On a class of wreath products of hypergroups and association schemes

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Abstract We characterize finite hypergroups S (in the sense of Frédéric Marty in Huitième Congrès des Mathématiciens, pp. 45–59, 1934) satisfying $|pq| = 1$ for any two elements p and q in S with $p \neq q^*$ in terms of wreath products. The result applies to association schemes of finite valency and provides a corresponding characterization in scheme theory. For association schemes S of finite valency satisfying the above condition, we provide a second characterization, a characterization in terms of the subconstituent algebra of S .

Keywords Groups · Association schemes · Hypergroups · Wreath products · Subconstituent algebras

Let S be a set, and let μ be a map from $S \times S$ to the power set of S . For any two elements p and q of S , we write pq instead of $\mu(p, q)$ and assume that pq is not empty.

For any two nonempty subsets P and Q of S , we define the *complex product* PQ to be the union of the sets pq with $p \in P$ and $q \in Q$. If one of the two factors in a complex product consists of a single element, say s , we write s instead of $\{s\}$ in that product.

Following (and generalizing) Frédéric Marty's terminology in [4], we call S a *hypergroup* (with respect to μ) if the following three conditions hold:¹

¹ It is worth mentioning that the notion of a hypergroup arises in many different ways in the literature. For fairly complete overviews on the different definitions of hypergroups that have been given since Marty, we refer to [1] and [3]. The definition we use in the present note is almost Marty's original definition. The only difference is that we do not assume that $pq = qp$ for any two hypergroup elements p and q .

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- H1 For any three elements p, q , and r in S , we have $p(qr) = (pq)r$.
 H2 The set S possesses an element e such that $se = \{s\}$ for each element s in S .
 H3 For each element s in S , there exists an element s^* in S such that $p \in rq^*$ and $q \in p^*r$ for any three elements p, q , and r in S satisfying $r \in pq$.

Each association scheme satisfies the above three conditions H1, H2, and H3 with respect to its complex multiplication; cf. [8, Lemma 1.3.1], [8, Lemma 1.3.3(i)], and [8, Lemma 1.3.3(ii)]. Thus, hypergroups generalize association schemes. In particular, they generalize groups.²

It is not so much the fact itself that each association scheme (and, therefore, each group) is a hypergroup which caught our interest in hypergroups, more interesting is the observation that many proofs in scheme theory can be divided clearly into a part which uses only hypergroup theoretic arguments and a part in which one has to deal with the underlying set on which the association scheme is defined. A first systematic attempt to separate the hypergroup axioms from scheme theory was made in [9]. Condition (a) of Theorem A seems to be a natural question within the framework of [9].

The hypergroups which correspond to groups are the hypergroups S in which $|pq| = 1$ for any two elements p and q in S .³ In the present article, we slightly weaken this latter condition by looking at hypergroups S where $|pq| = 1$ only for elements p and q in S with $p \neq q^*$. We prove the following theorem.

Theorem A *Let S be a finite hypergroup. Then the following conditions are equivalent:*

- For any two elements p and q in S with $p \neq q^*$, we have $|pq| = 1$.*
- The hypergroup S is a wreath product of finitely many finite hypergroups all of which are thin, are generated by an involution, or are generated by an idempotent element of S .*

The definition of a wreath product of hypergroups will be given in Section 3. The definition of a thin hypergroup, the one of an involution, and the one of an idempotent element of a hypergroup will be given in Section 4.

Since each association scheme is a hypergroup with respect to its complex multiplication, Theorem A applies to association schemes. More specifically, we shall prove the following (which generalizes part of [6, Theorem 1]).

Theorem B *Let S be an association scheme of finite valency. Then the following conditions are equivalent.*

- For any two elements p and q in S with $p \neq q^*$, we have $|pq| = 1$.*

²We also should mention that the set of the distinguished basis elements of a table algebra in the sense of [2] is a finite hypergroup with respect to the multiplication introduced in [2, Definition 1.4]. Of course, the converse is not true. In fact, each hypergroup element s satisfying $ss = \{s\}$, $s^* \neq s$, and $s^*s = \{1, s, s^*\}$ gives rise to a hypergroup with three elements which does not come from a table algebra.

³In Lemma 1.5, we shall see that these are exactly the hypergroups which contain “thin” elements only.

- (b) *The scheme S is a wreath product of finitely many finite schemes all of which are thin or generated by an involution.*
- (c) *For each nonprimary irreducible module W of the subconstituent algebra of S , we have $|\text{Supp}(W)| = 1$.*

As far as scheme theoretic terminology and notation is concerned, we refer to [8]. The subconstituent algebra of an association scheme was introduced in [7]. We shall give its definition at the beginning of Section 6. Section 6 is also the place where we define the primary irreducible module of the subconstituent algebra and the set $\text{Supp}(W)$.

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1 Preliminaries on hypergroups

In this section, the letter S stands for a hypergroup.

The first three results of this section were first proven in [9]. They are [9, Lemma 1.1.1], [9, Lemma 1.1.2], and [9, Lemma 1.1.4(i), (ii)]. In order to keep this note self-contained, we repeat the (short) proofs of these lemmas here.

An element e of S which satisfies $se = \{s\}$ for each element s in S is called a *neutral* element of S . Condition H2 guarantees the existence of neutral elements.

Lemma 1.1 *Let s be an element in S . Then the following hold:*

- (i) *For each neutral element e of S , we have $e \in s^*s$.*
- (ii) *We have $s^{**} = s$.*
- (iii) *For each neutral element e of S , we have $es = \{s\}$.*

Proof (i) Let e be a neutral element of S . Then, by definition, $s \in se$. Thus, by H3, $e \in s^*s$.

(ii) From (i) we know that $e \in s^*s$. Thus, by H3, $s \in s^{**}e$. On the other hand, by H2, $s^{**}e = \{s^{**}\}$. Thus, $s \in \{s^{**}\}$, and that means that $s^{**} = s$.

(iii) Let r be an element in es . Then, by H3, $e \in rs^*$. A second application of H3 yields $s^* \in r^*e$. On the other hand, by H2, $r^*e = \{r^*\}$. Thus, $s^* \in \{r^*\}$, and that means that $r^* = s^*$. Thus, by (ii), $r = r^{**} = s^{**} = s$. $\square$

Lemma 1.2 *The hypergroup S possesses exactly one neutral element.*

Proof Let c and d be neutral elements of S . Then, by Lemma 1.1(iii), $\{c\} = cd = \{d\}$. It follows that $c = d$. $\square$

For the remainder of this note, the uniquely determined neutral element of a hypergroup will be denoted by 1. Looking at H2 and H3, one easily verifies that $1^* = 1$.

Lemma 1.3 *Let p and q be elements in S . Then the following hold:*

- (i) *We have $1 \in pq$ if and only if $p = q^*$.*
- (ii) *Let r be an element in S satisfying $r \in pq$. Then $r^* \in q^*p^*$.*

Proof (i) Assume that $1 \in pq$. Then, by H3, $p \in 1 \cdot q^*$. On the other hand, by Lemma 1.1(iii), $1 \cdot q^* = \{q^*\}$. Thus, $p \in \{q^*\}$, and that means that $p = q^*$.

Conversely, assume that $p = q^*$. Then, by Lemma 1.1(i), $1 \in q^*q = pq$.

(ii) Applying H3 three times, we obtain from $r \in pq$ first $p \in rq^*$, then $q^* \in r^*p$, and, finally, $r^* \in q^*p^*$. $\square$

For each element s of S , we define

$$T_s := \{r \in S \mid sr = \{s\}\}.$$

Note that $T_1 = \{1\}$.

Lemma 1.4 *Let s be an element in S . Then the following hold:*

- (i) *Assume that $s \neq 1$. Then $T_s \subseteq s^*s \setminus \{s^*\}$.*
- (ii) *Let p and q be elements in T_s . Then $pq \subseteq T_s$.*

Proof (i) Let r be an element in T_s . Then, by definition, $sr = \{s\}$. In particular, $s \in sr$. Thus, by H3, $r \in s^*s$.

Assume that $s^* \in T_s$. Then, by definition, $ss^* = \{s\}$. On the other hand, by Lemma 1.1(i), (ii), $1 \in ss^*$. Thus, $s = 1$, contradicting our choice of s .

(ii) Since $p \in T_s$, $sp = \{s\}$. Similarly, as $q \in T_s$, $sq = \{s\}$. Thus, $spq = \{s\}$. It follows that $sr = \{s\}$ for each element r in pq . $\square$

Recall from Lemma 1.1(i) that $1 \in s^*s$ for each element s in S . An element s of S is called *thin* if $s^*s = \{1\}$. The set of all thin elements of S will be denoted by $O_\varnothing(S)$.

Lemma 1.5 *For any two elements p and q in $O_\varnothing(S)$, there exists an element r in $O_\varnothing(S)$ such that $pq = \{r\}$.*

Proof Let p and q be elements of $O_\varnothing(S)$. Then $p^*p = \{1\}$ and $q^*q = \{1\}$.

Let r and s be elements in pq . Then, referring to Lemma 1.3(ii), we obtain

$$r^*s \subseteq q^*p^*pq \subseteq q^*q = \{1\}.$$

Thus, by Lemma 1.3(i), $r^* = s^*$. It follows that $r = s$; cf. Lemma 1.1(ii). Thus, as $r^*s \subseteq \{1\}$, $r^*r \subseteq \{1\}$, and that means that $r \in O_\varnothing(S)$. $\square$

2 Closed subsets of hypergroups

In this section, the letter S stands for a hypergroup.

A nonempty subset R of S is called *closed* if $p^*q \subseteq R$ for any two elements p and q in R .

Note that $\{1\}$ and S are closed and that intersections of closed subsets are closed.

Lemma 2.1 *Let T and U be closed subsets of S . Then the following hold:*

- (i) *For any two elements p and q in S with $p \in TqU$, we have $TqU \subseteq TpU$.*
- (ii) *The set $\{TsU \mid s \in S\}$ is a partition on S .*

Proof (i) Let p and q be elements in S , and assume that $p \in TqU$. Then there exist elements t in T and u in U such that $p \in tqu$. Thus, tq possesses an element w such that $p \in wu$.

From $w \in tq$ we obtain $q \in t^*w$; cf. H3. Similarly, as $p \in wu$, $w \in pu^*$. Thus, as T and U are assumed to be closed,

$$q \in t^*w \subseteq t^*pu^* \subseteq TpU.$$

It follows that $TqU \subseteq TpU$.

(ii) For each element s in S , we have $s \in TsU$. Thus, S is equal to the union of the sets TsU with $s \in S$.

Let p and q be elements in S , assume that $TpU \cap TqU$ is not empty, and let r be an element in $TpU \cap TqU$. Since $r \in TpU$, $TrU \subseteq TpU$. From $r \in TqU$ we also obtain $TpU \subseteq TrU$; cf. (i). Thus, $TpU = TrU$. Similarly, $TqU = TrU$, so that $TpU = TqU$. $\square$

Lemma 2.2 *Let T be a closed subset of S . Let p, p', q , and q' be elements in S , and assume that $TpT = Tp'T$ and $TqT = Tq'T$. Then $\{TrT \mid r \in pTq\} = \{TrT \mid r \in p'Tq'\}$.*

Proof Let s be an element in S , and assume that $TsT \in \{TrT \mid r \in pTq\}$. Then there exists an element r in pTq such that $TsT = TrT$. It follows that

$$s \in TrT \subseteq TpTqT = Tp'Tq'T.$$

Thus, there exists an element r' in $p'Tq'$ such that $s \in Tr'T$. From $s \in Tr'T$ we obtain $TsT = Tr'T$; cf. Lemma 2.1(ii). Thus, as $r' \in p'Tq'$, $TsT \in \{TrT \mid r \in p'Tq'\}$.

So far, we have shown that $\{TrT \mid r \in pTq\} \subseteq \{TrT \mid r \in p'Tq'\}$. That $\{TrT \mid r \in p'Tq'\} \subseteq \{TrT \mid r \in pTq\}$ follows similarly. $\square$

Let T be a closed subset of S . For each element s in S , we define

$$s^T := TsT,$$

for each closed subset U of S with $T \subseteq U$, we set

$$U//T := \{u^T \mid u \in U\},$$

and, for any two elements p and q of S , we define

$$\mu_T(p^T, q^T) := \{r^T \mid r \in pTq\}.$$

Then, by Lemma 2.2, μ_T is a map from $S//T \times S//T$ to the power set of $S//T$. In the following, we shall write p^Tq^T instead of $\mu_T(p^T, q^T)$ whenever p and q are elements in S . Moreover, for each element s in S , we define $(s^T)^* := (s^*)^T$.

Lemma 2.3 *Let T be a closed subset of S . Then $S//T$ is a hypergroup (with respect to μ_T).*

Proof From H1 one easily obtains $p^T(q^T r^T) = (p^T q^T)r^T$ for any three elements p , q , and r in S .

For each element s in S , we obviously have $s^T 1^T = \{s^T\}$.

Let p , q , and r be elements in S , and assume that $r^T \in p^T q^T$. We first show that $p^T \in r^T (q^T)^*$.

From $r^T \in p^T q^T$ we obtain an element r' in pTq such that $r^T = r'^T$.

From $r' \in pTq$ we obtain an element t in T such that $r' \in ptq$. From $r' \in ptq$ we obtain $p \in r'q^*t^*$; cf. H3 and Lemma 1.3(ii). Thus, $r'q^*$ possesses an element p' such that $p \in p't^*$.

From $p' \in r'q^*$ we obtain $p' \in r'Tq^*$. Thus, by definition, $p'^T \in r'^T (q^T)^*$. From $p \in p't^*$ we obtain $TpT = Tp'T$; cf. Lemma 2.1(ii). Thus, $p^T = p'^T$. It follows that $p^T \in r'^T (q^T)^*$. Thus, as $r^T = r'^T$, $p^T \in r^T (q^T)^*$.

That $q^T \in (p^T)^* r^T$ follows similarly. □

Let R be a subset of S . We define $\langle R \rangle$ to be the intersection of all closed subsets of S which contain R (as a subset). If $R = \{r\}$ for some element r in S , we write $\langle r \rangle$ instead of $\langle R \rangle$. Note that $\langle R \rangle$ is closed. We call $\langle R \rangle$ the closed subset *generated by* R . We define R^* to be the set of all elements r^* with $r \in R$. If R is not empty, we set $R^0 := \{1\}$ and define inductively $R^n := R^{n-1}R$ for each positive integer n . The following lemma is [9, Lemma 1.6.1].

Lemma 2.4 *Let R be a nonempty subset of S . Then $\langle R \rangle$ is equal to the union of the sets $(R^* \cup R)^n$ with n a nonnegative integer.*

Proof Set $P := R^* \cup R$ and define Q to be the union of the sets P^n with n a nonnegative integer. We have to show that $\langle R \rangle = Q$.

Since $P^* = P$, $(P^n)^* = P^n$ for each nonnegative integer n ; cf. Lemma 1.3(ii). Thus, for any two nonnegative integers k and l ,

$$(P^k)^* P^l = P^k P^l = P^{k+l} \subseteq Q.$$

It follows that Q is closed. Thus, as $R \subseteq Q$, $\langle R \rangle \subseteq Q$.

Conversely, for each nonnegative integer n , we have

$$P^n \subseteq \langle P \rangle^n \subseteq \langle P \rangle = \langle R \rangle.$$

Therefore, $Q \subseteq \langle R \rangle$. □

Lemma 2.5 *Let s be an element in S . Then the following hold.*

- (i) *Let R be a nonempty subset of T_s satisfying $R^* \subseteq T_s$. Then $\langle R \rangle \subseteq T_s$.*
- (ii) *Assume that $(T_s)^* = T_s$. Then T_s is a closed subset of S .*

Proof (i) Let n be a positive integer, and let $s_1, \dots, s_n$ be elements in $R^* \cup R$. Then, by Lemma 1.4(ii), $s_1 \cdots s_n \subseteq T_s$. Thus, the claim follows from Lemma 2.4. $\square$

(ii) Setting $R = T_s$ in (i), this follows from (i). $\square$

Lemma 2.6 *Assume S to be finite. Then $O_{\vartheta}(S)$ is a closed subset of S .*

Proof Let s be an element in $O_{\vartheta}(S)$. Considering Lemma 1.5, we shall be done if we succeed in showing that $s^* \in O_{\vartheta}(S)$.

From Lemma 1.5 we obtain $|s^n| = 1$ for each positive integer n . Thus, as S is assumed to be a finite set, there exist positive integers k and l such that $k \leq l - 1$ and $s^k = s^l$. Let p denote the uniquely determined element in s^l . Then, as $s^k = s^l$, $s^k = \{p\}$. Thus, by Lemma 1.3(ii), $(s^*)^k = \{p^*\}$. It follows that

$$1 \in p^* p = (s^*)^k s^l = (s^*)^k s^k s^{l-k} = s^{l-k} = s^{l-k-1} s.$$

Thus, s^{l-k-1} possesses an element q such that $1 \in qs$. Thus, by Lemma 1.3(i), $q = s^*$. Thus, as $q \in s^{l-k-1}$,

$$s^* \in s^{l-k-1} \subseteq O_{\vartheta}(S);$$

cf. Lemma 1.5. $\square$

3 Homomorphisms of hypergroups

Let S and S' be hypergroups. A map ϕ from S to S' is called a *homomorphism* if $\phi(1) = 1$ and

$$\phi(pq) = \phi(p)\phi(q)$$

for any two elements p and q in S .

Lemma 3.1 *Let S and S' be hypergroups, let ϕ be a homomorphism from S to S' , and let s be an element in S . Then $\phi(s^*) = \phi(s)^*$.*

Proof By definition, $\phi(1) = 1$. On the other hand, by Lemma 1.1(i), $1 \in s^*s$. Thus,

$$1 = \phi(1) \in \phi(s^*s) = \phi(s^*)\phi(s).$$

Thus, by Lemma 1.3(i), $\phi(s^*) = \phi(s)^*$. $\square$

Let S and S' be hypergroups, and let ϕ be a homomorphism from S to S' . We define

$$\ker(\phi) := \{s \in S \mid \phi(s) = 1\}$$

and call this set the *kernel* of ϕ .

Lemma 3.2 *Let S and S' be hypergroups, let ϕ be a homomorphism from S to S' , and set $T := \ker(\phi)$. Then we have $p \in qT$ for any two elements p and q in S satisfying $\phi(p) = \phi(q)$.*

Proof Let p and q be elements in S , and assume that $\phi(p) = \phi(q)$. Then

$$1 \in \phi(q)^* \phi(p) = \phi(q^*) \phi(p) = \phi(q^* p);$$

cf. Lemmas 1.3(i) and 3.1. Thus, $q^* p$ possesses an element t such that $\phi(t) = 1$. From $t \in q^* p$ we obtain $p \in qt$. From $\phi(t) = 1$ we obtain $t \in T$. Thus, $p \in qT$. $\square$

A closed subset T of a hypergroup S will be called *normal* in S if $Ts \subseteq sT$ for each element s in S .⁴

A bijective homomorphism is called an *isomorphism*.

If there exists an isomorphism from a hypergroup S to a hypergroup S' , we say that S and S' are *isomorphic*, and we indicate this by writing $S \cong S'$.

In the following theorem, we shall write $\text{im}(\phi)$ to denote the image of the map ϕ .

Theorem 3.3 *Let S and S' be hypergroups, and let ϕ be a homomorphism from S to S' . Then the following hold:*

- (i) *The set $\ker(\phi)$ is a normal closed subset of S .*
- (ii) *The set $\text{im}(\phi)$ is a closed subset of S' .*
- (iii) *We have $S/\ker(\phi) \cong \text{im}(\phi)$.*

Proof (i) We first prove that $\ker(\phi)$ is closed. In order to do so, we fix elements p and q in $\ker(\phi)$ and an element r in p^*q . We have to show that $r \in \ker(\phi)$.

From $p \in \ker(\phi)$ we obtain $\phi(p) = 1$. Similarly, as $q \in \ker(\phi)$, $\phi(q) = 1$. Thus, referring to Lemma 3.1, we obtain

$$\phi(r) \in \phi(p^*q) = \phi(p^*)\phi(q) = \phi(p)^*\phi(q) = 1^*1 = \{1\}.$$

It follows that $\phi(r) = 1$, and that means that $r \in \ker(\phi)$.

Now we prove that $\ker(\phi)$ is normal in S . In order to do so, we set $T := \ker(\phi)$. Let t be an element in T , and let s be an element in S . We have to show that $ts \subseteq sT$.

Let r be an element in ts . Then

$$\phi(r) \in \phi(ts) = \phi(t)\phi(s) = 1 \cdot \phi(s) = \{\phi(s)\}.$$

Thus, $\phi(r) = \phi(s)$. Thus, by Lemma 3.2, $r \in sT$.

- (ii) Let v and w be elements in $\text{im}(\phi)$. We have to show that $v^*w \subseteq \text{im}(\phi)$.

Since $v \in \text{im}(\phi)$, S possesses an element p such that $\phi(p) = v$. Similarly, as $w \in \text{im}(\phi)$, S possesses an element q such that $\phi(q) = w$. Thus, by Lemma 3.1,

$$v^*w = \phi(p)^*\phi(q) = \phi(p^*)\phi(q) = \phi(p^*q) \subseteq \text{im}(\phi).$$

- (iii) Define $T := \ker(\phi)$.

⁴Let T be a normal closed subset of a hypergroup S , and let s be an element in S . Then $Ts^* \subseteq s^*T$. Thus, by Lemma 1.3(ii), $sT \subseteq Ts$. On the other hand, as T is assumed to be normal in S , $Ts \subseteq sT$. It follows that $Ts = sT$.

We first claim that, for any two elements p and q in S , $p^T = q^T$ if and only if $\phi(p) = \phi(q)$.

Assume first that $p^T = q^T$. Then, as T is normal in S , $p \in qT$, so that $p \in qt$ for some element t in T . Thus,

$$\phi(p) \in \phi(qt) = \phi(q)\phi(t) = \phi(q) \cdot 1 = \{\phi(q)\}.$$

It follows that $\phi(p) = \phi(q)$. Conversely, if $\phi(p) = \phi(q)$, then $p \in qT$; cf. Lemma 3.2. Thus, $TpT = TqT$, and that means that $p^T = q^T$.

Now we define $\psi(s^T) := \phi(s)$ for each element s in S . Then, as $p^T = q^T$ if and only if $\phi(p) = \phi(q)$ for any two elements p and q in S , ψ is an injective map from $S//T$ to $\text{im}(\phi)$. The definition of ψ also implies that ψ is surjective.

In order to show that ψ is a homomorphism, we pick elements p and q in S . Then

$$\begin{aligned}\psi(p^T q^T) &= \psi(\{r^T \mid r \in pTq\}) \\ &= \{\psi(r^T) \mid r \in pTq\} = \{\phi(r) \mid r \in pTq\}.\end{aligned}$$

For each element r in pTq , there exists an element t in T such that $r \in ptq$, and from $r \in ptq$ we obtain

$$\phi(r) \in \phi(ptq) = \phi(p)\phi(t)\phi(q) = \phi(p) \cdot 1 \cdot \phi(q) = \phi(pq).$$

Thus,

$$\psi(p^T q^T) = \phi(pq) = \phi(p)\phi(q) = \psi(p^T)\psi(q^T),$$

and we are done. $\square$

Theorem 3.4 *Let S be a hypergroup, let T and U be closed subsets of S , and assume that $T \subseteq U$. Then the following hold:*

- (i) *The set $U//T$ is closed in $S//T$.*
- (ii) *Assume that U is normal in S . Then $(S//T)//(U//T) \cong S//U$.*

Proof (i) Let p and q be elements in U . We have to show that $(p^T)^*q^T \subseteq U//T$.

Let r be an element in S , and assume that $r^T \in (p^T)^*q^T$. Then, by definition, $r \in Tp^*TqT \subseteq U$. It follows that $r^T \in U//T$.

(ii) For any two elements p and q in S with $TpT = TqT$, we have

$$UpU = (UT)p(TU) = U(TpT)U = U(TqT)U = (UT)q(TU) = UqU.$$

Thus, setting $\phi(s^T) := s^U$ for each element s in S , ϕ is a surjective map from $S//T$ to $S//U$.

Note also that

$$\phi(p^T q^T) = \phi(\{r^T \mid r \in pTq\}) = \{r^U \mid r \in pTq\}$$

and

$$\phi(p^T)\phi(q^T) = p^U q^U = \{r^U \mid r \in pUq\}$$

for any two elements p and q in S . Thus, as U is normal in S ,

$$\phi(p^T q^T) = \phi(p^T)\phi(q^T)$$

for any two elements p and q in S .

Note finally that, for each element s in S , $\phi(s^T) = 1^U$ if and only if $s \in U$. Thus, $\ker(\phi) = U//T$, and the claim follows from Theorem 3.3(iii). $\square$

Lemma 3.5 *Let S be a hypergroup, let T be a closed subset of S , and assume that $T \subseteq T_s$ for each element s in $S \setminus T$. Then T is normal in S .*

Proof For each element s in T , we have $Ts = T = sT$.

Let s be an element in $S \setminus T$. Then $T \subseteq T_s$ by hypothesis. Thus, by definition $sT = \{s\}$.

Since T is assumed to be closed, we obtain from $s \in S \setminus T$ also that $s^* \in S \setminus T$. Thus, $T \subseteq T_{s^*}$, and that means that $s^*T = \{s^*\}$. From this we obtain $Ts = \{s\}$; cf. Lemma 1.3(ii). Thus, as $sT = \{s\}$, $Ts = sT$. $\square$

Let S_1 and S_2 be hypergroups.

A hypergroup S is said to be a *wreath product* of S_1 and S_2 if it possesses a closed subset T such that

$$T \cong S_1, \quad S//T \cong S_2,$$

and $T \subseteq T_s$ for each element s in $S \setminus T$.

If a hypergroup S is a wreath product of S_1 and S_2 , we write $S = S_1 \wr S_2$.

Note that, by Lemma 3.5, the closed subset T that arises in the definition of a wreath product S is normal in S .

We also notice that a wreath product of association schemes in the sense of [5] is always a wreath product of hypergroups with respect to its complex multiplication. Conversely, if an association scheme S is a wreath product of association schemes S_1 and S_2 with respect to its complex multiplication, and if S_1 is uniquely determined by its structure constants, then S is a wreath product of the association schemes S_1 and S_2 in the sense of [5].

We shall now prove that wreath multiplication is associative.

Theorem 3.6 *For any three hypergroups S_1 , S_2 , and S_3 , we have $(S_1 \wr S_2) \wr S_3 = S_1 \wr (S_2 \wr S_3)$.*

Proof Set $S := (S_1 \wr S_2) \wr S_3$. Then, by definition, S possesses a closed subset U such that

$$U \cong S_1 \wr S_2, \quad S//U \cong S_3,$$

and $U \subseteq T_s$ for each element s in $S \setminus U$. Since $U \cong S_1 \wr S_2$, U possesses a closed subset T such that

$$T \cong S_1, \quad U//T \cong S_2,$$

and $T \subseteq T_u$ for each element u in $U \setminus T$.

Since $T \subseteq U$, we have $T \subseteq T_s$ for each element s in $S \setminus T$. Thus, as $T \cong S_1$, we shall be done if we succeed in showing that $S//T \cong S_2 \wr S_3$.

We saw already that $U//T \cong S_2$. Moreover, by Theorem 3.4(ii), $(S//T)/(U//T) \cong S//U \cong S_3$. (Recall that, by Lemma 3.5, U is normal in S .)

Let s be an element in $S \setminus U$, and let u be an element in U . Then, by definition,

$$s^T u^T = \{r^T \mid r \in sTu\}.$$

Since $s \in S \setminus U$, $U \subseteq T_s$. Thus, $sTu \subseteq sU = \{s\}$. It follows that $s^T u^T = \{s^T\}$. Thus, as u has been chosen arbitrarily in U , we have shown that $U//T \subseteq T_s T$. $\square$

4 Proof of Theorem A

In this section, the letter S stands for a hypergroup.

Recall that condition (a) of Theorem A says that $|pq| = 1$ for any two elements p and q in S with $p \neq q^*$. Within this section, we shall always mean condition (a) of Theorem A if we speak about condition (a).

Lemma 4.1 *Assume that S satisfies condition (a), and let T be a normal closed subset of S . Then $S//T$ satisfies condition (a).*

Proof Let p and q be elements in S , and assume that $p^T \neq (q^T)^*$. Then $p \neq q^*$. Thus, as we assume that S satisfies condition (a), there exists an element s in S such that $pq = \{s\}$. Thus, as T is assumed to be normal in S ,

$$p^T q^T = \{r^T \mid r \in pTq\} = \{s^T\},$$

so that $|p^T q^T| = 1$. $\square$

Lemma 4.2 *Assume that S is finite and satisfies condition (a). Then $O_\vartheta(S) \subseteq T_s$ for each element s in $S \setminus O_\vartheta(S)$.*

Proof Set $T := O_\vartheta(S)$, let s be an element in $S \setminus T$, and let t be an element in T . We have to show that $t \in T_s$, and that means that $st = \{s\}$.

Since S is assumed to be finite, T is closed; cf. Lemma 2.6. Thus, as $t \in T$, $t^* \in T$.

Since $s \in S \setminus T$ and $t^* \in T$, $s \neq t^*$. Thus, as S is assumed to satisfy condition (a), $st = \{r\}$ for some element r in S .

Assume, by way of contradiction, that $r \neq s$. Then $s^* \neq r^*$. Thus, as S is assumed to satisfy condition (a), $|s^* r| = 1$. Thus, as $t \in s^* r$, $s^* r = \{t\}$. Thus, as $st = \{r\}$, $s^* st = \{t\}$. It follows that

$$s^* s = s^* stt^* = tt^* = \{1\};$$

recall that $t^* \in T$. Thus, by definition, $s \in T$, contradiction. $\square$

Lemma 4.3 *Assume that S satisfies condition (a), let t be an element of S , and let s be an element in $S \setminus \langle t \rangle$. Then the following hold:*

- (i) We have $t^*t \subseteq T_s$.
- (ii) Assume that $t \in tt$. Then $\langle t \rangle \subseteq T_s$.

Proof (i) Since $s \in S \setminus \langle t \rangle$, $s \neq t$. Thus, as S is assumed to satisfy condition (a), $st^* = \{r\}$ for some element r in S .

From $r \in st^*$ we obtain $s \in rt$; cf. H3. If $r = t^*$, $s \in t^*t \subseteq \langle t \rangle$, contradiction. Thus, $r \neq t^*$, so that $rt = \{s\}$.

From $st^* = \{r\}$ and $rt = \{s\}$ we obtain $st^*t = \{s\}$. Thus, $t^*t \subseteq T_s$.

(ii) We are assuming that $t \in tt$. Thus, $t \in t^*t$ and $t^* \in t^*t$. Thus, by (i), $t \in T_s$ and $t^* \in T_s$. Thus, by Lemma 2.5(i), $\langle t \rangle \subseteq T_s$. $\square$

Lemma 4.4 Assume that S satisfies condition (a), and let s be an element in $S \setminus \{1\}$. Then $T_s = s^*s \setminus \{s^*\}$.

Proof From Lemma 1.4(i) we know that $T_s \subseteq s^*s \setminus \{s^*\}$. So it remains to show that $s^*s \setminus \{s^*\} \subseteq T_s$.

Let r be an element in $s^*s \setminus \{s^*\}$. From $r \neq s^*$ we obtain $s \neq r^*$. Thus, as S is assumed to satisfy condition (a), $|sr| = 1$. On the other hand, as $r \in s^*s$, $s \in sr$; cf. H3. Thus, $sr = \{s\}$, and that means that $r \in T_s$. $\square$

Lemma 4.5 Assume that S satisfies condition (a), let s be an element in S , and assume that $s \notin ss$. Then the following hold.

- (i) We have $T_s = s^*s$.
- (ii) The set T_s is a closed subset of S .

Proof (i) We are assuming that $s \notin ss$. Thus, by H3, $s \notin ss^*$. A second application of H3 now yields $s^* \notin s^*s$. Thus, the claim follows from Lemma 4.4.

(ii) From (i) we know that $T_s = s^*s$, from Lemma 1.3(ii) that $(s^*s)^* = s^*s$. Thus, $(T_s)^* = T_s$, so that the claim follows from Lemma 2.5(ii). $\square$

The hypergroup S is called *primitive* if S has exactly two closed subsets (which then must be $\{1\}$ and S .) It is called *thin* if each of its elements is thin.

Lemma 4.6 Assume that S is primitive and satisfies condition (a). Assume further that S is finite and that S possesses an element s such that $s \notin ss$. Then S is thin.

Proof We are assuming that S possesses an element s such that $s \notin ss$. Thus, by the definition of T_s , $s \notin T_s$. In particular, $T_s \neq S$.

On the other hand, according to Lemma 4.5(ii), T_s is closed. Thus, as S is assumed to be primitive, we must have $T_s = \{1\}$. Thus, by Lemma 4.5(i), $s^*s = \{1\}$, and that means that $s \in O_\vartheta(S)$.

Recall that S is assumed to be finite. Thus, by Lemma 2.6, $O_\vartheta(S)$ is a closed subset of S . Note also that $O_\vartheta(S) \neq \{1\}$, since $s \neq 1$ and $s \in O_\vartheta(S)$. Thus, as S is assumed to be primitive, $S = O_\vartheta(S)$, and that means that S is thin. $\square$

An element s of $S \setminus \{1\}$ will be called an *involution* if $\{1, s\}$ is closed.

Lemma 4.7 Assume that S is primitive and satisfies condition (a). Let s be an element in $S \setminus \{1\}$, and assume that $s \in ss$ and $s^* = s$. Then s is an involution and $S = \langle s \rangle$.

Proof We are assuming that $s^* = s$. Thus, by Lemma 4.4, $T_s = ss \setminus \{s\}$. It follows that $(T_s)^* = T_s$. Thus, by Lemma 2.5(ii), T_s is closed. Thus, as S is assumed to be primitive, $T_s = \{1\}$ or $T_s = S$.

Since $s \notin T_s$, $T_s \neq S$. It follows that $T_s = \{1\}$. Thus, as $T_s = ss \setminus \{s\}$, $ss = \{1\}$ or $ss = \{1, s\}$.

If $ss = \{1\}$, our hypothesis $s \in ss$ yields $s = 1$. Since this contradicts our choice of s , we must have $ss = \{1, s\}$. Thus, s is an involution. Thus, as S is assumed to be primitive, $S = \langle s \rangle$. $\square$

An element s of $S \setminus \{1\}$ will be called *idempotent* if $ss = \{s\}$ and $|\langle s \rangle| = 3$.

Note that, for each idempotent element s of S , $\langle s \rangle = \{1, s, s^*\}$.

Lemma 4.8 Assume that S is primitive and satisfies condition (a). Let s be an element in S , and assume $s \in ss$ and $s^* \neq s$. Then the following hold:

- (i) We have $T_s = \{1, s\}$.
- (ii) We have $s^*s = \{1, s, s^*\}$.
- (iii) The element s is idempotent, and $S = \langle s \rangle$.

Proof (i) We are assuming that $s \in ss$ and $s^* \neq s$. Thus, as S is assumed to satisfy condition (a), $ss = \{s\}$. This means that $s \in T_s$. Thus, as $1 \in T_s$, $\{1, s\} \subseteq T_s$.

Assume, by way of contradiction, that $T_s \not\subseteq \{1, s\}$. Then T_s possesses an element r different from 1 and s .

Since $r \in T_s$, $r \in s^*s \setminus \{s^*\}$; cf. Lemma 4.4. Thus, as $r \neq s$, $r \in s^*s \setminus \{s^*, s\}$. It follows that $r^* \in s^*s \setminus \{s^*, s\} \subseteq T_s$. Thus, by Lemma 2.5(i), $\langle r \rangle \subseteq T_s$.

Now recall that we are assuming S to be primitive. Thus, as $r \neq 1$, $\langle r \rangle = S$. It follows that $T_s = S$. This contradicts Lemma 4.4, where we saw that $s^* \notin T_s$.

(ii) Recall from Lemma 4.4 that $T_s = s^*s \setminus \{s^*\}$. Thus, by (i), $\{1, s\} = s^*s \setminus \{s^*\}$.

We are assuming that $s \in ss$. Thus, by H3, $s \in ss^*$. A second application of H3 now yields $s^* \in s^*s$. Thus, as $\{1, s\} = s^*s \setminus \{s^*\}$, we must have $s^*s = \{1, s, s^*\}$.

(iii) From $s^* \neq s$ we obtain $s \neq 1$. Thus, as S is assumed to be primitive, $S = \langle s \rangle$.

We are assuming that $s \in ss$ and $s^* \neq s$. Thus, as S is assumed to satisfy condition (a), $ss = \{s\}$.

From $s \in ss$ we also obtain $s^* \in s^*s^*$; cf. Lemma 1.3(ii). Thus, interchanging the roles of s and s^* in (ii) we obtain $ss^* = \{1, s, s^*\}$. Thus, $\langle s \rangle = \{1, s, s^*\}$. $\square$

We are now ready to prove Theorem A. For the convenience of the reader, we restate the theorem.

Theorem A Assume S to be finite. Then the following conditions are equivalent:

- (a) For any two elements p and q in S with $p \neq q^*$, we have $|pq| = 1$.
- (b) The hypergroup S is a wreath product of finitely many finite hypergroups all of which are thin or generated by an involution or by an idempotent element of S .

Proof (a) $\Rightarrow$ (b) Since S is assumed to be finite, we have $O_{\vartheta}(S) \subseteq T_s$ for each element s in $S \setminus O_{\vartheta}(S)$; cf. Lemma 4.2. Thus, by definition,

$$S \cong O_{\vartheta}(S) \wr S//O_{\vartheta}(S).$$

The fact that $O_{\vartheta}(S) \subseteq T_s$ for each element s in $S \setminus O_{\vartheta}(S)$ also implies that $O_{\vartheta}(S)$ is normal in S ; cf. Lemma 3.5. Thus, as S is assumed to satisfy (a), $S//O_{\vartheta}(S)$ satisfies (a); cf. Lemma 4.1.

Assume first that $O_{\vartheta}(S) \neq \{1\}$. Then, as $S//O_{\vartheta}(S)$ satisfies (a), induction allows us to conclude that $S//O_{\vartheta}(S)$ satisfies (b). Thus, as $S \cong O_{\vartheta}(S) \wr S//O_{\vartheta}(S)$, S satisfies condition (b); cf. Theorem 3.6.

Assume now that $O_{\vartheta}(S) = \{1\}$. There is nothing to show if $S = \{1\}$. Thus, we assume that $S \neq \{1\}$. Among the closed subsets of S different from $\{1\}$ we chose T minimal. Note that this implies that T is a primitive hypergroup satisfying (a).

Since $T \neq \{1\}$, $T \setminus \{1\}$ is not empty. We fix an element in $T \setminus \{1\}$ and call it t .

Since $O_{\vartheta}(S) = \{1\}$, $t \in tt$; cf. Lemma 4.6. Thus, by Lemmas 4.7 and 4.8(iii), $T = \langle t \rangle$, and t is an involution or an idempotent element of S .

From Lemma 4.3(ii) we also obtain $T \subseteq T_s$ for each element s in $S \setminus T$. Thus, by definition,

$$S = T \wr (S//T).$$

As before, the fact that $T \subseteq T_s$ for each element s in $S \setminus T$ also implies that T is normal in S ; cf. Lemma 3.5. Thus, as S is assumed to satisfy (a), $S//T$ satisfies (a); cf. Lemma 4.1.

Since $T \neq \{1\}$ and $S//T$ satisfies (a), induction allows us to conclude that $S//T$ satisfies (b). Thus, as $S \cong T \wr S//T$, S satisfies condition (b); cf. Theorem 3.6.

(b) $\Rightarrow$ (a) Assume that there exist hypergroups $S_1, \dots, S_n$ such that

$$S = S_1 \wr \dots \wr S_n$$

and, for each element i in $\{1, \dots, n\}$, S_i is thin or generated by an involution or by an idempotent element of S . Clearly we may assume that $S_i \neq \{1\}$ for each element i in $\{1, \dots, n\}$.

Since $S = S_1 \wr \dots \wr S_n$, S possesses a closed subset T such that

$$T \cong S_1, \quad S//T \cong S_2 \wr \dots \wr S_n,$$

and $T \subseteq T_s$ for each element s in $S \setminus T$. Note also that T is normal in S ; cf. Lemma 3.5.

Let p and q be elements in S with $p \neq q^*$. We have to show that $|pq| = 1$.

If $p, q \in T$, we obtain $|pq| = 1$ from $T \cong S_1$ together with the fact that S_i is thin or generated by an involution or by an idempotent element of S .

Assume that $p \in T$ and $q \notin T$. From $p \in T$ we obtain $p^* \in T$. From $q \notin T$ we obtain $q^* \notin T$, and that implies $T \subseteq T_{q^*}$. Thus, $p^* \in T_{q^*}$, and that means that $q^*p^* = \{q^*\}$. Thus, by Lemma 1.3(ii), $pq = \{q\}$.

If $p \notin T$ and $q \in T$, we obtain $q \in T_p$, and that means that $pq = \{p\}$.

Assume now that $p \notin T$ and $q \notin T$. Then $TpT = \{p\}$ and $TqT = \{q\}$. Thus, $TpT \neq Tq^*T$, and that means that $p^T \neq (q^T)^*$. Thus, as $T \neq \{1\}$, induction yields

$$p^T q^T = \{r^T\}$$

for some element r in S .

If $r \in T$, $TpT = Tq^*T$; cf. Lemma 1.3(i). Thus, as $TpT = \{p\}$ and $TqT = \{q\}$, $p = q^*$. Since this contradicts our choice of p and q , we conclude that $r \notin T$. Thus, $TrT = \{r\}$. Thus, as $p^T q^T = \{r^T\}$, $pq = \{r\}$. $\square$

5 Proof of Theorem B, Part I

In this short section, we deal with association schemes. We freely adopt the notation of [8].

Lemma 5.1 *Idempotent elements of association schemes do not have finite valency.*

Proof Let S be an association scheme, let s be an idempotent element of S , let y be an element in X , and let z be an element in ys .

Since s is idempotent, $ss = \{s\}$. Thus, as $z \in ys$, $zs \subseteq yss = ys$. Since $s \neq 1$, $z \notin zs$. Thus, $zs \subseteq ys \setminus \{z\}$. It follows that

$$n_s = |zs| \leq |ys| - 1 = n_s - 1,$$

so that n_s is not finite. $\square$

Proposition 5.2 *Let S be an association scheme on a finite set. Then the following conditions are equivalent:*

- (a) *For any two elements p and q in S with $p \neq q^*$, we have $|pq| = 1$.*
- (b) *The scheme S is a wreath product of finitely many finite schemes all of which are thin or generated by an involution.*

Proof Considering Theorem A, this follows immediately from Lemma 5.1. $\square$

6 Proof of Theorem B, Part II

In this section, the letter X stands for a finite set, the letter S for an association scheme on X . In our notation, we mostly follow [8].

Let C denote the field of the complex numbers, V the free C -module over X , and $\text{End}_C(V)$ the endomorphism ring of V . Then C is a subring of the center of $\text{End}_C(V)$.

For any subset Q of $\text{End}_C(V)$, we shall denote by $C[Q]$ the smallest subring of $\text{End}_C(V)$ which contains C and Q .

Let s be an element in S . Then, as X is a basis of V , there exists a uniquely defined element σ_s in $\text{End}_C(V)$ such that

$$x\sigma_s = \sum_{y \in xs} y$$

for each element x in X .

Note that $\{\sigma_s \mid s \in S\}$ is a linearly independent subset of $\text{End}_C(V)$ and that

$$\sigma_p \sigma_q = \sum_{r \in S} a_{pqr} \sigma_r$$

for any two elements p and q in S . In particular, $\{\sigma_s \mid s \in S\}$ is a basis of the subring

$$A := C[\{\sigma_s \mid s \in S\}]$$

of $\text{End}_C(V)$.⁵

For the remainder of this section, we fix an element in X and call it x .

Let s be an element in S . Then, as X is a basis of V , there exists a uniquely defined element ϵ_s in $\text{End}_C(V)$ such that

$$y\epsilon_s = \begin{cases} y & \text{if } (x, y) \in s, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\{\epsilon_s \mid s \in S\}$ is a linearly independent subset of $\text{End}_C(V)$ and that

$$\epsilon_p \epsilon_q = \delta_{pq} \epsilon_p$$

for any two elements p and q in S . In particular, $\{\epsilon_s \mid s \in S\}$ is a basis of the subring

$$E := C[\{\epsilon_s \mid s \in S\}]$$

of $\text{End}_C(V)$.⁶

We set

$$T := C[\{\sigma_s \mid s \in S\} \cup \{\epsilon_s \mid s \in S\}].$$

The ring T was introduced in [7] and is known as the subconstituent algebra of S with respect to x .

In [7, Lemma 3.4(i)] it was shown that T is semisimple. It is easy to see and well known that each irreducible T -module arises as submodule of V and that T is generated as a ring by the elements $\epsilon_p \sigma_r \epsilon_q$ with $p, q, r \in S$. We shall refer to these facts at various instances without further mention.

In the following, we set

$$\sigma := \sum_{s \in S} \sigma_s$$

and note that $\sigma \in A$.

⁵The algebra A is often called the adjacency algebra of S over C .

⁶The algebra E is usually called the dual adjacency algebra of S with respect to x .

Lemma 6.1 *Let p, q , and r be elements in S . Then the following hold:*

- (i) *We have $r \in pq$ if and only if $\epsilon_{p^*}\sigma_r\epsilon_q \neq 0$.*
- (ii) *We have $pq = \{r\}$ if and only if $\epsilon_{p^*}\sigma_r\epsilon_q = \epsilon_{p^*}\sigma\epsilon_q$.*

Proof (i) This follows immediately from the definition of ϵ_{p^*} , σ_r , and ϵ_q .

(ii) From (i) we obtain

$$\epsilon_{p^*}\sigma\epsilon_q = \sum_{s \in S} \epsilon_{p^*}\sigma_s\epsilon_q = \sum_{r \in pq} \epsilon_{p^*}\sigma_r\epsilon_q.$$

Thus, as $\{\epsilon_{p^*}\sigma_r\epsilon_q \mid r \in pq\}$ is linearly independent, $\epsilon_{p^*}\sigma_r\epsilon_q = \epsilon_{p^*}\sigma\epsilon_q$ if and only if $pq = \{r\}$. $\square$

Recall that condition (a) of Theorem B says that $|pq| = 1$ for any two elements p and q in S with $p \neq q^*$. For the remainder of this section, we shall always mean condition (a) of Theorem B if we speak about condition (a).

Lemma 6.2 *Condition (a) holds if and only if $\epsilon_{p^*}\sigma_r\epsilon_q = \epsilon_{p^*}\sigma\epsilon_q$ for any three elements p, q , and r in S with $p \neq q^*$ and $r \in pq$.*

Proof This follows immediately from Lemma 6.1(ii). $\square$

In the following, we set

$$j := \sum_{y \in X} y$$

and note that $j \in V$.

We also notice that jE is an irreducible T -module with basis $\{j\epsilon_s \mid s \in S\}$. The irreducible T -module jE is called the *primary module* of T .

For each irreducible T -module W , we define

$$\text{Supp}(W) := \{s \in S \mid W\epsilon_s \neq \{0\}\}$$

and note that $\text{Supp}(jE) = S$.

Proposition 6.3 *The following conditions are equivalent:*

- (a) *For any two elements p and q in S with $p \neq q^*$, we have $|pq| = 1$.*
- (b) *For each nonprimary irreducible T -module W , we have $|\text{Supp}(W)| = 1$.*

Proof (a) $\Rightarrow$ (b) Let W be an irreducible T -module different from jE . We may (and shall) assume that W is a submodule of V .

Let u be an element in $\text{Supp}(W)$. Then, by definition, $W\epsilon_u \neq \{0\}$. We first claim that $W\epsilon_u$ is a T -module.

Note that, for each element w in W , $w \in W\epsilon_u$ if and only if $w\epsilon_u = w$.

Let w be an element in $W\epsilon_u$, and let p, q , and r be elements in S . Then, as T is generated as a ring by the elements $\epsilon_{p^*}\sigma_r\epsilon_q$ with $p, q, r \in S$, it suffices to show that $w\epsilon_{p^*}\sigma_r\epsilon_q \in W\epsilon_u$.

If $u \neq p^*$, we have $w\epsilon_{p^*}\sigma_r\epsilon_q = w\epsilon_u\epsilon_{p^*}\sigma_r\epsilon_q = 0$, and we are done. Thus, we may assume that $u = p^*$. By Lemma 6.1(i), we may also assume that $r \in pq$.

Assume that $u \neq q$. Then, as $u = p^*$ and $r \in pq$, $\epsilon_{p^*}\sigma_r\epsilon_q = \epsilon_{p^*}\sigma\epsilon_q$; cf. Lemma 6.2. Now recall that $w(\epsilon_{p^*}\sigma) \in W$ and that $(w\epsilon_{p^*})\sigma \in jE$. Thus, as $W \neq jE$, $w\epsilon_{p^*}\sigma = 0$. It follows that $w\epsilon_{p^*}\sigma_r\epsilon_q = w\epsilon_{p^*}\sigma\epsilon_q = 0$, and we are done.

Assume that $u = q$. Then $w\epsilon_{p^*}\sigma_r\epsilon_q = (w\epsilon_{p^*}\sigma_r\epsilon_q)\epsilon_u$, so that $w\epsilon_{p^*}\sigma_r\epsilon_q \in W\epsilon_u$.

We have seen that $W\epsilon_u$ is a T -module. Thus, as $W\epsilon_u \neq \{0\}$, the irreducibility of W forces $W\epsilon_u = W$. It follows that $W\epsilon_s = \{0\}$ for each element s in $S \setminus \{u\}$, and that means that $\text{Supp}(W) = \{u\}$.

(b) $\Rightarrow$ (a) Let p and q be elements of S with $p \neq q^*$, and let r be an element in pq . We have to show that $pq = \{r\}$.

Let W be an irreducible T -module different from jE . Thus, as $p \neq q^*$, we have $W\epsilon_{p^*} = \{0\}$ or $W\epsilon_q = \{0\}$. It follows that

$$W\epsilon_{p^*}\sigma_t\epsilon_q = \{0\}$$

for each element t in S . As a consequence,

$$W\epsilon_{p^*}\sigma\epsilon_q = \{0\}.$$

Now we determine the action of $\epsilon_{p^*}\sigma_r\epsilon_q$ on the T -submodule jE of V . Recall that $\{j\epsilon_s \mid s \in S\}$ is a basis of jE . For each element s in S , we have

$$(j\epsilon_s)(\epsilon_{p^*}\sigma_r\epsilon_q) = \delta_{sp^*}j\epsilon_{p^*}\sigma_r\epsilon_q = \delta_{sp^*}a_{p^*rq}j\epsilon_q.$$

Thus, as σ is defined to be the sum of the endomorphisms σ_s with $s \in S$, we obtain from [8, Lemma 1.1.3(iii)] that

$$(j\epsilon_s)(\epsilon_{p^*}\sigma\epsilon_q) = \delta_{sp^*}n_{p^*}j\epsilon_q$$

for each element s in S .

Thus,

$$\epsilon_{p^*}\sigma\epsilon_q = \frac{n_{p^*}}{a_{p^*rq}}\epsilon_{p^*}\sigma_r\epsilon_q.$$

On the other hand,

$$\epsilon_{p^*}\sigma\epsilon_q = \epsilon_{p^*} \sum_{s \in S} \sigma_s \epsilon_q = \sum_{s \in S} \epsilon_{p^*}\sigma_s \epsilon_q = \sum_{s \in pq} \epsilon_{p^*}\sigma_s \epsilon_q.$$

Thus, as $\{\epsilon_{p^*}\sigma_s \epsilon_q \mid s \in pq\}$ is a linearly independent subset of $\text{End}(V)$, we must have $pq = \{r\}$. $\square$

We conclude this section by mentioning that Theorem B is a consequence of Propositions 5.2 and 6.3.

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