k = w}} (-1)^k,$$

where $x_i \notin W_c(X)$ if $i > 0$, and $1 \leq k \leq \ell(x, w)$.

Proof The statement follows immediately from Theorem 3.7 and the well known fact that $P_{x,w}(0) = 1$ for all $x, w \in W(X)$ such that $x \leq w$ (see, e.g., [3, Proposition 5.1.5]). $\square$

By [47, Proposition 3.8.5], Corollary 3.10 asserts that $D_{x,w}(0)$ equals the Möbius function $\mu(\hat{0}, w)$ in the poset $\{y \in W(X) \setminus W_c(X) : y \in [x, w]\} \cup \{\hat{0}\}$. This suggests the study of the partial order induced on $W(X) \setminus W_c(X)$ by the Bruhat order.

Lemma 3.11 *Let $x \in W_c(X)$, $w \notin W_c(X)$ be such that $x < w$. If $\ell(x, w) = 1$ then $D_{x,w}(q) = -1$. If $\ell(x, w) = 2$, then*

$$D_{x,w}(q) = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k = 1, \\ -1 & \text{if } k = 0, \end{cases}$$

with $k \stackrel{\text{def}}{=} |\{y \notin W_c(X) : x < y < w\}|$.

Proof By Lemma 2.4, $P_{x,w}(q) = 1$ for all $x, w \in W(X)$ such that $\ell(x, w) \leq 2$. If $\ell(x, w) = 1$, then Theorem 3.7 implies $D_{x,w}(q) = -P_{x,w}(q) = -1$. If $\ell(x, w) = 2$, then the interval $[x, w]$ is isomorphic to the boolean lattice B_2 (see, e.g., [3, Lemma 2.7.3]), so $k \in \{0, 1, 2\}$. Moreover, $P_{x,y}(q) = P_{y,w}(q) = 1$ for all $y \notin W_c(X)$ such that $x < y < w$, since $\ell(x, y) = \ell(y, w) = 1$. From Theorem 3.7 we get

$$\begin{aligned} D_{x,w}(q) &= \sum \left((-1)^k \prod_{i=1}^k P_{x_{i-1}, x_i}(q) \right) \\ &= -P_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} P_{x,y}(q) P_{y,w}(q) \\ &= -1 + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} 1 \\ &= -1 + k, \end{aligned}$$

and the statement follows. $\square$

We end this section by deriving from Theorem 3.7 a closed formula for the polynomials $D_{x,w}(q)$ indexed by elements $x \in W_c(X)$ and $w \notin W_c(X)$ such that $([x, w] \cap (W(X) \setminus W_c(X))) \cup \{x\} = [x, w] \cong B_{l(x,w)}$. In type A it is easy to realize this case. Let $x \in W(A_n)$. Recall that x is said to be a Coxeter element if $s_{\sigma(1)} \cdots s_{\sigma(n)}$ is a reduced expression for x , for some $\sigma \in S_n$. It is clear that a Coxeter element is always a fully commutative element.

Theorem 3.12 *Let $s_1 s_2 \cdots s_n \cdots s_2 s_1$ be a reduced expression for $w \in W(A_n)$ and let $x \in W(A_n)$ be a Coxeter element. Then the following hold:*

- (i) $x \leq w$;
- (ii) $[x, w] \cong B_{\ell(x,w)}$;
- (iii) $([x, w] \cap (W(A_n) \setminus W_c(A_n))) \cup \{x\} = [x, w]$.

Proof

(i) We find a reduced expression for x which is a subexpression of

$$s_1 s_2 \cdots s_n \cdots s_2 s_1.$$

From [44, Theorem 1.5] there is a bijection between the set of Coxeter elements and the acyclic orientations of the Coxeter graph A_n . Let A_n^x be the acyclic orientation of the graph A_n associated to x . We say that s_i is on the left (respectively, on the right) of s_{i+1} in $x = s_{\sigma(1)} \cdots s_{\sigma(n)}$ if $s_i \longrightarrow s_{i+1}$ (respectively, $s_i \longleftarrow s_{i+1}$) in A_n^x . Therefore we are able to produce a reduced expression for x from A_n^x in the following way: set $x_n := s_n$ and juxtapose s_{n-1} to the left (respectively, to the right) of x_n if $s_{n-1} \longrightarrow s_n$ (respectively, $s_{n-1} \longleftarrow s_n$). Set $x_{n-1} := s_{n-1} x_n$ (respectively, $x_n s_{n-1}$). Repeat the same process with x_{n-1} and s_{n-2} , and so on. The process ends when we get x_1 . In fact, x_1 is a reduced expression for x and x_1 is, by construction, a subexpression of w . Hence (i) follows from Theorem 3.3.

(ii) By Theorem 3.3, every element $y \in [x, w]$ admits (at least) one reduced expression that is a subexpression of $s_1 s_2 \cdots s_n \cdots s_2 s_1$. Let $r(y)$ be one of these reduced expressions. Observe that the reduced expression x_1 obtained in (i) is a possible choice for $r(x)$. Consider the map $\phi : [x, w] \rightarrow \mathcal{A}$, with $\mathcal{A} = \{(\alpha_1, \dots, \alpha_{n-1}) : \alpha_i \in \{1, 2\}\}$, such that $\phi(y) = (\alpha_1, \dots, \alpha_{n-1})$ if and only if $r(y)$ has α_i occurrences of the generator s_i . By [40, Corollary 3.3], the map ϕ is well-defined. We claim that ϕ is a bijection.

First, we prove the surjectivity. Fix $(\alpha_1, \dots, \alpha_{n-1}) \in \mathcal{A}$. We describe an algorithm to construct (a reduced expression $r(y)$ for) an element $y \in W(A_n)$ such that $y \in [x, w]$ and $\phi(y) = (\alpha_1, \dots, \alpha_{n-1})$ in the following way: set $y_n := s_n$. If $\alpha_{n-1} = 2$ then set $y_{n-1} := s_{n-1} y_n s_{n-1}$. Otherwise, proceed as in the proof of point (i), that is, juxtapose s_{n-1} to the left (respectively, to the right) of y_n if $s_{n-1} \longrightarrow s_n$ (respectively, $s_{n-1} \longleftarrow s_n$) and set $y_{n-1} := s_{n-1} y_n$ (respectively, $y_n s_{n-1}$). Repeat the same process with y_{n-1} and α_{n-2} , and so on. The process ends when we get y_1 . In fact, $r(x) = x_1$ is a subexpression of y_1 by construction. Next, we show that y_1 is a reduced expression. Observe that if y_j is reduced then $y_j s_{j-1} > y_j$ and $s_{j-1} y_j > y_j$, since there is no occurrence of s_{j-1} in y_j . Now, we proceed by contradiction to prove that $s_{j-1} y_j s_{j-1}$ is reduced. Suppose $\ell(s_{j-1} y_j s_{j-1}) < \ell(y_j) + 2$, i.e., $\ell(s_{j-1} y_j s_{j-1}) \leq \ell(y_j)$. By applying Lemma 3.4 with $x = s_{j-1} y_j s_{j-1}$ and $w = y_j s_{j-1}$ we get $s_{j-1} y_j = y_j s_{j-1}$, and $s_{j-1} y_j, y_j s_{j-1}$ are both reduced expressions. Hence, [40, Lemma 3.1] implies that s_{j-1} commutes with each generator in y_j , which is absurd, since $s_j \leq y_j$ by construction. Therefore

$$\ell(s_{j-1} y_j s_{j-1}) = \ell(y_j) + 2.$$

We conclude that y_1 is a reduced expression by induction on $n - i$, with $i = 0 \cdots n - 1$.

Denote by $y \in W(A_n)$ the element that admits y_1 as a reduced expression. Then y has the desired properties.

For the injectivity we proceed by contradiction. Suppose that $u, v \in [x, w]$ are such that $\phi(u) = \phi(v) = (\alpha_1, \dots, \alpha_{n-1})$, with $u \neq v$. Denote by $r(u)$ (respectively, $r(v)$) the reduced expression of u (respectively, v) obtained by applying the algorithm described above. Then $u \neq v$ implies $r(u) \neq r(v)$, that is, there exists an index $i \in$

$[n-1]$ such that $\alpha_i = 1$ and the position of the factor s_i in $r(u)$ and $r(v)$ is different. Denote by j be the minimum among these indices. Therefore, for every $h < j$ such that $\alpha_h = 1$, s_h appears on the same side in $r(u)$ as $r(v)$. Suppose that $\alpha_{j+1} = 1$ and, for instance, that

$$r(u) = y_1 s_j \widehat{s_{j+1}} \cdots s_n \cdots s_{j+1} \widehat{s_j} y_2,$$

where $y_1 \leq s_1 s_2 \cdots s_{j-1}$ and $y_2 \leq s_{j-1} s_{j-2} \cdots s_1$. Then

$$r(v) = y_1 \widehat{s_j} s_{j+1} \cdots s_n \cdots \widehat{s_{j+1}} s_j y_2 \quad \text{or} \quad r(v) = y_1 \widehat{s_j} \widehat{s_{j+1}} \cdots s_n \cdots s_{j+1} s_j y_2.$$

In both cases, $r(v)$ is a reduced expression such that s_j is on the right of s_{j+1} . On the other hand, $r(u)$ is a reduced expression of u such that s_j is on the left of s_{j+1} . Hence, Theorem 3.3 implies that x admits a reduced expression that is a subexpression of $r(u)$ and a (possibly different) reduced expression that is a subexpression of $r(v)$. This is a contradiction, since x is uniquely determined by the relations $s_i \longrightarrow s_{i+1}$ or $s_i \longleftarrow s_{i+1}$. The same conclusion holds if we consider different deletions of s_j and s_{j+1} . In the case $\alpha_{j+1} = 2$, we may assume that

$$r(u) = y_1 s_j s_{j+1} \cdots s_n \cdots s_{j+1} \widehat{s_j} y_2 \quad \text{and} \quad r(v) = y_1 \widehat{s_j} s_{j+1} \cdots s_n \cdots s_{j+1} s_j y_2.$$

Observe that $r(v)$ (respectively, $r(u)$) is a reduced expression such that s_j is on the right (respectively, on the left) of s_{j+1} . Therefore, we reach the same contradiction that we obtained in the previous case.

(iii) Let $y \in (x, w]$ and $\phi(y) = (\alpha_1, \dots, \alpha_{n-1})$. Let j be the maximum of the $i \in [n-1]$ such that $\alpha_i = 2$. Then $r(y)$ contains the substring $s_j s_{j+1} s_j$, so $y \notin W_c(X)$. $\square$

Corollary 3.13 *Let $s_1 s_2 \cdots s_n \cdots s_2 s_1$ be a reduced expression for $w \in W(A_n)$ and let $x \in W(A_n)$ be a Coxeter element. Then $D_{x,w}(q) = \varepsilon_x \varepsilon_w$.*

Proof If $n = 1$ then $x = w$ and the statement follows trivially. Suppose $n > 1$. If $u, v \in W(A_n)$ are such that $[u, v] \simeq B_{\ell(u,v)}$, then $P_{u,v}(q) = 1$ (see [6, Corollary 4.12]). Therefore, Theorem 3.12 implies that $P_{u,v}(q) = 1$, for all $u, v \in [x, w]$. Hence, from Theorem 3.7 and Theorem 3.12 we achieve

$$D_{x,w}(q) = \sum \left((-1)^k \prod_{i=1}^k P_{x_{i-1}, x_i}(q) \right) = \sum (-1)^k,$$

where the sum runs over all the chains $x = x_0 < x_1 < \cdots < x_k = w$ such that $1 \leq k \leq \ell(x, w)$. Therefore, $D_{x,w}(q)$ equals the alternating sum

$$\sum_{k=1}^{\ell(x,w)} (-1)^k c_k,$$

where c_k denotes the number of chains $x = x_0 < x_1 < \cdots < x_k = w$. By [47, Proposition 3.8.5], we get

$$\sum_{k=1}^{\ell(x,w)} (-1)^k c_k = \mu(x, w),$$

where μ denotes the Möbius function on the poset induced by the Bruhat order on $[x, w]$. On the other hand, if $\mathcal{P} = (\mathcal{P}, \leq)$ is a boolean poset and $U, V \in \mathcal{P}$ are such that $U \leq V$, then $\mu_{\mathcal{P}}(U, V) = (-1)^{|V-U|}$, where $|V-U|$ denotes the length of the interval $[U, V]$ (see, e.g., [47, Example 3.8.3]). Finally, observe that the length of a Bruhat interval $[x, w]$ is $\ell(x, w)$ (see, e.g., [3, Theorem 2.2.6]) and the statement follows. $\square$

4 The involution ι

In this section we study the family of polynomials $\{a_{x,w}(q)\}_{x,w \in W_c(X)} \subseteq \mathbb{Z}[q]$ which express the involution ι in terms of the t -basis (see Proposition 2.10). More precisely, we obtain, using the results in Sect. 3, a non-recursive formula for these polynomials, an expression for their constant term and symmetry properties.

Proposition 4.1 *Let X be such that (2.4) holds. Let $x, w \in W_c(X)$ be such that $x \leq w$. Then*

$$a_{x,w}(q) = \varepsilon_x \varepsilon_w R_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} \varepsilon_y \varepsilon_w R_{y,w}(q) \left(\sum (-1)^k \prod_{i=1}^k P_{x_{i-1}, x_i}(q) \right),$$

where the second sum runs over all the chains $x = x_0 < \dots < x_k = y$ such that $x_i \notin W_c(X)$ if $i > 0$.

Proof From Theorem 2.1 and Proposition 2.10 we get

$$a_{x,w}(q) = \sum_{x \leq y \leq w} \varepsilon_y \varepsilon_w R_{y,w}(q) D_{x,y}(q), \quad (4.1)$$

for all $x, w \in W_c(X)$ such that $x \leq w$. Since $D_{x,w}(q) = \delta_{x,w}$ if $x, w \in W_c(X)$, we have

$$a_{x,w}(q) = \varepsilon_x \varepsilon_w R_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} \varepsilon_y \varepsilon_w R_{y,w}(q) D_{x,y}(q), \quad (4.2)$$

so the statement follows immediately from Theorem 3.7. $\square$

Proposition 4.1 allows us to compute the constant term of the polynomial $a_{x,w}(q)$.

Corollary 4.2 *For all $x, w \in W_c(X)$ such that $x \leq w$ we have*

$$a_{x,w}(0) = \sum (-1)^k,$$

where the sum runs over all the chains $x = x_0 < x_1 < \dots < x_{k+1} = w$ such that $x_i \notin W_c(X)$ if $1 \leq i \leq k$, and $0 \leq k \leq \ell(x, w) - 1$.

Proof The statement follows from the fact that $R_{x,w}(0) = \varepsilon_x \varepsilon_w$ (see [3, Proposition 5.1.3]) and from Proposition 4.1. $\square$

Again, we deduce from Proposition 4.1 the following symmetry properties of the polynomials $\{a_{x,w}(q)\}_{x,w \in W_c(X)}$.

Corollary 4.3 *Let $x, w \in W_c(X)$. Then we have*

- (i) $a_{x,w}(q) = a_{x^{-1},w^{-1}}(q)$;
- (ii) $a_{x,w}(q) = a_{w_0 x w_0, w_0 w w_0}(q)$.

Proof By Lemma 3.8 and by (4.1) we get

$$\begin{aligned} a_{x^{-1},w^{-1}}(q) &= \sum_{x^{-1} \leq y \leq w^{-1}} \varepsilon_y \varepsilon_{w^{-1}} R_{y,w^{-1}}(q) D_{x^{-1},y}(q) \\ &= \sum_{x^{-1} \leq z^{-1} \leq w^{-1}} \varepsilon_{z^{-1}} \varepsilon_{w^{-1}} R_{z^{-1},w^{-1}}(q) D_{x^{-1},z^{-1}}(q) \\ &= \sum_{x \leq z \leq w} \varepsilon_z \varepsilon_w R_{z,w}(q) D_{x,z}(q) \\ &= a_{x,w}(q), \end{aligned}$$

where we used Corollary 3.9(i) and the property $R_{x,w}(q) = R_{x^{-1},w^{-1}}(q)$, for all $x, w \in W(X)$ (see, e.g., [3, §5, Exercise 10.(a)]). The same holds for $a_{w_0 x w_0, w_0 w w_0}(q)$, using Corollary 3.9(ii) and [3, §5, Exercise 10.(b)]. $\square$

Lemma 4.4 *Let $x, w \in W_c(X)$ be such that $x < w$. If $\ell(x, w) = 1$ then $a_{x,w}(q) = 1 - q$. If $\ell(x, w) = 2$, then*

$$a_{x,w}(q) = \begin{cases} q^2 - 1 & \text{if } k = 2, \\ q^2 - q & \text{if } k = 1, \\ q^2 - 2q + 1 & \text{if } k = 0, \end{cases}$$

with $k \stackrel{\text{def}}{=} |\{y \notin W_c(X) : x < y < w\}|$.

Proof By Lemma 2.4, $R_{x,w}(q) = (q-1)^{\ell(x,w)}$ for all $x, w \in W(X)$, $x \leq w$ such that $\ell(x, w) \leq 2$. If $\ell(x, w) = 1$, then (4.2) implies $a_{x,w}(q) = \varepsilon_x \varepsilon_w R_{x,w}(q) = 1 - q$. If $\ell(x, w) = 2$, then the interval $[x, w]$ is isomorphic to the boolean lattice B_2 (see, e.g., [3, Lemma 2.7.3]). Moreover $R_{y,w}(q) = q - 1$ and $D_{x,y}(q) = -1$ for all $y \notin W_c(X)$ such that $x < y < w$, since $\ell(x, y) = \ell(y, w) = 1$ (see Lemma 3.11). From Equation (4.2) we get

$$a_{x,w}(q) = \varepsilon_x \varepsilon_w R_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} \varepsilon_y \varepsilon_w R_{y,w}(q) D_{x,y}(q)$$

$$\begin{aligned}
&= (q-1)^2 + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} (q-1) \\
&= (q-1)^2 + k(q-1),
\end{aligned}$$

and the statement follows. $\square$

5 The polynomials $L_{x,w}$

In this section we study the polynomials $\{L_{x,w}(q^{-\frac{1}{2}})\}_{x,w \in W_c(X)}$ which play the same role, in $TL(X)$, as the Kazhdan–Lusztig polynomials in $\mathcal{H}(X)$. More precisely, using the results in the previous section, we obtain a non-recursive formula, symmetry properties and expressions for the constant term for these polynomials.

Theorem 5.1 *Let X be such that (2.4) holds. For all elements $x, w \in W_c(X)$ such that $x < w$ we have*

$$L_{x,w}(q^{-\frac{1}{2}}) = q^{\frac{\ell(x)-\ell(w)}{2}} \sum \left((-1)^k \prod_{i=1}^{k+1} P_{x_{i-1}, x_i}(q) \right),$$

where the sum runs over all the chains $x = x_0 < x_1 < \cdots < x_{k+1} = w$ such that $x_i \notin W_c(X)$ if $1 \leq i \leq k$, and $0 \leq k \leq \ell(x, w) - 1$.

Proof On the one hand, from Proposition 2.8 and the definition of the t -basis we get

$$\begin{aligned}
\sigma(C'_w) &= q^{-\frac{\ell(w)}{2}} \sum_{y \leq w} P_{y,w}(q) \sigma(T_y) \\
&= q^{-\frac{\ell(w)}{2}} \sum_{y \leq w} P_{y,w}(q) \left(\sum_{\substack{x \in W_c(X) \\ x \leq y}} D_{x,y}(q) t_x \right) \\
&= q^{-\frac{\ell(w)}{2}} \sum_{\substack{x \in W_c(X) \\ x \leq w}} \left(\sum_{x \leq y \leq w} D_{x,y}(q) P_{y,w}(q) \right) t_x.
\end{aligned}$$

On the other hand, by definition,

$$c_w = \sum_{\substack{x \in W_c(X) \\ x \leq w}} q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}) t_x.$$

Therefore, by (2.4),

$$q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}) = q^{-\frac{\ell(w)}{2}} \sum_{x \leq y \leq w} D_{x,y}(q) P_{y,w}(q).$$

Since $D_{x,y}(q) = \delta_{x,y}$ if $y \in W_c(X)$ and $x \leq y$, we achieve

$$L_{x,w}(q^{-\frac{1}{2}}) = q^{\frac{\ell(x)-\ell(w)}{2}} \left(P_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} D_{x,y}(q) P_{y,w}(q) \right). \quad (5.1)$$

Combining (5.1) and Theorem 3.7 we get

$$\begin{aligned} L_{x,w}(q^{-\frac{1}{2}}) &= q^{\frac{\ell(x)-\ell(w)}{2}} \left(P_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} \sum_{\substack{x=x_0 < \dots \\ \dots < x_k=y}} \left((-1)^k \prod_{i=1}^k P_{x_{i-1},x_i}(q) \right) P_{y,w}(q) \right) \\ &= q^{\frac{\ell(x)-\ell(w)}{2}} \left(\sum_{\substack{x=x_0 < \dots \\ \dots < x_{k+1}=w}} (-1)^k \prod_{i=1}^{k+1} P_{x_{i-1},x_i}(q) \right), \end{aligned}$$

where $x_i \notin W_c(X)$ if $1 \leq i \leq k$, as desired. $\square$

Theorem 5.1 shows that the L -polynomials depend only on the Kazhdan–Lusztig polynomials and the poset structure induced by the Bruhat order on $\{x, w\} \cup ((x, w) \setminus (x, w)_c)$, where $(x, w)_c = \{y \in (x, w) : y \in W_c(X)\}$.

In the same way that Corollary 3.9 follows from Theorem 3.7 we deduce from Theorem 5.1 the following symmetry properties of the L -polynomials, whose proof we omit.

Corollary 5.2 *For all $x, w \in W_c(X)$ such that $y \leq w$, we have*

- (i) $L_{x,w}(q^{-\frac{1}{2}}) = L_{x^{-1},w^{-1}}(q^{-\frac{1}{2}})$;
- (ii) $L_{x,w}(q^{-\frac{1}{2}}) = L_{w_0 x w_0, w_0 w w_0}(q^{-\frac{1}{2}})$.

Lemma 5.3 *Let $x, w \in W_c(X)$ be such that $x < w$. If $\ell(x, w) = 1$ then $L_{x,w}(q^{-\frac{1}{2}}) = q^{-\frac{1}{2}}$. If $\ell(x, w) = 2$, then*

$$L_{x,w}(q^{-\frac{1}{2}}) = \begin{cases} -q^{-1} & \text{if } k = 2, \\ 0 & \text{if } k = 1, \\ q^{-1} & \text{if } k = 0, \end{cases}$$

with $k \stackrel{\text{def}}{=} |\{y \notin W_c(X) : x < y < w\}|$.

Proof If $\ell(x, w) = 1$, then $P_{x,w}(q) = 1$ (see Lemma 2.4). Equation (5.1) then implies that $L_{x,w}(q^{-\frac{1}{2}}) = q^{-\frac{1}{2}} P_{x,w}(q) = q^{-\frac{1}{2}}$. If $\ell(x, w) = 2$, then from (5.1) and Lemma 3.11 we get

$$L_{x,w}(q^{-\frac{1}{2}}) = q^{\frac{\ell(x)-\ell(w)}{2}} \left(P_{x,w}(q) + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} D_{x,y}(q) P_{y,w}(q) \right)$$

$$\begin{aligned}
&= q^{-1} \left(1 + \sum_{\substack{y \notin W_c(X) \\ x < y < w}} (-1) \right) \\
&= q^{-1} (1 - k),
\end{aligned}$$

and the statement follows. $\square$

The previous calculations show that $\{L_{x,w}(q^{-\frac{1}{2}})\} \not\subseteq \mathbb{N}[q^{-\frac{1}{2}}]$.

Acknowledgements I would like to thank Prof. Francesco Brenti for introducing me to this topic and for many useful conversations.

References

1. Billey, S.C., Jockusch, W., Stanley, R.P.: Some combinatorial properties of Schubert polynomials. *J. Algebr. Comb.* **2**, 345–374 (1993)
2. Billey, S.C., Warrington, G.S.: Kazhdan–Lusztig polynomials for 321-hexagon-avoiding permutations. *J. Algebr. Comb.* **13**, 111–136 (2001)
3. Björner, A., Brenti, F.: *Combinatorics of Coxeter Groups*. Graduate Texts in Mathematics, vol. 231. Springer, Berlin (2005)
4. Boe, B.D.: Kazhdan–Lusztig polynomials for Hermitian symmetric spaces. *Trans. Am. Math. Soc.* **309**, 279–294 (1988)
5. Brenti, F.: A combinatorial formula for Kazhdan–Lusztig polynomials. *Invent. Math.* **118**, 371–394 (1994)
6. Brenti, F.: Combinatorial properties of the Kazhdan–Lusztig R -polynomials for S_n . *Adv. Math.* **126**, 21–51 (1997)
7. Brenti, F.: Lattice paths and Kazhdan–Lusztig polynomials. *J. Am. Math. Soc.* **11**, 229–259 (1998)
8. Brenti, F., Caselli, F., Marietti, M.: Special matchings and Kazhdan–Lusztig polynomials. *Adv. Math.* **202**, 555–601 (2006)
9. Brenti, F., Simion, R.: Explicit formulae for some Kazhdan–Lusztig polynomials. *J. Algebr. Comb.* **11**, 187–196 (2000)
10. Caselli, F.: Proof of two conjectures of Brenti and Simion on Kazhdan–Lusztig polynomials. *J. Algebr. Comb.* **18**, 171–187 (2003)
11. Cautis, S., Jackson, D.M.: The matrix of chromatic joins and the Temperley–Lieb algebra. *J. Comb. Theory, Ser. A* **89**, 109–155 (2003)
12. Delanoy, E.: Combinatorial invariance of Kazhdan–Lusztig polynomials on intervals starting from the identity. *J. Algebr. Comb.* **24**, 437–463 (2006)
13. Deodhar, V.V.: On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials. *J. Algebra* **111**, 483–506 (1987)
14. Deodhar, V.V.: A combinatorial setting for questions in Kazhdan–Lusztig theory. *Geom. Dedic.* **36**, 95–119 (1990)
15. Du, J.: Global IC bases for quantum linear groups. *J. Pure Appl. Algebra* **114**, 25–37 (1996)
16. Dyer, M.J.: On coefficients of q in Kazhdan–Lusztig polynomials. In: *Algebraic groups and Lie groups*. Austral. Math. Soc. Lect. Ser., vol. 9, pp. 189–194 (1997)
17. Fan, C.K.: A Hecke algebra quotient and properties of commutative elements of a Weyl group. Ph.D. thesis, M.I.T (1995)
18. Fan, C.K.: A Hecke algebra quotient and some combinatorial applications. *J. Algebr. Comb.* **5**, 175–189 (1996)
19. Fan, C.K., Green, R.M.: Monomials and Temperley–Lieb algebras. *J. Algebra* **190**, 498–517 (1997)
20. Graham, J.J.: Modular representations of Hecke algebras and related algebras. Ph.D. thesis, University of Sydney (1995)
21. Green, R.M.: Generalized Temperley–Lieb algebras and decorated tangles. *J. Knot Theory Ramif.* **7**, 155–171 (1998)

22. Green, R.M.: Generalized Jones traces and Kazhdan–Lusztig bases. *J. Pure Appl. Algebra* **211**, 744–772 (2007)
23. Green, R.M.: Leading coefficients of Kazhdan–Lusztig polynomials and fully commutative elements. *J. Algebr. Comb.* **30**, 165–171 (2009)
24. Green, R.M., Losonczy, J.: Canonical bases for Hecke algebra quotients. *Math. Res. Lett.* **6**, 213–222 (1999)
25. Green, R.M., Losonczy, J.: A projection property for Kazhdan–Lusztig bases. *Int. Math. Res. Not.* **1**, 23–34 (2000)
26. Green, R.M., Losonczy, J.: Fully commutative Kazhdan–Lusztig cells. *Ann. Inst. Fourier* **51**, 1025–1045 (2001)
27. Humphreys, J.E.: *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics, vol. 29. Cambridge University Press, Cambridge (1990)
28. Incitti, F.: On the combinatorial invariance of Kazhdan–Lusztig polynomials. *J. Comb. Theory, Ser. A* **113**, 1332–1350 (2006)
29. Jones, B.C.: Leading coefficients of Kazhdan–Lusztig polynomials for Deodhar elements. *J. Algebr. Comb.* **29**, 229–260 (2009)
30. Jones, V.F.R.: A polynomial invariant for knots via von Neumann algebras. *Bull. Am. Math. Soc.* **12**, 103–111 (1985)
31. Jones, V.F.R.: Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126**, 335–388 (1987)
32. Kauffman, L., Thomas, R.: Temperley–Lieb algebras and the four-color theorem. *Combinatorica* **23**, 653–667 (2003)
33. Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. *Invent. Math.* **53**, 165–184 (1979)
34. Lascoux, A.: Polynômes de Kazhdan–Lusztig pour les variétés de Schubert vexillaires. *C. R., Math.* **321**, 667–670 (1995)
35. Lascoux, A., Schützenberger, M.-P.: Polynômes de Kazhdan & Lusztig pour les grassmanniennes. Young tableaux and Schur functors in algebra and geometry (Toruń, 1980) **87**, 249–266
36. Lascoux, A., Schützenberger, M.-P.: Treillis et bases des groupes de Coxeter. *Electron. J. Combin.* **3**, R27 (1996)
37. Levy, D.: Structure of Temperley–Lieb algebras and its application to 2D statistical models. *Phys. Rev. Lett.* **64**, 499–502 (1990)
38. Losonczy, J.: The Kazhdan–Lusztig basis and the Temperley–Lieb quotient in type D. *J. Algebra* **233**, 1–15 (2000)
39. Marietti, M.: Closed product formulas for certain R -polynomials. *Eur. J. Comb.* **23**, 57–62 (2002)
40. Marietti, M.: Boolean elements in Kazhdan–Lusztig theory. *J. Algebra* **295**, 1–26 (2006)
41. Martin, P.P.: Temperley–Lieb algebra, group theory and the Potts model. *J. Phys. A* **3**, 577–591 (1988)
42. Martin, P.P.: Temperley–Lieb algebras and the long distance properties of statistical mechanical models. *J. Phys. A* **23**, 7–30 (1990)
43. Polo, P.: Construction of arbitrary Kazhdan–Lusztig polynomials in symmetric groups. *Represent. Theory* **3**, 90–104 (1999)
44. Shi, J.-Y.: The enumeration of Coxeter elements. *J. Algebr. Comb.* **6**, 161–171 (1997)
45. Shi, J.-Y.: Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups. *Proc. Am. Math. Soc.* **131**, 3371–3378 (2003)
46. Shi, J.-Y.: Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups. II. *Proc. Am. Math. Soc.* **133**, 2525–2531 (2005)
47. Stanley, R.P.: *Enumerative Combinatorics, Vol. 1*. Cambridge Studies in Advanced Mathematics, vol. 49. Cambridge University Press, Cambridge (1997)
48. Stembridge, J.R.: On the fully commutative elements of Coxeter groups. *J. Algebr. Comb.* **5**, 353–385 (1996)
49. Temperley, H.N.V., Lieb, E.H.: Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the “percolation” problem. *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.* **322**, 251–280 (1971)