On changing highest weight theories for finite W -algebras

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Abstract A highest weight theory for a finite W -algebra $U(\mathfrak{g}, e)$ was developed in Brundan et al. (Int. Math. Res. Not. 15:rnn051, 2008). This leads to a strategy for classifying the irreducible finite dimensional $U(\mathfrak{g}, e)$ -modules. The highest weight theory depends on the choice of a parabolic subalgebra of $\mathfrak{g}$ leading to different parameterizations of the finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules. We explain how to construct an isomorphism preserving bijection between the parameterizing sets for different choices of parabolic subalgebra when $\mathfrak{g}$ is of type A , or when $\mathfrak{g}$ is of types C or D and e is an even multiplicity nilpotent element.

Keywords Finite W -algebras · Representation theory

1 Introduction

Let $U(\mathfrak{g}, e)$ be the finite W -algebra associated to the nilpotent element e in a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Finite W -algebras were introduced to the mathematical literature by Premet in [16] and have subsequently attracted a lot of interest, see for example the recent survey [14]. In [4] a highest weight theory for $U(\mathfrak{g}, e)$ is developed. The key theorem required for the highest weight theory, [4, Theorem 4.3], says that there is a subquotient of $U(\mathfrak{g}, e)$ isomorphic to $U(\mathfrak{g}_0, e)$, where $\mathfrak{g}_0$ is a minimal Levi subalgebra of $\mathfrak{g}$ containing e . This allows a definition of Verma modules by inducing finite dimensional irreducible $U(\mathfrak{g}_0, e)$ -modules. These Verma modules have irreducible heads and all finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules can be realized in this manner.

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At present the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules is unknown, except in some special cases. In [5] Brundan and Kleshchev classified these modules in the case that $\mathfrak{g}$ is of type A . In [1] the first author found the classification in the case that $\mathfrak{g}$ is classical and e is a *rectangular* nilpotent. In [2] the authors classified the finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules with integral central character in the case that $\mathfrak{g}$ is classical and e is an *even multiplicity* nilpotent. All of these classifications can be stated nicely in terms of the highest weight theory.

One particular feature of this highest weight theory is that it requires the choice of $\mathfrak{q}$, a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{g}_0$ as a Levi subalgebra. For a finite dimensional irreducible $U(\mathfrak{g}_0, e)$ -module V , we denote the Verma module corresponding to V and $\mathfrak{q}$ by $M(V, \mathfrak{q})$ and write $L(V, \mathfrak{q})$ for its irreducible head. Let $\mathfrak{q}, \mathfrak{q}'$ be two parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{g}_0$ as a Levi subalgebra, and let V, V' be two finite dimensional irreducible $U(\mathfrak{g}_0, e)$ -modules. It is a natural to ask: when is $L(V, \mathfrak{q}) \cong L(V', \mathfrak{q}')$? The main purpose of this note is to answer this question for the cases where the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules is known.

For the cases that we consider e is of standard Levi type, so by a result of Kostant in [10, Sect. 2], we see that $U(\mathfrak{g}_0, e)$ is isomorphic to $S(\mathfrak{t})^{W_0}$, where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}_0$ and W_0 is the Weyl group of $\mathfrak{g}_0$ with respect to $\mathfrak{t}$. This isomorphism leads to a nice description of finite dimensional irreducible modules for $U(\mathfrak{g}_0, e)$ in terms of *tables* associated to e , as explained in Sects. 4 and 5. Our main results are Theorems 4.3 and 5.7, which give a combinatorial explanation of how to pass from a table parameterizing a finite dimensional irreducible $U(\mathfrak{g}, e)$ -module corresponding to a choice of parabolic subalgebra $\mathfrak{q}$ to one corresponding to a different choice of parabolic subalgebra $\mathfrak{q}'$. This combinatorics is given by the row swapping operations on tables defined in [2, Sect. 4].

The proofs of our main results depend crucially on the relationship between finite dimensional $U(\mathfrak{g}, e)$ modules and primitive ideals of $U(\mathfrak{g})$ with associated variety $\overline{G \cdot e}$ proved by Losev in [11, 12]; this is discussed in Sect. 2.5. A connection between modules for $U(\mathfrak{g}, e)$ and certain Whittaker modules for $U(\mathfrak{g})$ predicted in [4, Conjecture 5.3] and verified by [13, Theorem 4.1] and [2, Proposition 3.12] is also of importance; this is explained in Sect. 2.6. We make vital use of a theorem of Joseph, [9, Théorème 1], which, in the case $\mathfrak{g}$ is of type A , tells us when two weights of $\mathfrak{t}$ correspond to the same primitive ideal. Other important ingredients are the notion of “Levi subalgebras” of $U(\mathfrak{g}, e)$ established in [2, Sect. 3], and the description of the component group action for the case of rectangular nilpotent elements from [1, Theorem 1.3].

We now give a brief outline of the structure of this paper. In Sect. 2, we give a recollection of the theory of finite W -algebras that we require later in the paper. We prove two general results about changing highest weight theories in Sect. 3. The main content of the paper is Sects. 4 and 5, in which we prove Theorems 4.3 and 5.7. In both of these sections we explain how *tables* are used to describe the highest weight theory and the combinatorics of tables required for changing between different highest weight theories.

2 Review of finite W -algebras

Throughout this paper we work over the field of complex numbers $\mathbb{C}$; though all of our results remain valid over any algebraically closed field of characteristic 0. As a convention throughout this paper, by a “module” we mean a finitely generated left module.

In this paper, we often consider twisted modules. Let A be an algebra and G a group that acts on A . Given an A -module M and $g \in G$, the twisted module $g \cdot M$ is equal to M as a vector space with action defined by “ $am = (g^{-1} \cdot a)m$ ” for $a \in A$ and $m \in M$.

2.1 Definition of $U(\mathfrak{g}, e)$

Below we recall the definition of $U(\mathfrak{g}, e)$ via non-linear Lie algebras; we refer the reader to [4, Sect. 2.2] for more details.

Let G be a connected reductive algebraic group over $\mathbb{C}$, let $\mathfrak{g}$ be the Lie algebra of G , and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$. Let $(\cdot | \cdot)$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, and define $\chi \in \mathfrak{g}^*$ by $\chi(x) = (e | x)$. Also let $\tilde{G}$ be a possibly disconnected algebraic group with identity component equal to G which preserves the form $(\cdot | \cdot)$. For example in Sect. 5 below we consider the case where $\tilde{G} = \mathrm{O}_{2n}$ and $G = \mathrm{SO}_{2n}$.

Given a subgroup A of G with Lie algebra $\mathfrak{a}$ and $x \in \mathfrak{g}$, we write A^x for the centralizer of x in A and $\mathfrak{a}^x$ for the centralizer of x in $\mathfrak{a}$. For $g \in G$ and $x \in \mathfrak{g}$, we write $g \cdot x$ for the image of x under the adjoint action of g .

Fix an $\mathfrak{sl}_2$ -triple (e, h, f) in $\mathfrak{g}$. We choose a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $h \in \mathfrak{t}$ and $\mathfrak{t}^e$ is a maximal toral subalgebra of $\mathfrak{g}^e$. We write $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{C}$ for the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. Let $\Phi \subseteq \mathfrak{t}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Given $\alpha \in \Phi$, we write $\alpha^\vee \in \mathfrak{t}$ for the corresponding coroot. Let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

Let

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i),$$

be a *good grading* for grading for e compatible with $\mathfrak{t}$, i.e. $e \in \mathfrak{g}(2)$, $\mathfrak{g}^e \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$ and $\mathfrak{t} \subseteq \mathfrak{g}(0)$. Good gradings for e are classified in [6]; see also [3]. The standard example of a good grading is the *Dynkin grading*, which is given by $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. The good grading is given by the $\mathrm{ad} h'$ -eigenspace decomposition for some $h' \in \mathfrak{g}$; this follows from the fact that all derivations of the derived subalgebra of $\mathfrak{g}$ are inner. By [3, Lemma 19], we have $h' - h \in \mathfrak{t}^e$.

We define the following subspaces of $\mathfrak{g}$:

$$\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}(j), \quad \mathfrak{n} = \bigoplus_{j < 0} \mathfrak{g}(j), \quad \mathfrak{h} = \mathfrak{g}(0), \quad \mathfrak{k} = \mathfrak{g}(-1).$$

In particular, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{h}$ and $\mathfrak{n}$ is the nilradical of the opposite parabolic.

We define a symplectic form $\langle \cdot | \cdot \rangle$ on $\mathfrak{k}$ by $\langle x | y \rangle = \chi([y, x])$. Let $\mathfrak{k}^{\text{ne}} = \{x^{\text{ne}} \mid x \in \mathfrak{k}\}$ be a “neutral” copy of $\mathfrak{k}$. We write $x^{\text{ne}} = x(-1)^{\text{ne}}$ for any element $x \in \mathfrak{g}$. Now make $\mathfrak{k}^{\text{ne}}$ into a non-linear Lie algebra with non-linear Lie bracket defined by $[x^{\text{ne}}, y^{\text{ne}}] = \langle x | y \rangle$ for $x, y \in \mathfrak{k}$. Note that $U(\mathfrak{k}^{\text{ne}})$ is isomorphic to the Weyl algebra associated to $\mathfrak{k}$ and the form $\langle \cdot | \cdot \rangle$.

We view $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k}^{\text{ne}}$ as a non-linear Lie algebra with bracket obtained by extending the brackets already defined on $\mathfrak{g}$ and $\mathfrak{k}^{\text{ne}}$ to all of $\tilde{\mathfrak{g}}$, and declaring $[x, y^{\text{ne}}] = 0$ for $x \in \mathfrak{g}, y \in \mathfrak{k}$. Then $U(\tilde{\mathfrak{g}}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{k}^{\text{ne}})$. Also let $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{k}^{\text{ne}}$; this is a subalgebra of $\tilde{\mathfrak{g}}$ whose universal enveloping algebra is identified with $U(\mathfrak{p}) \otimes U(\mathfrak{k}^{\text{ne}})$.

We define $\tilde{\mathfrak{n}}_{\chi} = \{x - x^{\text{ne}} - \chi(x) \mid x \in \mathfrak{n}\}$. By the PBW theorem for $U(\tilde{\mathfrak{g}})$ we have a direct sum decomposition $U(\tilde{\mathfrak{g}}) = U(\tilde{\mathfrak{p}}) \oplus U(\tilde{\mathfrak{g}})\tilde{\mathfrak{n}}_{\chi}$. We write $\text{Pr} : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{p}})$ for the projection along this direct sum decomposition. We define the *finite W-algebra*

$$U(\mathfrak{g}, e) = U(\tilde{\mathfrak{p}})^{\mathfrak{n}} = \{u \in U(\tilde{\mathfrak{p}}) \mid \text{Pr}([x - x^{\text{ne}}, u]) = 0 \text{ for all } x \in \mathfrak{n}\}.$$

It is a subalgebra of $U(\tilde{\mathfrak{p}})$ by [4, Theorem 2.4].

We write $\text{Irr}_0 U(\mathfrak{g}, e)$ for the set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules. For a finite dimensional irreducible $U(\mathfrak{g}, e)$ -module L , we denote its isomorphism class by $[L] \in \text{Irr}_0 U(\mathfrak{g}, e)$.

2.2 Central characters

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$ and $Z(\mathfrak{g}, e)$ denote the center of $U(\mathfrak{g}, e)$. It is easy to see that the restriction of the linear map $\text{Pr} : U(\tilde{\mathfrak{g}}) \rightarrow U(\tilde{\mathfrak{p}})$ defines an injective algebra homomorphism $\text{Pr} : Z(\mathfrak{g}) \hookrightarrow Z(\mathfrak{g}, e)$. As explained in the footnote to [17, Question 5.1], this map is also surjective, so it is an algebra isomorphism

$$\text{Pr} : Z(\mathfrak{g}) \xrightarrow{\sim} Z(\mathfrak{g}, e).$$

We view $Z(\mathfrak{g})$ as a subalgebra of $U(\mathfrak{g}, e)$ -module via Pr . Given a $U(\mathfrak{g}, e)$ -module V , we say V is of *central character* $\psi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ if $zv = \psi(z)v$ for all $z \in Z(\mathfrak{g})$ and $v \in V$.

2.3 The component group action

We write $H = G(0)$ for the Levi subgroup of G with Lie algebra $\mathfrak{h} = \mathfrak{g}(0)$, so $H = G^{h'}$ (recall that the good grading of $\mathfrak{g}$ is the $\text{ad } h'$ -eigenspace decomposition). The argument in the proof of [8, Proposition 5.9] shows that the component group of the centralizer of e in G , denoted by $C(e) = G^e / (G^e)^{\circ}$, is naturally isomorphic to $H^e / (H^e)^{\circ}$. From now on we identify $C(e) = H^e / (H^e)^{\circ}$.

One can check that the adjoint action of H^e on $\mathfrak{g}$ gives rise to a well-defined action of H^e on $U(\mathfrak{g}, e)$. It was proved by Premet in [17, Sect. 2.5] that there is an embedding

$$\theta : \mathfrak{h}^e \hookrightarrow U(\mathfrak{g}, e); \quad (2.1)$$

see also [4, Theorem 3.3]. Moreover, the adjoint action of $\mathfrak{h}^e$ on $U(\mathfrak{g}, e)$ through this embedding coincides with differential of the action of H^e on $U(\mathfrak{g}, e)$.

We write $\text{Prim}_0 U(\mathfrak{g}, e)$ for the set of primitive ideals of $U(\mathfrak{g}, e)$ of finite codimension. The set $\text{Prim}_0 U(\mathfrak{g}, e)$ identifies naturally with $\text{Irr}_0 U(\mathfrak{g}, e)$. The action of H^e on $U(\mathfrak{g}, e)$ induces an action on $\text{Prim}_0 U(\mathfrak{g}, e)$. Since the action of $\mathfrak{h}^e$ of $U(\mathfrak{g}, e)$ coincides with the differential of the action of H^e , we see that the action of H^e on $\text{Prim}_0 U(\mathfrak{g}, e)$ factors through $C(e)$. So we obtain an action of $C(e)$ on $\text{Prim}_0 U(\mathfrak{g}, e)$, and thus on $\text{Irr}_0 U(\mathfrak{g}, e)$.

Next we note that this action can also be described in terms of twisting the action of $U(\mathfrak{g}, e)$ on its finite dimensional irreducible modules by elements of $C(e)$. Let $c \in C(e)$ and $\dot{c} \in H^e$ be a lift of c , and let L be finite dimensional irreducible $U(\mathfrak{g}, e)$ -module. Up to isomorphism $\dot{c} \cdot L$ only depends on c , and we define

$$c \cdot [L] = [\dot{c} \cdot L]. \quad (2.2)$$

It is straightforward to see that the actions of $C(e)$ on isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules via twisting and via the action of $C(e)$ on primitive ideals are the same.

Now let $\tilde{H} = \tilde{G}^{h'}$ be the centralizer of h' in $\tilde{G}$, and let $\tilde{H}^e$ be the centralizer of e in $\tilde{H}$. Then since $\tilde{H}^e$ fixes e and h' and preserves the form $(\cdot | \cdot)$, $\tilde{H}^e$ also acts on $U(\mathfrak{g}, e)$. The content of the previous two paragraphs remains valid if we replace $C(e)$ with $\tilde{C}(e) = \tilde{H}^e / (\tilde{H}^e)^\circ$.

2.4 Skryabin's equivalence

Skryabin's equivalence relates the category $U(\mathfrak{g}, e)\text{-mod}$ of finitely generated $U(\mathfrak{g}, e)$ -modules to a certain category of generalized Whittaker modules for $U(\mathfrak{g})$. To state this equivalence, we require the Whittaker model definition of $U(\mathfrak{g}, e)$, which is outlined below.

Let $\mathfrak{l}$ be a Lagrangian subspace of $\mathfrak{k}$ with respect to the symplectic form $\langle \cdot | \cdot \rangle$. Then define $\mathfrak{m} = \bigoplus_{j \leq -2} \mathfrak{g}(j) \oplus \mathfrak{l}$ and $\mathfrak{m}_\chi = \{x - \chi(x) \mid x \in \mathfrak{m}\} \subseteq U(\mathfrak{g})$. Then $\mathcal{Q}_\chi \cong U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$ is a left $U(\mathfrak{g})$ -module. There is a natural isomorphism $\text{End}_{U(\mathfrak{g})}(\mathcal{Q}_\chi)^{\text{op}} \xrightarrow{\sim} U(\mathfrak{g}, e)$, by [4, Theorem 2.4]. The algebra $\text{End}_{U(\mathfrak{g})}(\mathcal{Q}_\chi)^{\text{op}}$ is the Whittaker model definition of the finite W -algebra associated to $\mathfrak{g}$ and e .

Now $\mathcal{Q}_\chi$ is naturally a right $\text{End}_{U(\mathfrak{g})}(\mathcal{Q}_\chi)^{\text{op}}$ -module and thus can be viewed as a right $U(\mathfrak{g}, e)$ -module. Therefore, we can define the $U(\mathfrak{g})$ -module $\mathcal{S}(M) = \mathcal{Q}_\chi \otimes_{U(\mathfrak{g}, e)} M$ for $M \in U(\mathfrak{g}, e)\text{-mod}$. Let $\text{Wh}(\mathfrak{g}, \mathfrak{m}_\chi)$ be the category of finitely generated $U(\mathfrak{g})$ -modules on which $\mathfrak{m}_\chi$ acts locally nilpotently. For $M \in U(\mathfrak{g}, e)\text{-mod}$ it is easy to check that $\mathcal{S}(M) \in \text{Wh}(\mathfrak{g}, \mathfrak{m}_\chi)$. Skryabin's equivalence from [18] says that the functor

$$\mathcal{S}: U(\mathfrak{g}, e)\text{-mod} \rightarrow \text{Wh}(\mathfrak{g}, \mathfrak{m}_\chi)$$

is an equivalence of categories. A quasi-inverse is given by the functor

$$N \mapsto N^{\mathfrak{m}_\chi} = \{n \in N \mid xn = \chi(x)n \text{ for all } x \in \mathfrak{m}\}$$

for $N \in \text{Wh}(\mathfrak{g}, \mathfrak{m}_\chi)$.

2.5 Losev's map between ideals

In [11] Losev constructs a map $\cdot^\dagger$ from the set of ideals of $U(\mathfrak{g}, e)$ to the set of ideals of $U(\mathfrak{g})$. By [11, Theorem 1.2.2], this map restricts to a surjection

$$I \mapsto I^\dagger: \quad \text{Prim}_0 U(\mathfrak{g}, e) \twoheadrightarrow \text{Prim}_e U(\mathfrak{g}), \quad (2.3)$$

where $\text{Prim}_e U(\mathfrak{g})$ denotes the primitive ideals of $U(\mathfrak{g})$ with associated variety equal to $\overline{G \cdot e}$. For a definition of associated varieties, see for example [8, Sect. 9].

Furthermore, in [12, Theorem 1.2.2] Losev proves that the fibres of the map in (2.3) are precisely the $C(e)$ -orbits in $\text{Prim}_0 U(\mathfrak{g}, e)$, for the action of $C(e)$ explained in Sect. 2.3.

By [11, Theorem 1.2.2], the map $\cdot^\dagger$ restricted to $\text{Prim}_0 U(\mathfrak{g}, e)$ can be described as follows. Let $I \in \text{Prim}_0 U(\mathfrak{g}, e)$ and let L be a finite dimensional irreducible $U(\mathfrak{g}, e)$ -module with $\text{Ann}_{U(\mathfrak{g}, e)}(L) = I$. Then

$$I^\dagger = \text{Ann}_{U(\mathfrak{g})}(\mathcal{S}(L)).$$

In [11, Theorem 1.2.2], it is proved that if L is an irreducible $U(\mathfrak{g}, e)$ -module with central character $\psi: Z(\mathfrak{g}) \rightarrow \mathbb{C}$, then $\text{Ann}_{U(\mathfrak{g}, e)}(L)^\dagger \cap Z(\mathfrak{g}) = \ker \psi$, where $Z(\mathfrak{g})$ is viewed as a subalgebra of $U(\mathfrak{g}, e)$ as in Sect. 2.2. Thus $\cdot^\dagger$ preserves central characters.

2.6 Review of highest weight theory

Highest weight theory for finite W -algebras was introduced in [4, Sect. 4]. In this paper we restrict to the case where e is of standard Levi type, as defined below.

We let $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [t, x] = 0 \text{ for all } t \in \mathfrak{t}^e\}$ be the centralizer of $\mathfrak{t}^e$ in $\mathfrak{g}$. Then $\mathfrak{g}_0$ is a Levi subalgebra of $\mathfrak{g}$ and e is a distinguished nilpotent element of $\mathfrak{g}_0$. We restrict to the case that e is of *standard Levi type*, which means that e is regular nilpotent in $\mathfrak{g}_0$. We write Φ_0 for the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{t}$.

We can form the $\mathfrak{t}^e$ -weight space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$, where $\Phi^e \subseteq (\mathfrak{t}^e)^*$ and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{t}^e\}$. Then Φ^e is a restricted root system; see [3, Sects. 2 and 3] for information on restricted root systems.

We choose a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. The parabolic subalgebra $\mathfrak{q}$ gives a system Φ_+^e of positive roots in Φ^e , namely, $\Phi_+^e = \{\alpha \in \Phi^e \mid \mathfrak{g}_\alpha \subseteq \mathfrak{q}\}$. The highest weight theory explained below depends on this choice of $\mathfrak{q}$, and this dependency is the main topic of study in this article.

Note that $\mathfrak{t}^e \subseteq \mathfrak{h}^e$ embeds in $U(\mathfrak{g}, e)$ via the map θ from (2.1). Therefore, we have a $\mathfrak{t}^e$ -weight space decomposition

$$U(\mathfrak{g}, e) = U(\mathfrak{g}, e)_0 \oplus \bigoplus_{\alpha \in \mathbb{Z}\Phi^e \setminus \{0\}} U(\mathfrak{g}, e)_\alpha.$$

The zero weight space $U(\mathfrak{g}, e)_0$ is a subalgebra of $U(\mathfrak{g}, e)$ and we define $U(\mathfrak{g}, e)_{\#}$ to be the left ideal of $U(\mathfrak{g}, e)$ generated by $U(\mathfrak{g}, e)_{\alpha}$ for $\alpha \in \Phi_+^e$. Then $U(\mathfrak{g}, e)_{0,\#} = U(\mathfrak{g}, e)_0 \cap U(\mathfrak{g}, e)_{\#}$ is a two sided ideal of $U(\mathfrak{g}, e)_0$ so we can form the quotient $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\#}$.

By [4, Theorem 4.3], there is an isomorphism

$$U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\#} \cong U(\mathfrak{g}_0, e). \quad (2.4)$$

This isomorphism is central to the development of the highest weight theory since it is used to define Verma modules, as we explain below.

Since e is regular in $\mathfrak{g}_0$, we find that $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ is a Borel subalgebra of $\mathfrak{g}_0$; we write $\mathfrak{b}_0 = \mathfrak{p}_0$ and $\Phi_0^+ \subseteq \Phi_0$ for the system of positive roots corresponding to $\mathfrak{b}_0$. Then we set $\mathfrak{b}_q = \mathfrak{b}_0 \oplus \mathfrak{q}_u$, where $\mathfrak{q}_u$ denotes the nilradical of our parabolic $\mathfrak{q}$, so that $\mathfrak{b}_q$ is a Borel subalgebra of $\mathfrak{g}$. We also need another Borel subalgebra, $\tilde{\mathfrak{b}}_q = \tilde{\mathfrak{b}}_0 \oplus \mathfrak{q}_u$ where $\tilde{\mathfrak{b}}_0$ is the opposite Borel to $\mathfrak{b}_0$ in $\mathfrak{g}_0$. We let ρ_q and $\tilde{\rho}_q$ denote the half sum of the positive roots corresponding to $\mathfrak{b}_q$ and $\tilde{\mathfrak{b}}_q$, respectively.

Since e is regular in $\mathfrak{g}_0$, a result of Kostant in [10, Sect. 2] gives that $U(\mathfrak{g}_0, e) \cong S(\mathfrak{t})^{W_0}$, where W_0 denotes the Weyl group of $\mathfrak{g}_0$ with respect to $\mathfrak{t}$. An explicit isomorphism

$$\xi_{-\tilde{\rho}_q}: U(\mathfrak{g}_0, e) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}$$

is given in [4, Lemma 5.1], where $\xi_{-\tilde{\rho}_q}$ is the composition of the natural projection $U(\mathfrak{b}_0) \rightarrow S(\mathfrak{t})$ with the shift $S_{-\tilde{\rho}_q}: S(\mathfrak{t}) \rightarrow S(\mathfrak{t})$, where $S_{-\tilde{\rho}_q}(t) = t - \tilde{\rho}_q(t)$ for $t \in \mathfrak{t}$.

The finite dimensional irreducible modules for $S(\mathfrak{t})^{W_0}$ are all 1-dimensional and are indexed by the set $\mathcal{L} = \mathfrak{t}^*/W_0$ of W_0 -orbits in $\mathfrak{t}^*$. Given $\Lambda \in \mathcal{L}$ we let V_{Λ} be the $U(\mathfrak{g}_0, e)$ -module corresponding to Λ through $\xi_{-\tilde{\rho}_q}$. We define the Verma module

$$M(\Lambda, q) = (U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_{\#}) \otimes_{U(\mathfrak{g}_0, e)} V_{\Lambda}, \quad (2.5)$$

where $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_{\#}$ is viewed as a right $U(\mathfrak{g}_0, e)$ -module via the isomorphism from (2.4).

By [4, Theorem 4.5], $M(\Lambda, q)$ has a unique maximal submodule and we write $L(\Lambda, q)$ for the irreducible quotient. Moreover, any finite dimensional irreducible $U(\mathfrak{g}, e)$ -module is isomorphic to $L(\Lambda, q)$ for some $\Lambda \in \mathcal{L}$, and $L(\Lambda, q) \cong L(\Lambda', q)$ if and only if $\Lambda = \Lambda'$.

Let $\Psi: Z(\mathfrak{g}) \rightarrow S(\mathfrak{t})^W$ be the Harish–Chandra isomorphism defined by

$$z \equiv S_{\rho_q}(\Psi(z)) \pmod{U(\mathfrak{g})\mathfrak{b}_{q,u}},$$

where $\mathfrak{b}_{q,u}$ denotes the nilradical of $\mathfrak{b}_q$. Under this isomorphism, the central character of $L(\Lambda, q)$ corresponds to the W -orbit in $\mathfrak{t}^*$ containing Λ by [4, Corollary 4.8].

We let $\mathcal{L}_q^+ = \{\Lambda \in \mathcal{L} \mid L(\Lambda, q) \text{ is finite dimensional}\}$. So this set parameterizes the isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules. For a different choice of parabolic subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$, we obtain another subset $\mathcal{L}_{q'}^+$ of $\mathcal{L}$ that parameterizes the isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules. Thus there is a bijection $f: \mathcal{L}_q^+ \xrightarrow{\sim} \mathcal{L}_{q'}^+$ such that

$L(\Lambda, \mathfrak{q}) \cong L(f(\Lambda), \mathfrak{q}')$. The main theorems of this paper are Theorems 4.3 and 5.7, which give a combinatorial description of this bijection in certain cases.

Given a $U(\mathfrak{g}, e)$ -module V we say that $v \in V$ is a highest weight vector for (the parabolic subalgebra) $\mathfrak{q}$ if $uv = 0$ for all $u \in U(\mathfrak{g}, e)_{\neq}$, and v is an eigenvector for every element of $U(\mathfrak{g}, e)_0$. In this case $\langle v \rangle$ has the structure of a $U(\mathfrak{g}_0, e)$ -module, which is isomorphic to V_Λ for some $\Lambda \in \mathcal{L}$, and we say that v is of highest weight Λ . Since $L(\Lambda, \mathfrak{q})$ is irreducible, it has a unique, up to scalar multiplication, highest weight vector for $\mathfrak{q}$ (of highest weight Λ). Given another parabolic subalgebra $\mathfrak{q}'$ with Levi factor $\mathfrak{g}_0$, we can define highest weight vectors for $\mathfrak{q}'$ analogously.

As explained in Sect. 2.3, there is an action of $\tilde{C}(e)$ on the set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules given by (2.2). This gives an action of $\tilde{C}(e)$ on $\mathcal{L}_q^+$ defined by

$$c \cdot [L(\Lambda, \mathfrak{q})] = [L(c \cdot \Lambda, \mathfrak{q})],$$

for $c \in \tilde{C}(e)$ and $\Lambda \in \mathcal{L}_q^+$. To be clear, here we are defining an action of $\tilde{C}(e)$ on a subset of $\mathfrak{t}^*/W_0$. In some cases it is possible to define a more natural action of $\tilde{C}(e)$ on $\mathfrak{t}^*/W_0$, however in general these actions are not compatible.

Next in (2.6) we state a relationship between the highest weight theory and the map $\cdot^\dagger$ from Sect. 2.5. This is due to an equivalence of categories between an analogue of the BGG category $\mathcal{O}$ for $U(\mathfrak{g}, e)$ and a certain category of generalized Whittaker modules for $U(\mathfrak{g})$, which was predicted in [4, Conjecture 5.3]. This conjecture was verified by [13, Theorem 4.1], but in the setting of highest weight theory defined in a different way. In [2, Proposition 3.12] it is shown that the Verma modules defined in the different highest weight theories coincide, thus completing the verification of [4, Conjecture 5.3].

Let $\Lambda \in \mathcal{L}^+$ and take $\lambda \in \Lambda$ such that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_0^+$. Let $L(\lambda, \mathfrak{b}_q)$ be the irreducible highest weight $U(\mathfrak{g})$ -module with highest weight $\lambda - \rho_q$ with respect to the Borel subalgebra $\mathfrak{b}_q$. Then using [4, Conjecture 5.3], [15, Theorem 5.1] and [11, Theorem 1.2.2], we obtain

$$\text{Ann}_{U(\mathfrak{g}, e)}(L(\Lambda, \mathfrak{q}))^\dagger = \text{Ann}_{U(\mathfrak{g})}(L(\lambda, \mathfrak{b}_q)). \quad (2.6)$$

2.7 Parabolic highest weight theories

We end this section by briefly discussing a “parabolic generalization” of the highest weight theory from [2, Sect. 3]. To do this we first recall that a subalgebra $\mathfrak{s}$ of $\mathfrak{t}^e$ is called a *full subalgebra* if $\mathfrak{s}$ is equal to the centre of the Levi subalgebra $\mathfrak{g}^\mathfrak{s} = \{x \in \mathfrak{g} \mid [t, x] = 0 \text{ for all } t \in \mathfrak{s}\}$ of $\mathfrak{g}$.

By [2, Theorem 3.2], there is an isomorphism generalizing that of (2.4) between a subquotient of $U(\mathfrak{g}, e)$ and $U(\mathfrak{g}^\mathfrak{s}, e)$. This is obtained by taking $\mathfrak{s}$ -weight spaces in $U(\mathfrak{g}, e)$ rather than $\mathfrak{t}^e$ weight spaces. To define parabolic Verma modules, we need to use the parabolic subalgebra $\mathfrak{q}_\mathfrak{s}$ which has $\mathfrak{g}^\mathfrak{s}$ as its Levi factor and contains $\mathfrak{q}$. Then given an irreducible finite dimensional module V for $U(\mathfrak{g}^\mathfrak{s}, e)$ we can define a parabolic Verma module $M_\mathfrak{s}(V, \mathfrak{q}_\mathfrak{s})$ for $U(\mathfrak{g}, e)$, which has an irreducible head $L_\mathfrak{s}(V, \mathfrak{q}_\mathfrak{s})$ as in [2, Sect. 3.3]. We denote these modules by $M_\mathfrak{s}(V, \mathfrak{q})$ and $L_\mathfrak{s}(V, \mathfrak{q})$.

The version of (2.4) in the case “ $\mathfrak{g} = \mathfrak{g}^s$ ” allows us to define Verma modules for $U(\mathfrak{g}^s, e)$. For $\Lambda \in \mathcal{L} = \mathfrak{t}^*/W_0$ we can define the Verma module $M^s(\Lambda, \mathfrak{q})$ for $U(\mathfrak{g}^s, e)$ in analogy to (2.5), see [2, Sect. 3.3] for details. We write $L^s(\Lambda, \mathfrak{q})$ for the irreducible head of $M^s(\Lambda, \mathfrak{q})$.

The important point for us is the transitivity result [2, Proposition 3.6]. This says that if $\Lambda \in \mathcal{L}_q^+$ (so that $L(\Lambda, \mathfrak{q})$ is finite dimensional), then $L^s(\Lambda, \mathfrak{q})$ is finite dimensional and

$$L_s(L^s(\Lambda, \mathfrak{q}), \mathfrak{q}) \cong L(\Lambda, \mathfrak{q}). \quad (2.7)$$

3 Generalities about changing height weight theories

In this section we prove two general results about changing highest weight theories. In Sect. 3.1 we prove Theorem 3.1, which says how to pass between highest weight theories up to the action of $C(e)$. Then in Sect. 3.2 we prove Proposition 3.3, which deals with the case where the parabolic subalgebras are conjugate under the action of the restricted Weyl group W^e .

3.1 Changing the highest weight theory up to the action of $C(e)$

Let $\mathfrak{q}$ and $\mathfrak{q}'$ be parabolic subalgebras of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. Let $\mathfrak{b}_{\mathfrak{q}'}$ be the Borel subalgebra of $\mathfrak{g}$ given by $\mathfrak{b}_{\mathfrak{q}'} = \mathfrak{b}_0 \oplus \mathfrak{q}'_u$, where $\mathfrak{q}'_u$ is the nilradical of $\mathfrak{q}'$. Define $\rho_{\mathfrak{q}'}$ to be the half sum of the positive roots determined by $\mathfrak{b}_{\mathfrak{q}'}$. Let $\Lambda \in \mathcal{L}_q^+$ and let $\Lambda' \in \mathcal{L}_{q'}^+$. Take $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$ with $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ and $\langle \lambda', \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_0^+$. We denote the highest weight $U(\mathfrak{g})$ -module with highest weight $\lambda' - \rho_{\mathfrak{q}'}$ with respect to $\mathfrak{b}_{\mathfrak{q}'}$ by $L(\lambda', \mathfrak{b}_{\mathfrak{q}'})$. Finally let $w \in W$ be such that $w \cdot \mathfrak{b}_{\mathfrak{q}'} = \mathfrak{b}_{\mathfrak{q}}$.

Theorem 3.1 *In the notation given above we have $[L(\Lambda, \mathfrak{q})]$ and $[L(\Lambda', \mathfrak{q}')] lying in the same $C(e)$ -orbit if and only if $\text{Ann}_{U(\mathfrak{g})} L(\lambda, \mathfrak{b}_{\mathfrak{q}}) = \text{Ann}_{U(\mathfrak{g})} L(w\lambda', \mathfrak{b}_{\mathfrak{q}})$.$*

Proof From (2.6) we have

$$\text{Ann}_{U(\mathfrak{g}, e)} L(\Lambda, \mathfrak{q})^\dagger = \text{Ann}_{U(\mathfrak{g})} L(\lambda, \mathfrak{b}_{\mathfrak{q}}).$$

Similarly, we have

$$\text{Ann}_{U(\mathfrak{g}, e)} L(\Lambda', \mathfrak{q}')^\dagger = \text{Ann}_{U(\mathfrak{g})} L(\lambda', \mathfrak{b}_{\mathfrak{q}'}).$$

Also, if $\dot{w} \in N_G(\mathfrak{t})$ is a lift of w , then $\dot{w} \cdot L(\lambda', \mathfrak{b}_{\mathfrak{q}'}) \cong L(w\lambda', \mathfrak{b}_{\mathfrak{q}})$, because the highest weight vector with respect to $\mathfrak{b}_{\mathfrak{q}'}$ in $L(\lambda', \mathfrak{b}_{\mathfrak{q}'})$ is a highest weight vector of highest weight $w\lambda'$ with respect to $\mathfrak{b}_{\mathfrak{q}}$ in $\dot{w} \cdot L(\lambda', \mathfrak{b}_{\mathfrak{q}'})$. Thus

$$\text{Ann}_{U(\mathfrak{g})} L(\lambda', \mathfrak{b}_{\mathfrak{q}'}) = \dot{w}^{-1} \cdot \text{Ann}_{U(\mathfrak{g})} L(w\lambda', \mathfrak{b}_{\mathfrak{q}}) = \text{Ann}_{U(\mathfrak{g})} L(w\lambda', \mathfrak{b}_{\mathfrak{q}}),$$

where the last inequality holds since twisting an annihilator with an inner automorphism preserves the annihilator. So, recalling the discussion from Sect. 2.5, we see that the theorem follows from [12, Theorem 1.2.2]. $\square$

3.2 Changing highest weight theory with the restricted Weyl group

The restricted Weyl group $W^e = N_{G^e}(\mathfrak{t}^e)/Z_{G^e}(\mathfrak{t}^e)$ is defined in [3, Sect. 3], where $N_{G^e}(\mathfrak{t}^e)$ is the normalizer of $\mathfrak{t}^e$ in G^e and $Z_{G^e}(\mathfrak{t}^e)$ is the centralizer of $\mathfrak{t}^e$ in G^e .

As in Sect. 2.3 we let $H = G(0)$ be the Levi subgroup of G with Lie algebra $\mathfrak{h} = \mathfrak{g}(0)$. Also we let R be the unipotent subgroup of G with Lie algebra $\bigoplus_{j>1} \mathfrak{g}(j)$. Then we have a Levi decomposition $G^e = H^e \ltimes R^e$; this can be proved using the argument in [8, Proposition 5.9]. We see that $N_{G^e}(\mathfrak{t}^e) = N_{H^e}(\mathfrak{t}^e) \ltimes Z_{R^e}(\mathfrak{t}^e)$, because $\mathfrak{t}^e \subseteq \mathfrak{h}^e$. This leads to an isomorphism $W^e \cong N_{H^e}(\mathfrak{t}^e)/Z_{H^e}(\mathfrak{t}^e)$.

We can view W^e naturally as a subgroup of $\mathrm{GL}(\mathfrak{t}^e)$. Viewing W as a subgroup of $\mathrm{GL}(\mathfrak{t})$, we note that W_0 centralizes $\mathfrak{t}^e$ and $N_W(W_0)$ normalizes $\mathfrak{t}^e$. Thus $N_W(W_0)$ can be viewed as a subgroup of $\mathrm{GL}(\mathfrak{t}^e)$. Thanks to [3, Lemma 14], we have $W^e = N_W(W_0)/W_0$ as subgroups of $\mathrm{GL}(\mathfrak{t}^e)$.

An element of $Z_{G^e}(\mathfrak{t}^e)$ normalizes any parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$, and any element of $N_{G^e}(\mathfrak{t}^e)$ normalizes $\mathfrak{g}_0$. Therefore, W^e acts on the set of parabolic subalgebras of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. Below we explain how to pass between different highest weight theories corresponding to parabolic subalgebras that are conjugate by W^e .

The adjoint action of H^e on $U(\mathfrak{g}, e)$, explained in Sect. 2.3 restricts to an action of $N_{H^e}(\mathfrak{t}^e)$ on $U(\mathfrak{g}, e)$. Thus we can twist $U(\mathfrak{g}, e)$ -modules by elements of $N_{H^e}(\mathfrak{t}^e)$. The adjoint action of $N_{H^e}(\mathfrak{t}^e)$ on $\mathfrak{g}$ also gives rise to an action of $N_{H^e}(\mathfrak{t}^e)$ on $U(\mathfrak{g}_0, e)$. Thus we can twist $U(\mathfrak{g}_0, e)$ -modules by elements of $N_{H^e}(\mathfrak{t}^e)$. Let G_0 be the centralizer of $\mathfrak{t}^e$ in G so the Lie algebra of G_0 is $\mathfrak{g}_0$; we note that $Z_{G^e}(\mathfrak{t}^e)$ is the centralizer of e in G_0 . Now e is regular in $\mathfrak{g}_0$, and $Z_{H^e}(\mathfrak{t}^e)$ is a Levi factor of $Z_{G^e}(\mathfrak{t}^e)$, thus $Z_{H^e}(\mathfrak{t}^e)$ is equal to the centre of G_0 ; this follows from standard results about the centralizers of regular nilpotent elements. Therefore, we see that the action of $Z_{H^e}(\mathfrak{t}^e)$ on $U(\mathfrak{g}_0, e)$ is trivial and thus we can twist $U(\mathfrak{g}_0, e)$ -modules by elements of W^e . Hence, we obtain an action of W^e on $\mathcal{L} = \mathfrak{t}^*/W_0$. From the proof of [3, Lemma 14], we see that through the isomorphism $W^e \cong N_W(W_0)/W_0$ this action coincides with the natural action of $N_W(W_0)/W_0$ on $\mathfrak{t}^*/W_0$.

For the remainder of this subsection we fix $\mathfrak{q}$ a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. Let $\Lambda \in \mathcal{L}_{\mathfrak{q}}^+$, let v_+ be the highest weight vector in $L(\Lambda, \mathfrak{q})$ for $\mathfrak{q}$, and let $h \in N_{H^e}(\mathfrak{t}^e)$. In $h \cdot L(\Lambda, \mathfrak{q})$, we see that v_+ is a highest weight vector for $\mathfrak{q}' = h \cdot \mathfrak{q}$. Therefore, if $h \in Z_{H^e}(\mathfrak{t}^e)$, then v_+ is a highest weight vector for $\mathfrak{q}$. Since the action of $Z_{H^e}(\mathfrak{t}^e)$ on $U(\mathfrak{g}_0, e)$ is trivial we thus see that $h \cdot L(\Lambda, \mathfrak{q}) \cong L(\Lambda, \mathfrak{q})$. Hence, we obtain an action of W^e on $\mathrm{Irr}_0 U(\mathfrak{g}, e)$.

The following lemma is immediate from the discussion above.

Lemma 3.2 *There are actions of W^e on $\mathcal{L}$ and $\mathrm{Irr}_0 U(\mathfrak{g}, e)$. For $w \in W^e$ and $\Lambda \in \mathcal{L}_{\mathfrak{q}}^+$, we have*

$$w \cdot [L(\Lambda, \mathfrak{q})] = [L(w \cdot \Lambda, w \cdot \mathfrak{q})].$$

Let $\mathfrak{k} = \mathfrak{g}^h$. We note that $\mathfrak{k}^e$ is reductive, see [8, Proposition 5.9], and that $\mathfrak{k}^e \subseteq \mathfrak{h}^e$, because $h' - h \in \mathfrak{t}^e$. Also $\mathfrak{t}^e$ is a maximal toral subalgebra of $\mathfrak{k}^e$. We decompose $\mathfrak{k}^e$ in

to $\mathfrak{t}^e$ -weight spaces

$$\mathfrak{k}^e = \mathfrak{k}_0^e \oplus \bigoplus_{\alpha \in (\Phi^e)^\circ} \mathfrak{k}_\alpha^e,$$

where $(\Phi^e)^\circ \subseteq \Phi^e$ is the root system of $\mathfrak{k}^e$ with respect to $\mathfrak{t}^e$. Then $(\Phi^e)_+^\circ = (\Phi^e)^\circ \cap \Phi_+^e$ is a system of positive roots of positive roots in $(\Phi^e)^\circ$. As in [3, Sect. 3] we define Z^e to be the stabilizer in W^e of the dominant chamber in $\mathbb{R}\Phi_e$ determined by $(\Phi^e)_+^\circ$.

Let $(W^e)^\circ = N_{(H^e)^\circ}(\mathfrak{t}^e)/Z_{(H^e)^\circ}(\mathfrak{t}^e)$. By [3, Lemma 15], we find that $W^e \cong Z^e \ltimes (W^e)^\circ$. Moreover, the inclusion $N_{H^e}(\mathfrak{t}^e) \hookrightarrow H^e$ induces an isomorphism

$$\iota: Z^e \xrightarrow{\sim} H^e/(H^e)^\circ Z_{H^e}(\mathfrak{t}^e).$$

Also $H^e/(H^e)^\circ Z_{H^e}(\mathfrak{t}^e)$ is a quotient of the component group $C(e)$ via the natural map

$$\kappa: C(e) \twoheadrightarrow H^e/(H^e)^\circ Z_{H^e}(\mathfrak{t}^e).$$

Let $z \in Z^e$ and let $c \in C(e)$ such that $\iota(z) = \kappa(c)$. Then by the definitions of the actions, we have $[L(c \cdot \Lambda, q)] = c \cdot [L(\Lambda, q)] = z \cdot [L(\Lambda, q)]$, for $\Lambda \in \mathcal{L}_q^+$. Since $N_{(H^e)^\circ}(\mathfrak{t}^e) \subseteq (H^e)^\circ$, we have $[w \cdot L(\Lambda, q)] = [L(\Lambda, q)]$ for any $w \in (W^e)^\circ$ and $\Lambda \in \mathcal{L}_q^+$.

Putting together the discussion above we arrive at the following proposition.

Proposition 3.3 *Let $\mathfrak{q}, \mathfrak{q}'$ be parabolic subalgebras of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$ and with $w \cdot \mathfrak{q}' = \mathfrak{q}$ for some $w \in W^e$, and let $\Lambda \in \mathcal{L}_q^+$. Write $w = zv \in W^e$, where $z \in Z^e$ and $v \in (W^e)^\circ$, and let $c \in C(e)$ such that $\iota(z) = \kappa(c)$. Then*

$$[L(\Lambda, q)] = [L(c \cdot \Lambda, \mathfrak{q}')].$$

In Sect. 5.8 we require the restricted Weyl group $\tilde{W}^e = N_{\tilde{G}^e}(\mathfrak{t}^e)/Z_{\tilde{G}^e}(\mathfrak{t}^e)$ for $\tilde{G}$. It is easy to check that everything above holds with $\tilde{W}^e$ in place of W^e .

4 Changing highest weight theories in type A

The goal of this section is to prove Theorem 4.3, which explains how to construct the bijection between parameterizing sets of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules for different highest weight theories when $\mathfrak{g}$ is of type A. First we recall the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules in Sect. 4.1. Next, in Sect. 4.2, we recall some definitions from [2, Sect. 4] regarding frames and tables, which give the combinatorics for the description of the highest weight theories. Finally, in Sect. 4.4, we state and prove Theorem 4.3.

4.1 The classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules

We let $\mathfrak{g} = \mathfrak{gl}_n$ and let $\{e_{i,j} \mid 1 \leq i, j \leq n\}$ be the standard basis of $\mathfrak{g}$. Write $(\cdot | \cdot)$ for the trace form on $\mathfrak{g}$. Let $\mathfrak{t}$ be the maximal toral subalgebra of diagonal matrices. Define

$\epsilon_i \in \mathfrak{t}^*$ to be dual to $e_{i,i}$. The Weyl group W of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is the symmetric group S_n .

We recall that nilpotent G -orbits are parameterized by partitions of n . Also we recall that the centralizer in G of any nilpotent element $e \in \mathfrak{g}$ is connected, so that $C(e)$ is trivial.

To define $U(\mathfrak{g}, e)$ we require a good grading for e . Good gradings for e were classified in [6] using *pyramids*. A pyramid is a finite collection of boxes in the plane such that:

- the boxes are arranged in connected rows;
- each box is 2 units by 2 units;
- each box is centred at a point in $\mathbb{Z}^2$;
- if a box centred at (i, j) is not in the bottom row then there is a box in the pyramid centred at $(i, j-2)$ or there are two boxes in the pyramid centred at $(i-1, j-2)$ and $(i+1, j-2)$.

For example

(4.1)

is a pyramid.

Let $\mathbf{p} = (p_1 \geq p_2 \geq \dots \geq p_m)$ be a partition of n and let P be a pyramid with row lengths given by $\mathbf{p}$.

The *coordinate table* of P is obtained by filling the boxes in P with entries $1, \dots, n$ filled in from top to bottom and from left to right and is denoted K . For example if P is the pyramid in (4.1) then the coordinate table of P is

(4.2)

Define $e = \sum e_{i,j} \in \mathfrak{g}$ where we sum over all i, j such that j is the right neighbour of i in K , so e is a nilpotent element of $\mathfrak{g}$ with Jordan type $\mathbf{p}$. In the example above we have

$$e = e_{2,3} + e_{4,5} + e_{5,6} + e_{6,7}.$$

For $i = 1, \dots, n$, we write $\text{col}(i)$ for the x -coordinate of the center of the box in K containing i . Let

$$\mathfrak{g}(k) = \langle e_{i,j} \mid \text{col}(j) - \text{col}(i) = k \rangle.$$

Then $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k)$ is a good grading for e and all good gradings for e occur in this way; we refer to [6, Sect. 4] and [3, Sect. 6] for more information on good gradings for $\mathfrak{gl}_n$. Now the finite W -algebra $U(\mathfrak{g}, e)$ can be defined as in Sect. 2.1.

For $i = 1, \dots, n$ we write $\text{row}(i)$ for the row of K in which i appears where we label the rows of K with $1, \dots, m$ from top to bottom. Then we have

$$\mathfrak{g}_0 = \langle e_{i,j} \mid \text{row}(i) = \text{row}(j) \rangle,$$

and

$$\mathfrak{b}_0 = \langle e_{i,j} \mid \text{row}(i) = \text{row}(j) \text{ and } \text{col}(i) \leq \text{col}(j) \rangle.$$

We take

$$\mathfrak{q} = \langle e_{i,j} \mid \text{row}(i) \leq \text{row}(j) \rangle,$$

as our choice of parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$.

We say that P is *justified* if the boxes are aligned in columns. We let $\text{Tab}(P)$ denote the set of fillings of P with complex numbers.

We define the *left justification* of P to be the diagram $l(P)$ obtained from P by left justifying the rows; given $A \in \text{Tab}(P)$, we define $l(A) \in \text{Tab}(l(P))$ similarly. For example, if

$$A = \begin{array}{cccc} & & 5 & \\ & -1 & 3 & \\ -3 & 1 & 1 & 4 \end{array}$$

then

$$l(A) = \begin{array}{cccc} 5 & & & \\ -1 & 3 & & \\ -3 & 1 & 1 & 4 \end{array}$$

The *row equivalence class* of $A \in \text{Tab}(P)$ is obtained by taking all possible permutations of the entries in the rows of A ; we write $\bar{A}$ for the row equivalence class of A . We write $\text{Row}(P)$ for the set of row equivalence classes of elements in $\text{Tab}(P)$.

For justified P , we say $A \in \text{Tab}(P)$ is *column strict* if the entries are strictly decreasing down columns. In this definition, and for the rest of this paper, we are using the partial order on $\mathbb{C}$ where $a \leq b$ if $b - a \in \mathbb{Z}_{\geq 0}$.

To each $A \in \text{Tab}(P)$ we associate a weight $\lambda_A = \sum a_i \epsilon_i \in \mathfrak{t}^*$, where a_i is the number in the box of A which occupies the same position as i in K . For example, with K and A as above we have

$$\lambda_A = 5\epsilon_1 - \epsilon_2 + 3\epsilon_3 - 3\epsilon_4 + \epsilon_5 + \epsilon_6 + 4\epsilon_7.$$

Let Λ_A be the W_0 -orbit of λ_A . We note that W_0 is isomorphic to $S_{p_1} \times \dots \times S_{p_m}$ and the action of W_0 on $\mathfrak{t}^*$ corresponds to W_0 acting on tables by permuting entries in rows. Thus Λ_A corresponds to the row equivalence class $\bar{A}$ of A . We write $L(\bar{A})$ for the highest weight irreducible $U(\mathfrak{g}, e)$ -module $L(\Lambda_A, \mathfrak{q})$, as defined in Sect. 2.6.

Now we are ready to state the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules, as discovered by Brundan and Kleshchev in [5].

Theorem 4.1 [5, Theorem 7.9]

$$\{L(\overline{A}) \mid \overline{A} \in \text{Row}(P), \overline{l(A)} \text{ contains an element which is column strict}\}$$

is a complete set of pairwise distinct isomorphism classes of finite dimensional simple $U(\mathfrak{g}, e)$ -modules.

4.2 Frames and tables

We recall some definitions about frames and tables, for more details see [2, Sect. 4]. We note that we use different notation for row swapping here.

A *box diagram* is a finite connected collection of boxes arranged in rows in the plane. Note that the symmetric group S_m acts naturally on the set of box diagrams with m rows by permuting rows. We number the rows in a box diagram from top to bottom. The pyramids from the previous subsection are box diagrams. A *frame* is a box diagram which is S_m -conjugate to a pyramid, where m is the number of rows in the pyramid. Given a frame F with m rows and $\sigma \in S_m$ we write $\sigma \cdot F$ for the image of F under the action of σ .

A frame is called *justified* if the boxes are aligned in columns. Given a frame F , the *left justification* of F is the frame $l(F)$ obtained from F by left justifying the rows.

A frame filled with complex numbers is called a *table*. Given a table A , the *frame of* A is obtained by removing the numbers in the boxes. Let F be a frame with m rows. We write $\text{Tab}(F)$ for the set of all tables with frame F . For $1 \leq i, j \leq m$, we write A_i for the i th row of $A \in \text{Tab}(F)$, and we write A_j^i for the table formed by rows i to j from A , for $i < j$. For $A \in \text{Tab}(F)$, we write $l(A) \in \text{Tab}(l(F))$ for the *left justification* of A .

For example,

is a frame,

and

$$l(A) = \begin{array}{|c|c|c|c|} \hline & 3 & 3 & 5 & 5 \\ \hline 4 & & & & \\ \hline 1 & 2 & & & \\ \hline \end{array} \in \text{Tab}(l(F)).$$

Suppose F is justified. We say $A \in \text{Tab}(F)$ is *column strict* if the entries are strictly decreasing down columns. The *row equivalence class* of $A \in \text{Tab}(F)$, denoted by $\bar{A}$, is obtained by taking all possible permutations of the entries in the rows of A . We write $\text{Row}(F)$ for the set of row equivalence classes of elements in $\text{Tab}(F)$.

A *tableau* is a column strict left justified table A such that the row lengths are weakly increasing from bottom to top, and such that if a lies to the left of b in the same row of A , then $a \not\geq b$. The *shape* of a tableau A with m rows is the partition $\mathbf{p} = (p_1, \dots, p_m)$, where p_i is the length of the i th row of A .

4.3 Row swapping and the Robinson–Schensted algorithm

Fix a frame F with m rows, let $1 \leq k < m$ and write $s_k = (k, k+1) \in S_m$. An important notion for us is row swapping in tables, as defined in [2, Sect. 4.3]. We now define $s_k \star$ the row swapping operation which takes as input $\bar{A} \in \text{Row}(F)$ and outputs $s_k \star \bar{A} \in \text{Row}(s_k \cdot F)$. Let A be an element of $\bar{A}$. First, if $l(A_{k+1}^k)$ does not contain an element which is column strict then we say that $s_k \star \bar{A}$ is undefined. Otherwise, let $c_1, c_2, \dots, c_s$ be the entries of A_k and let $d_1, d_2, \dots, d_t$ be the entries of A_{k+1} . We split into two cases.

Case 1: $s < t$ We choose $e_1, \dots, e_s$ from $d_1, \dots, d_t$ so that $e_i < c_i$ and $\sum_{i=1}^s c_i - e_i$ is minimal. Then $e_1, \dots, e_s$ form the entries of row $k+1$ in $s_k \star \bar{A}$, while the remaining entries in A_{k+1} are added to $c_1, \dots, c_s$ to form the entries of row k in $s_k \star \bar{A}$.

Case 2: $s > t$ We choose $e_1, \dots, e_t$ from $c_1, \dots, c_s$ so that $e_i > d_i$ and $\sum_{i=1}^t e_i - d_i$ is minimal. Then $e_1, \dots, e_t$ form the entries of row k in $s_k \star \bar{A}$, while the remaining elements from row k are added to $d_1, \dots, d_t$ to form the entries of row $k+1$ in $s_k \star \bar{A}$.

In the example above we have

$$\begin{array}{|c|c|c|c|} \hline & 5 & & \\ \hline 3 & 3 & 4 & 5 \\ \hline & 1 & 2 & \\ \hline \end{array} \in s_1 \star \bar{A}.$$

We finish this subsection with a brief discussion of the Robinson–Schensted algorithm. Given $A \in \text{Tab}(F)$, we write $\text{word}(A)$ for the sequence of complex numbers created by listing the entries in A row by row from left to right, top to bottom. In the example above we have $\text{word}(A) = (5, 3, 3, 4, 5, 1, 2)$. The Robinson–Schensted

algorithm is a process that takes as input a sequence of complex numbers and outputs a tableau. For a table A , we write $\text{RS}(A)$ for the output of the Robinson–Schensted algorithm with input $\text{word}(A)$. For $\bar{A} \in \text{Row}(F)$ we write $\text{RS}(\bar{A})$ to denote the row equivalence class of $\text{RS}(A)$, where $A \in \bar{A}$ is chosen so that if a is to the left of b in a row of A , then $a \not\geq b$. We refer the reader to [7] or [2, Sect. 4.2] for an explanation of the Robinson–Schensted algorithm.

An important point for us is [2, Lemma 4.8], which we recall below. In fact the lemma below is a little bit stronger than *loc. cit.*, but is straightforward to deduce.

Lemma 4.2 *Let F be a frame with m rows, $1 \leq k \leq m$, $\bar{A} \in \text{Row}(F)$ and $\bar{A}' \in \text{Row}(s_k \cdot F)$ such that $s_k \star \bar{A}$ is defined. Then $\text{RS}(\bar{A}') = \text{RS}(\bar{A})$ if and only if $\bar{A}' = s_k \star \bar{A}$.*

4.4 Changing highest weight theories for $U(\mathfrak{g}, e)$

We use the notation from Sect. 4.1. In particular, $\mathbf{p} = (p_1 \geq p_2 \geq \cdots \geq p_m)$ is a partition of n , P is a pyramid with m rows and row lengths given by the partition $\mathbf{p}$, and K is the coordinate table of P .

For $\sigma \in S_m$, we recall that $\sigma \cdot P$ is the frame obtained from P by swapping rows according to σ ; we define $\sigma \cdot K$ similarly. For K as in (4.2) and $\sigma = (123) \in S_3$, we have

$$\sigma \cdot K = \begin{array}{|c|c|c|c|} \hline 4 & 5 & 6 & 7 \\ \hline & & 1 & \\ \hline 2 & 3 & & \\ \hline \end{array}$$

Let $\sigma \in S_m$ and $i = 1, \dots, n$. We write $\text{row}_\sigma(i) = \sigma(\text{row}(i))$ for the row of $\sigma \cdot K$ that contains i . We can define $e, \mathfrak{g}(k), \mathfrak{g}_0$ and $\mathfrak{b}_0$ from $\sigma \cdot K$ in exactly the same way as we defined them from K . We define

$$\mathfrak{q}_\sigma = \langle e_{i,j} \mid \text{row}_\sigma(i) \leq \text{row}_\sigma(j) \rangle.$$

Then $\mathfrak{q}_\sigma$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$. Moreover, it is easy to see that any parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$ occurs in this way for some $\sigma \in S_m$. To shorten notation from now we write $\mathfrak{b} = \mathfrak{b}_q$ and $\mathfrak{b}_\sigma = \mathfrak{b}_{q_\sigma}$.

For $B \in \text{Tab}(\sigma \cdot F)$ we define $\lambda_{B,\sigma} = \sum b_i \epsilon_i \in \mathfrak{t}^*$, where b_i is the number in the box of B which occupies the same position as i in $\sigma \cdot K$. Let $\Lambda_{B,\sigma}$ be the W_0 -orbit of $\lambda_{B,\sigma}$. We write $L_\sigma(\bar{B})$ for the highest weight irreducible $U(\mathfrak{g}, e)$ -module $L(\Lambda_{B,\sigma}, \mathfrak{q}_\sigma)$. We define

$$\text{Row}^+(\sigma \cdot F) = \{ \bar{B} \in \text{Row}(\sigma \cdot F) \mid L_\sigma(\bar{B}) \text{ is finite dimensional} \},$$

and

$$\mathcal{X}^+(F) = \bigcup_{\sigma \in S_m} \text{Row}^+(\sigma \cdot F).$$

Below we state our theorem which tells us how to change between different highest weight theories. For the statement we require the $\star$ -action of S_m on $\mathcal{X}^+(F)$. To define this let $\sigma \in S_m$ and $\bar{B} \in \text{Row}^+(\sigma \cdot F)$. Write σ as a product of simple reflections $\sigma = s_{i_1} \cdots s_{i_l}$ and define

$$\sigma \star \bar{B} = s_{i_1} \star (s_{i_2} \star (\cdots (s_{i_l} \star \bar{B}) \cdots)). \quad (4.3)$$

Theorem 4.3

- (i) The $\star$ -action of S_m on $\mathcal{X}^+(P)$ is well defined.
- (ii) Let $\sigma, \tau \in S_m$, and $\bar{B} \in \text{Row}^+(\sigma \cdot F)$, $\bar{B}' \in \text{Row}^+(\tau \sigma \cdot F)$. Then $L_\sigma(\bar{B}) \cong L_{\tau\sigma}(\bar{B}')$ if and only if $\bar{B}' = \tau \star \bar{B}$.

Before proving the theorem we give a technical remark, which is required in the proof.

Remark 4.4 An alternative proof of Theorem 4.1 now follows from [2, Proposition 3.12] and the arguments in the proof of [4, Corollary 5.6]. This is based on an argument first showing that $L(\bar{A})$ is finite dimensional if and only if the shape of $\text{RS}(\bar{A})$ is $\mathbf{p}$; this requires [9, Corollary 3.3]. Then it is an easy combinatorial argument to show that the shape of $\text{RS}(A)$ is $\mathbf{p}$ if and only if $\bar{L}(\bar{A})$ contains an element which is column strict. These arguments are also valid, though the combinatorial argument is a bit more complicated, if we use “upside-down pyramids”, for which the row lengths are decreasing from top to bottom, instead of pyramids.

Proof of Theorem 4.3 First we have to give some more notation. For $1 \leq k \leq m$, we define $t_k \in \mathfrak{t}$ by $t_k = \sum_{j|\text{row}(j)=k} e_{jj}$. Then we have $\mathfrak{t}^e = \langle t_1, \dots, t_m \rangle$. Next for $1 \leq k \leq m$ we define

$$\mathfrak{s}_{k,l} = \{t_j \mid 1 \leq j \leq m, j \neq k, l\} \cup \{t_k + t_l\}.$$

Then $\mathfrak{s}_{k,l}$ is a full subalgebra of $\mathfrak{t}^e$ and we have

$$\mathfrak{g}^{\mathfrak{s}_{k,l}} = \langle e_{i,j} \mid \text{row}(i) = \text{row}(j) \text{ or } \{\text{row}(i), \text{row}(j)\} = \{k, k+1\} \rangle.$$

Therefore,

$$\mathfrak{g}^{\mathfrak{s}_{k,l}} \cong \left(\bigoplus_{j \neq k,l} \mathfrak{gl}_{p_j} \right) \oplus \mathfrak{gl}_{p_k+p_l},$$

and the finite W -algebra $U(\mathfrak{g}^{\mathfrak{s}_{k,l}}, e)$ decomposes as a tensor product

$$U(\mathfrak{g}^{\mathfrak{s}_{k,l}}, e) \cong \left(\bigotimes_{j \neq k,l} U(\mathfrak{gl}_{p_j}, e_j) \right) \otimes U(\mathfrak{gl}_{p_k+p_l}, e_k + e_l),$$

where e_j is the projection of e in $\mathfrak{gl}_{p_j}$.

Now we show that for $\bar{B} \in \mathcal{X}^+(F)$ and $1 \leq k < m$, we see that $s_k \star \bar{B}$ is defined. Let $B \in \bar{B}$ be such that if a is to the left of b in a row of B then $a \not\geq b$.

Let $\sigma \in S_m$ such that $\overline{B} \in \text{Row}(\sigma \cdot F)$ and $L_\sigma(\overline{B})$ is finite dimensional, and let $k' = \sigma^{-1}(k)$ and $l = \sigma^{-1}(k+1)$. In this case $L_\sigma(\overline{B}) = L(\Lambda_{B,\sigma}, q_\sigma)$ is finite dimensional so as explained before (2.7), we have $L^{\mathfrak{s}_{k',l}}(\Lambda_{B,\sigma}, q_\sigma)$ is finite dimensional. Now $L^{\mathfrak{s}_{k',l}}(\Lambda_{B,\sigma}, q_\sigma)$ is a tensor product of irreducible highest weight modules for each of the finite W -algebras in the tensor product decomposition of $U(\mathfrak{g}^{\mathfrak{s}_{k',l}}, e)$. We consider the tensor factor corresponding to $U(\mathfrak{gl}_{p_{k'}+p_l}, e_{k'} + e_l)$. Under the natural identifications, we see that up to some central shift (due to the difference between “ ρ for $\mathfrak{g}$ and ρ for $\mathfrak{gl}_{p_{k'}+p_l}$ ”) this tensor factor is the highest weight $U(\mathfrak{gl}_{p_{k'}+p_l}, e_{k'} + e_l)$ -module labelled by B_{k+1}^k ; the central shift corresponds to a constant being added to all the entries in B_{k+1}^{k+1} . Thus by Theorem 4.1 and Remark 4.4 we find that B_{k+1}^k is justified row equivalent to column strict. Hence, $s_k \star \overline{B}$ is defined.

Next we show that $L_{s_k \sigma}(s_k \star \overline{B}) \cong L_\sigma(\overline{B})$. To do this we use Theorem 3.1 and a result of Joseph which tells us when two highest weight $U(\mathfrak{g})$ -modules have the same annihilator. Given $\lambda = \sum_{i=1}^n a_i \epsilon_i \in \mathfrak{t}^*$, we define $\text{RS}(\lambda)$ to be the output of the Robinson–Schensted algorithm applied to $\text{word}(\lambda) = (a_1, \dots, a_n)$. Then [9, Théorème 1] says that for $\lambda, \mu \in \mathfrak{t}^*$ we have

$$\text{Ann } L(\lambda, \mathfrak{b}) = \text{Ann } L(\mu, \mathfrak{b}) \quad \text{if and only if} \quad \overline{\text{RS}(\lambda)} = \overline{\text{RS}(\mu)}. \quad (4.4)$$

We see that $W = S_n$ acts on words of length n and elements of $\mathfrak{t}^*$ in the usual way. Let $w_\sigma \in S_n$ be the permutation such that $w_\sigma \cdot \text{word}(K) = \text{word}(\sigma \cdot K)$ and define $w_{s_k \sigma}$ similarly. So we have $\mathfrak{b}_\sigma = w_\sigma \cdot \mathfrak{b}$ and $\mathfrak{b}_{s_k \sigma} = w_{s_k \sigma} \cdot \mathfrak{b}$.

Let $s_k \star B$ denote an element in $s_k \star \overline{B}$ such that if a, b lie in the same row as $s_k \star B$ then $a \not\geq b$. Now we have $\text{RS}(\overline{B}) = \text{RS}(w_\sigma^{-1} \cdot \lambda_{B,\sigma})$ and $\text{RS}(s_k \star \overline{B}) = \text{RS}(w_{s_k \sigma}^{-1} \cdot \lambda_{s_k \star B, s_k \sigma})$. By Lemma 4.2, we have $\text{RS}(\overline{B}) = \text{RS}(s_k \star \overline{B})$, so we get $\text{RS}(w_\sigma^{-1} \cdot \lambda_{B,\sigma}) = \text{RS}(w_{s_k \sigma}^{-1} \cdot \lambda_{s_k \star B, s_k \sigma})$, which implies that $\text{Ann } L(w_\sigma^{-1} \cdot \lambda_{B,\sigma}, \mathfrak{b}) = \text{Ann } L(w_{s_k \sigma}^{-1} \cdot \lambda_{s_k \star B, s_k \sigma}, \mathfrak{b})$ by (4.4). Hence, by Theorem 3.1 we find that $[L_\sigma(\overline{B})]$ and $[L_{s_k \sigma}(\overline{B})]$ lie in the same orbit of $C(e)$ in $\text{Irr}_0 U(\mathfrak{g}, e)$. We note that the condition imposed on B means that $\langle \lambda_{B'}, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_0^+$, and similarly for $s_k \star B$, so that we can apply Theorem 3.1. Since $C(e)$ is trivial, we have $L_\sigma(\overline{B}) \cong L_{s_k \sigma}(s_k \star \overline{B})$.

Now let $\tau \in S_m$ then by writing τ as a product of simple reflections we can define $\tau \star \overline{B}$ as in (4.3). By induction we see that each of the row swapping operations is defined, and that

$$L_\sigma(\overline{B}) \cong L_{\tau \sigma}(\tau \star \overline{B}). \quad (4.5)$$

Then we also see that $\tau \star \overline{B}$ does not depend on the choice of the expression of τ in terms of simple reflection, because $L_\sigma(\overline{B}) \cong L_{\tau \sigma}(\tau \star \overline{B})$. This means that the $\star$ -action is a well-defined action of S_m on $\mathcal{X}^+(F)$ giving (i). Then (ii) is just (4.5). $\square$

We state the following corollary, which is an immediate consequence of Theorem 4.3 and Lemma 4.2.

Corollary 4.5 *Let $\sigma, \tau \in S_m$ and $\overline{B} \in \text{Row}^+(\sigma \cdot F)$, $\overline{B}' \in \text{Row}^+(\tau \sigma \cdot F)$. Then $L_\sigma(\overline{B}) \cong L_{\tau \sigma}(\overline{B})$ if and only if $\text{RS}(\overline{B}) = \text{RS}(\overline{B}')$.*

As an example of the $\star$ -action, we give $\sigma \star \bar{A}$, where

$$A = \begin{array}{|c|c|c|} \hline & 4 & \\ \hline & -2 & \\ \hline -3 & 1 & 3 \\ \hline -4 & -1 & 2 \\ \hline \end{array}$$

and $\sigma = (123)$. Then we have

$$\begin{array}{|c|c|c|} \hline -2 & 1 & 4 \\ \hline & 3 & \\ \hline & -3 & \\ \hline -4 & -1 & 2 \\ \hline \end{array} \in \sigma \star \bar{A}.$$

Note that A is row equivalent to column strict, yet $\sigma \star \bar{A}$ does not contain any column strict elements. So for different highest weight theories the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -module is not just that $L_\sigma(\bar{A})$ is finite dimensional if and only if $\bar{A}$ contains a column strict table.

Remark 4.6 Let $\sigma, \tau \in S_m$ and suppose that $\tau\sigma \cdot F = \sigma \cdot F$, i.e. τ permutes rows $\sigma \cdot F$ of the same length. Then as explained in [3, Sect. 6], there exists an element of the restricted Weyl group $w \in W^e$ such that $w \cdot q_\sigma = q_{\tau\sigma}$. Thus for any $\bar{B} \in \text{Row}^+(\sigma \cdot F)$, we have $L_\sigma(\bar{B}) \cong L_{\tau\sigma}(\bar{B})$ by Proposition 3.3. This can also be easily verified by noting that a row swapping operation on two rows of the same length is trivial.

5 Changing highest weight theories associated to even multiplicity finite W -algebras

In this section we prove Theorem 5.7, which tells us how to pass between different highest weight theories when $\mathfrak{g}$ is of type C or D and e is even multiplicity. We recall some definitions from [2, Sect. 4] regarding s-frames and s-tables in Sect. 5.1. Then we use s-tables to give the notation for finite W -algebras in Sect. 5.2 and the combinatorics for the description of the highest weight theories in Sect. 5.3. In the remaining subsections we review the classification of finite dimensional $U(\mathfrak{g}, e)$ -modules from [2, Sect. 5] and describe the bijection between parameterizing sets for different highest weight theories.

In this section we often consider sets of the form $\{1, 2, \dots, l, -l, \dots, -2, -1\}$ and we use the unconventional total order on this set given by $1 \leq 2 \leq \dots \leq l \leq -l \leq \dots \leq -2 \leq -1$.

5.1 s-Frames and s-tables

The combinatorics for the highest weight theories for finite W -algebras associated to even multiplicity nilpotent elements in classical Lie algebra algebras involves a skew-symmetric version of tables called s-tables. Below we review the terminology for s-frames and s-tables from [2, Sect. 4.4].

We define an *s-frame* to be a frame where the boxes are arranged symmetrically around the origin. We say that an s-frame is a *symmetric pyramid* if the row lengths weakly decrease from the centre outwards; we note that a symmetric pyramid is uniquely determined by its row lengths. In this paper we only consider s-frames which have an even number of rows.

An example of an s-frame (which is not a symmetric pyramid) is

We define an *s-table* to be an s-frame for which every box is filled with a complex number. Furthermore we require that the boxes be filled skew-symmetrically with respect to the centre. Given an s-frame F , we write $\text{sTab}(F)$ for the set of s-tables with frame F . We write $\overline{A}^s = \overline{A} \cap \text{sTab}(F)$ for the set of s-tables row equivalent to A . For example

$$\begin{array}{|c|c|c|c|} \hline & -7 & 3 & \\ \hline -8 & -4 & 2 & 5 \\ \hline -5 & -2 & 4 & 8 \\ \hline & -3 & 7 & \\ \hline \end{array} \in \text{sTab}^{\leq}(F), \quad (5.1)$$

where F is its s-frame. A piece of notation that we require later is as follows. Given a sign $\phi \in \{\pm\}$, we define

$$\text{sTab}_{\phi}(F) = \begin{cases} \{A \in \text{sTab}(F) \mid A \text{ has all entries in } \mathbb{Z} \text{ or all entries in } \frac{1}{2} + \mathbb{Z}\} & \text{if } \phi = +; \\ \{A \in \text{sTab}(F) \mid A \text{ has all entries in } \mathbb{Z}\} & \text{if } \phi = -. \end{cases}$$

The subset of $\text{sTab}_{\phi}(F)$ consisting of s-tables with entries weakly increasing along rows is denoted by $\text{sTab}_{\phi}^{\leq}(F)$.

Let F be an s-frame and $A \in \text{sTab}(F)$. By assumption, F has an even number of rows, say $2m$. We label the rows of F and A with $1, \dots, m, -m, \dots, -1$ from bottom

to top. Given $i = \pm 1, \dots, \pm m$ we write A_i for row of A labelled by i , and for $i > 0$ we write A_{-i}^i for the s-table obtained by removing rows $\pm 1, \dots, \pm(i-1)$. The table obtained from A by removing all boxes below the central point is denoted by A_+ . For example if A is the table above, then

$$A_+ = \begin{array}{|c|c|c|c|} \hline & -7 & 3 & \\ \hline -8 & -4 & 2 & 5 \\ \hline \end{array}$$

Finally in this subsection we generalize the row swapping procedure to s-tables. As above let F be an s-frame with $2m$ rows, and let $A \in \text{sTab}_{\phi}^{\leq}(F)$, where $\phi \in \{\pm\}$. Let $k = 1, \dots, m-1$. Then we define

$$\bar{s}_k \star A = s_{-k} \star (s_k \star A),$$

where s_k swaps rows k and $k+1$ as defined in Sect. 4.2, and s_{-k} swaps rows $-(k+1)$ and $-k$ using the same rules. Here we define the row swapping operations directly on elements of $\text{sTab}_{\phi}^{\leq}(F)$ rather than on row equivalence classes, because there is a unique element of $\text{sTab}_{\phi}^{\leq}(F)$ in any row equivalence class. We note that $s_k \star A$ is defined if and only if $s_{-k} \star A$ is defined, and that the operators s_k and s_{-k} commute. Also we note that when s_k is defined, the action of s_{-k} is “dual” to that of s_k , so $\bar{s}_k \star A$ is an s-table.

5.2 Notation for even multiplicity finite W -algebras

For the rest of this section, we fix a sign $\phi \in \{\pm\}$. As a shorthand we say that an integer l is ϕ -even if $\phi = +$ and l is even or $\phi = -$ and l is odd; we define ϕ -odd similarly.

We specify coordinates for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$. Let $V = \mathbb{C}^{2n}$ be the $2n$ -dimensional vector space with standard basis $\{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ and non-degenerate bilinear form $(\cdot, \cdot)$ defined by $(e_i, e_j) = 0$ if i and j have the same sign, and $(e_i, e_{-j}) = \delta_{i,j}$, $(e_{-i}, e_j) = \phi \delta_{i,j}$ for $i, j = 1, \dots, n$. Let $\tilde{G} = G_{2n}^{\phi} = \{x \in \text{GL}_{2n} \mid (xv| xv') = (v|v') \text{ for all } v, v' \in V\}$, and $\mathfrak{g} = \mathfrak{g}_{2n}^{\phi} = \{x \in \mathfrak{gl}_{2n} \mid (xv|v') = -(v| xv') \text{ for all } v, v' \in V\}$ be the Lie algebra of $\tilde{G}$. So $\tilde{G} = \text{O}_{2n}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$ if $\phi = +$, and $\tilde{G} = \text{Sp}_{2n}$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$ if $\phi = -$. We write G for the identity component group of $\tilde{G}$, so $G = \tilde{G}$ in the type C case, and $G = \text{SO}_{2n}$ in the type D case. We let $(\cdot|\cdot)$ be the trace form on $\mathfrak{g}$.

Let $\{e_{i,j} \mid i, j = 1, \dots, n, -n, \dots, -1\}$ be the standard basis of $\mathfrak{gl}_{2n}$, and define $f_{i,j} = e_{i,j} - \eta_{i,j} e_{-j,-i}$ where $\eta_{i,j} = 1$ if i and j have the same sign and $\eta_{i,j} = \phi$ if i and j have different signs. Then the standard basis of $\mathfrak{g}$ is $\{f_{i,j} \mid i < -j\}$ if $\phi = +$ and $\{f_{i,j} \mid i \leq -j\}$ if $\phi = -$, where we use the order given by $1 \leq 2 \leq \dots \leq n \leq -n \leq \dots \leq -2 \leq -1$. Let $\mathfrak{t} = \langle f_{i,i} \mid i = 1, \dots, n \rangle$ be the standard Cartan subalgebra of $\mathfrak{g}$ of diagonal matrices. We define $\{\epsilon_i \mid i = 1, \dots, n\}$ to be the basis of $\mathfrak{t}^*$ dual to $\{f_{i,i} \mid i = 1, \dots, n\}$.

We recall that nilpotent $\tilde{G}$ -orbits in $\mathfrak{g}$ are parameterized by partitions $\mathbf{p}$, such that each ϕ -even part of $\mathbf{p}$ has even multiplicity when $\mathfrak{g} = \mathfrak{so}_{2n}$. For $\mathfrak{g} = \mathfrak{so}_{2n}$, we also

recall that a nilpotent $\tilde{G}$ -orbit parameterized by $\mathbf{p}$ is a single G -orbit unless all parts of $\mathbf{p}$ are even and of even multiplicity. In this latter case, where we say that $\mathbf{p}$ is *very even*, the $\tilde{G}$ -orbit parameterized by $\mathbf{p}$ splits into two G -orbits.

We recall the structure of the component group $\tilde{C}(e)$ of the centralizer of e in $\tilde{G}$. Suppose $e \in \mathfrak{g}$ lies in the nilpotent $\tilde{G}$ -orbit corresponding to the partition $\mathbf{p}$. Then $\tilde{C}(e) \cong \mathbb{Z}_2^d$, where d is the number of distinct ϕ -odd parts of $\mathbf{p}$, see for example [8, Sect. 3.13]. We note that $C(e)$ is equal to $\tilde{C}(e)$ unless $\mathfrak{g} = \mathfrak{so}_{2n}$ and $\mathbf{p}$ has an odd part, in which case $C(e)$ has index 2 in $\tilde{C}(e)$.

For the remainder of this section we fix an even multiplicity partition $\mathbf{p} = (p_1^2, \dots, p_r^2)$ of $2n$, where $p_i \geq p_{i+1}$ for each i . The *symmetric pyramid* of $\mathbf{p}$ is the symmetric pyramid with row lengths given by $\mathbf{p}$ as defined in Sect. 5.1; we write $P = P_{\mathbf{p}}$ for this s-frame. The table with frame P and with boxes filled by $1, \dots, n, -n, \dots, -1$ from left to right and top to bottom is called the *coordinate pyramid* of $\mathbf{p}$ and denoted by $K = K_{\mathbf{p}}$. For example

$$K = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline -5 & -4 & -3 \\ \hline & -2 & -1 \\ \hline \end{array}$$

is a coordinate table.

We define the nilpotent element $e \in \mathfrak{g}$ with Jordan type $\mathbf{p}$ by $e = \sum f_{i,j}$, where we sum over all i, j such that i and j are positive and j is in the box immediately to the right of i in K . We write $\text{col}(i)$ for the x -coordinate of the box in K containing i and we define $h = \sum_{i=1}^n -\text{col}(i) f_{i,i}$. For example, if K is as above, we have $e = f_{1,2} + f_{3,4} + f_{4,5}$ and $h = -f_{1,1} + f_{2,2} - 2f_{3,3} + 2f_{5,5}$. Then the $\text{ad } h$ eigenspace decomposition gives the Dynkin grading

$$\mathfrak{g}(k) = \langle f_{i,j} \mid \text{col}(j) - \text{col}(i) = k \rangle.$$

The finite W -algebra $U(\mathfrak{g}, e)$ can now be defined as in Sect. 2.1.

We do not consider other good gradings for e here, as there are not many non-Dynkin good gradings, so it is not particularly advantageous to do so; we refer the reader to [6, Sects. 5 and 6] and [3, Sects. 6 and 7] for more information on good gradings for classical Lie algebras.

5.3 Highest weight theories for $U(\mathfrak{g}, e)$

We now discuss highest weight theories for $U(\mathfrak{g}, e)$. We continue to use the notation from the previous subsection; in particular, P is the symmetric pyramid of $\mathbf{p}$ and K is the coordinate pyramid of $\mathbf{p}$. First we consider the highest weight theory for a particular choice $\mathfrak{q}$ of parabolic subalgebra, then we give the notation for other choices of parabolic subalgebra.

For $i = \pm 1, \dots, \pm n$ we write $\text{row}(i)$ for the row of K in which i appears; recall that rows in P are labelled with $-m, \dots, -1, 1, \dots, m$ from bottom to top. Then we have

$$\mathfrak{g}_0 = \langle f_{i,j} \mid \text{row}(i) = \text{row}(j) \rangle,$$

and

$$\mathfrak{b}_0 = \langle f_{i,j} \mid \text{row}(i) = \text{row}(j) \text{ and } \text{col}(i) \leq \text{col}(j) \rangle$$

Let

$$\mathfrak{q} = \langle f_{i,j} \mid \text{row}(i) \leq \text{row}(j) \rangle,$$

which is a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$; here we are using the ordering $1 \leq 2 \leq \dots \leq m \leq -m \leq \dots \leq -2 \leq -1$.

To each $A \in \text{sTab}_{\phi}^{\leq}(F)$ we associate a weight $\lambda_A = \sum a_i \epsilon_i \in \mathfrak{t}^*$, where a_i is the number in the box of A which occupies the same position as i in K . For example, with K as above and

$$A = \begin{array}{|c|c|c|} \hline & 2 & 7 \\ \hline -3 & 1 & 4 \\ \hline -4 & -1 & 3 \\ \hline & -7 & -2 \\ \hline \end{array} \quad (5.2)$$

we have

$$\lambda_A = -3\epsilon_1 + \epsilon_2 + 4\epsilon_3 + 2\epsilon_4 + 7\epsilon_5.$$

Let Λ_A be the W_0 -orbit of λ_A . We note that W_0 is isomorphic to $S_{p_1} \times \dots \times S_{p_m}$ and the action of W_0 on $\mathfrak{t}^*$ corresponds to W_0 acting on tables by permuting entries in rows. Thus Λ_A corresponds to the row equivalence class $\overline{A}^s$ of A . We write $L(A)$ for the highest weight irreducible $U(\mathfrak{g}, e)$ -module $L(\Lambda_A, \mathfrak{q})$, as defined in Sect. 2.6. Later, in Theorem 5.5, we state the main theorem from [2], which determines when $L(A)$ is finite dimensional.

We note that the restriction to tables in $\text{sTab}_{\phi}(P)$ corresponds to the central character of $L(A)$ being integral. Also as we use tables in $\text{sTab}_{\phi}^{\leq}(P)$ there is no need to use the row equivalence class in the notation for $L(A)$.

We now give the notation for highest weight theories corresponding to other choices of parabolic subalgebra. Let W_m denote the Weyl group of type B_m acting on $\{\pm 1, \dots, \pm m\}$ in the usual way. We write $\overline{S}_m$ for the subgroup of W_m isomorphic to S_m consisting of the permutations with no sign changes. The standard generators of W_m are denoted by $r, \overline{s}_1, \dots, \overline{s}_{n-1}$, where r is the transposition $(n, -n)$ and $\overline{s}_1, \dots, \overline{s}_{n-1}$ are the standard generators of $\overline{S}_m$, so $\overline{s}_i = (i, i+1)(-i, -i-1)$. Given $\sigma \in \overline{S}_m$ we write $\underline{\sigma}$ for the corresponding element of S_m .

For $\sigma \in W_m$, we define $\sigma \cdot P$ to be the frame obtained from P by permuting rows according to σ and define $\sigma \cdot K$ similarly. For example for K as in (4.2) and

$\sigma = (1, -2)(2, -1) \in W_2$, we have

$$\sigma \cdot K = \begin{array}{|c|c|c|} \hline -5 & -4 & -3 \\ \hline -2 & -1 & \\ \hline 1 & 2 & \\ \hline 3 & 4 & 5 \\ \hline \end{array}$$

Let $\sigma \in W_m$ and $i = \pm 1, \dots, \pm n$. We write $\text{row}_\sigma(i)$ for the row of $\sigma \cdot K$ that contains i . We can define e , $\mathfrak{g}(k)$, $\mathfrak{g}_0$ and $\mathfrak{b}_0$ from $\sigma \cdot K$ in exactly the same way as we defined them from K . We define

$$\mathfrak{q}_\sigma = \langle e_{i,j} \mid \text{row}_\sigma(i) \leq \text{row}_\sigma(j) \rangle.$$

Then $\mathfrak{q}_\sigma$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$. Moreover, it is easy to see that any parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{g}_0$ occurs in this way for some $\sigma \in W_m$.

For $B \in \text{sTab}_\phi^\leq(\sigma \cdot P)$ we define $\lambda_{B,\sigma} = \sum b_i \epsilon_i \in \mathfrak{t}^*$, where b_i is the number in the box of B which occupies the same position as i in $\sigma \cdot K$. Let $\Lambda_{B,\sigma}$ be the W_0 -orbit of $\lambda_{B,\sigma}$. We write $L_\sigma(B)$ for the highest weight irreducible $U(\mathfrak{g}, e)$ -module $L(\Lambda_{B,\sigma}, \mathfrak{q}_\sigma)$. We define

$$\text{sTab}_\phi^+(\sigma \cdot P) = \{B \in \text{sTab}_\phi^\leq(\sigma \cdot P) \mid L_\sigma(B) \text{ is finite dimensional}\}$$

and

$$\mathcal{X}_\phi^+(P) = \bigcup_{\sigma \in W_m} \text{sTab}_\phi^+(\sigma \cdot P).$$

5.4 Changing the highest weight theory “in the top half”

In this section we begin to show how to pass between different highest weight theories, where the change involves permuting rows according to an element of $\overline{S}_m \subseteq W_m$, i.e. permuting rows in the top half. In the statement we use the $\star$ -action of $\overline{S}_m$ on $\mathcal{X}_\phi^+(P)$ defined by extending the row swapping operations $\overline{s}_k$ from Sect. 5.1 in analogy to (4.3).

Proposition 5.1

- (i) The $\star$ -action of $\overline{S}_m$ on $\mathcal{X}_\phi^+(P)$ is well defined.
- (ii) Let $\sigma \in W_m$, $\tau \in \overline{S}_m$, and $B \in \text{sTab}_\phi^+(\sigma \cdot P)$, $B' \in \text{sTab}_\phi^+(\tau \sigma \cdot F)$. Then $L_\sigma(B) \cong L_{\tau\sigma}(B')$ if and only if $B' = \tau \star B$.

Proof First let $\sigma, \tau \in \overline{S}_m$, and $B \in \text{sTab}_\phi^+(\sigma \cdot P)$. We set $t = \sum_{\text{row}_\sigma(i) > 0} f_{i,i}$ and $\mathfrak{s} = \langle t \rangle$, which is a full subalgebra of $\mathfrak{t}^e$. Then $\mathfrak{g}^\mathfrak{s} \cong \mathfrak{gl}_n$ and we see that $(\mathfrak{q}_\sigma)^\mathfrak{s} = (\mathfrak{q}_{\tau\sigma})^\mathfrak{s}$.

Now $L_\sigma(B) = L(\Lambda_B, \mathfrak{q}_\sigma)$ is finite dimensional, so, as explained before (2.7), we have $L^\mathfrak{s}(\Lambda_B, \mathfrak{q}_\sigma)$ is a finite dimensional $U(\mathfrak{g}^\mathfrak{s}, e)$ -module. We see that up to some

central shift (due to the different root systems for $\mathfrak{g}$ and $\mathfrak{gl}_n$) $L^s(\Lambda_B, q_\sigma)$ is isomorphic to the highest weight $U(\mathfrak{gl}_n, e)$ -module $L_{\underline{\sigma}}(B_+)$; this central shift corresponds to adding a constant to all entries in B_+ .

Now using Theorem 4.3, the table $\underline{\tau} \star B_+$ is defined. Clearly $\tau \star B$ is the s -table with $(\tau \star B)_+ = \underline{\tau} \star B_+$, so, in particular, it is well defined. Again by Theorem 4.3, we have $L_{\underline{\sigma}}(B_+) \cong L_{\underline{\tau}\underline{\sigma}}(\underline{\tau} \star B_+)$, which implies that $L^s(\Lambda_B, q_\sigma) \cong L^s(\Lambda_{\tau \star B}, q_{\tau\sigma})$. Thus as $(q_\sigma)_s = (q_{\tau\sigma})_s$, we get $M_s(L^s(\Lambda_B, q_\sigma)) \cong M_s(L^s(\Lambda_{\tau \star B}, q_{\tau\sigma}))$, which means that $L_\sigma(B) \cong L_{\tau\sigma}(\tau \star B)$. In particular, $L_{\tau\sigma}(\tau \star B)$ is finite dimensional.

In the case that $\sigma \in W_m \setminus \bar{S}_m$, all of the arguments above go through with a minor complication regarding the identification $\mathfrak{g}^s \cong \mathfrak{gl}_n$. $\square$

5.5 The component group action

In order to state the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules in Theorem 5.5 we need to recall the action of the component group $\tilde{C}(e)$ on $\mathcal{X}_\phi^+(P)$ from [2, Sect. 5.3]. In fact we complete the verification that we do get the true action of $\tilde{C}(e)$, see [2, Remark 5.9]. The component group action is also required for Theorem 5.7, where we complete the description of how to pass between different highest weight theories.

The description of the action depends on the notion of the $\sharp$ -element of a list of complex numbers. Given a list $(a_1, \dots, a_{2k+1})$ of complex numbers let $\{(a_1^{(i)}, \dots, a_{2k+1}^{(i)}) \mid i \in I\}$ be the set of all permutations of this list which satisfy $a_{2j-1}^{(i)} + a_{2j}^{(i)} > 0$ for each $j = 1, \dots, k$. Assuming that such rearrangements exist, we define the $\sharp$ -element of $(a_1, \dots, a_{2k+1})$ to be the unique maximal element of the set $\{a_{2k+1}^{(i)} \mid i \in I\}$. On the other hand, if no such rearrangements exist, we say that the $\sharp$ -element of $(a_1, \dots, a_{2k+1})$ is undefined. For example, the $\sharp$ -element of $(-3, -1, 2)$ is -3 , whereas the $\sharp$ -element of $(-3, -2, 1)$ is undefined. We abuse notation somewhat by saying that the $\sharp$ -element of a list of numbers with an even number of elements is the $\sharp$ -element of that list with 0 inserted.

We begin by considering the case where $\mathbf{p} = (n^2)$, n is ϕ -odd, and n is odd if $\mathfrak{g} = \mathfrak{so}_{2n}$. In this case we have $\tilde{C}(e) \cong \mathbb{Z}_2 = \langle c \rangle$, and we define an operation of c on $\text{sTab}_\phi^+(P)$ as follows. Let $A \in \text{sTab}_\phi^+(P)$ and let $a_1, \dots, a_n$ be the entries of row 1 of A . By [1, Theorem 1.2] the $\sharp$ -element of $a_1, \dots, a_n$ is defined; let a be this number. We declare that $c \cdot A \in \text{sTab}_\phi^-(P)$ is the s -table obtained from A by replacing one occurrence of a in row 1 with $-a$, and one occurrence of $-a$ in row -1 with a . Then [1, Theorem 1.3] says that $c \cdot L(A) = L(c \cdot A)$; in particular, $c \cdot A \in \text{sTab}_\phi^+(P)$.

An example of this action is

$$c \cdot \begin{array}{|c|c|} \hline 1 & 2 \\ \hline -2 & -1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline -2 & 1 \\ \hline -1 & 2 \\ \hline \end{array}$$

Now we define an operation of c on $\mathcal{X}^+(P)$ for $\mathbf{p}$ any even multiplicity partition. Let $B \in \mathcal{X}^+(P)$ and let $\sigma \in W_m$ such that $B \in \text{sTab}_\phi^+(\sigma \cdot P)$. Suppose that the length of row m in B is ϕ -even, then we define $c \cdot B = B$. Next suppose that the length of

row m in B is ϕ -odd. Below we justify that $c \cdot B_{-m}^m$ is defined. This allows us to define $c \cdot B$ to be the table obtained from B by replacing rows m and $-m$ by $c \cdot B_{-m}^m$.

To justify that $c \cdot B_{-m}^m$ is defined we let $t_\sigma = \sum_{i|\text{row}_\sigma(i) \neq m} f_{i,i}$ and $\mathfrak{s}_\sigma = \langle t_\sigma \rangle$, which is a full subalgebra of $\mathfrak{t}^e$. The Levi subalgebra $\mathfrak{g}^{\mathfrak{s}_\sigma}$ is isomorphic to $\mathfrak{gl}_{n-p_\sigma(m)} \oplus \mathfrak{g}_{2p_\sigma(m)}^\phi$ and the finite W -algebra $U(\mathfrak{g}^{\mathfrak{s}_\sigma}, e)$ decomposes as a tensor product

$$U(\mathfrak{g}^{\mathfrak{s}_j}, e) \cong U(\mathfrak{gl}_{n-p_\sigma(m)}, e'_\sigma) \otimes U(\mathfrak{g}_{2p_\sigma(m)}^\phi, e_\sigma),$$

where e'_σ and e_σ denote the projections of e into $\mathfrak{gl}_{n-p_\sigma(m)}$ and $\mathfrak{g}_{2p_\sigma(m)}^\phi$, respectively. As explained before (2.7), we find that $L_\sigma^{\mathfrak{s}_\sigma}(B)$ is finite dimensional. Also $L_\sigma^{\mathfrak{s}_\sigma}(B)$ is the tensor product of irreducible highest weight modules for $U(\mathfrak{gl}_{n-p_\sigma(m)}, e_0)$ and $U(\mathfrak{g}_{2p_\sigma(m)}^\phi, e_1)$. The tensor factor that is a $U(\mathfrak{g}_{2p_\sigma(m)}^\phi, e_j)$ -module is the highest weight module labelled by B_{-m}^m . Therefore, we see that $c \cdot B_{-m}^m$ is defined by [1, Theorem 1.2] as above.

Next we describe the action of $\tilde{C}(e)$ on $\mathcal{X}^+(P)$. Let $i_1 < \dots < i_d$ be minimal such that $p_{i_1}, \dots, p_{i_d}$ are the distinct parts of $\mathbf{p} = (p_1^2 \geq p_2^2 \geq \dots \geq p_r^2)$ that are ϕ -odd. Then we can choose generators $c_1, \dots, c_d$ for $\tilde{C}(e) \cong \mathbb{Z}_2^d$ corresponding to $p_{i_1}, \dots, p_{i_d}$. A lift of the element c_j in H^e fixes all basis vectors e_k , except those where $\text{row}(k) = \pm i_j$. If l is in the same column as k with $\text{row}(k) = i_j$ with $\text{row}(l) = -i_j$, then up to a sign the lift of c_j exchanges e_k and e_l . Explicit formulas for the lift of c_j can be found in [2, Sect. 5.3], these can be deduced from the explicit description of centralizers given in [8, Sect. 3].

Let $j = 1, \dots, d$, below we give the action of c_j . Let $B \in \mathcal{X}_\phi^+(P)$ and let $\sigma \in W_m$ such that $B \in \text{sTab}_\phi^+(\sigma \cdot P)$. Let $\tau \in \bar{S}_m$ be the permutation

$$\tau = \bar{s}_m \bar{s}_{m-1} \cdots \bar{s}_{|\sigma(i_j)|}.$$

We consider $L_{\tau\sigma}(\tau \star B)$, which is isomorphic to $L_\sigma(B)$ by Proposition 5.1. From the formula for the lift of c_j given in [2, Sect. 5.3], we see that c_j is in the subgroup of $\tilde{G}$ isomorphic to $\tilde{G}_{2p_{\tau\sigma(1)}}^\phi$ corresponding to the direct summand $\mathfrak{g}_{2p_{\tau\sigma(1)}}^\phi$ of $\mathfrak{g}^{\mathfrak{s}_{\tau\sigma}}$. Therefore, by [2, Lemma 3.15] and [1, Theorem 6.1] we find that

$$c_j \cdot [L_{\tau\sigma}(\tau \star B)] = [L_{\tau\sigma}(c \cdot (\tau \star B))].$$

This leads us to define

$$c_j \cdot B = \tau^{-1} \star (c \cdot (\tau \star B)). \quad (5.3)$$

Then by Proposition 5.1 we obtain the following.

Proposition 5.2 *In the notation given above we have*

$$c_j \cdot [L_\sigma(B)] = [L_\sigma(\tau^{-1} \star (c \cdot (\tau \star B)))].$$

The following is an immediate consequence of Proposition 5.2.

Corollary 5.3 *The operation of the elements of $\tilde{C}(e)$ on $\mathcal{X}_\phi^+(P)$ is a $\tilde{C}(e)$ group action.*

We refer the reader to [2, Sect. 5.3] for some examples of applications of the operators c_j .

Remark 5.4 We chose i_j to be minimal for definiteness. Let i'_j be such that $p_{i'_j} = p_{i_j}$. Then there is a lift of c_j which acts in the way described above except with i'_j in place of i_j . The arguments above all go through with i'_j in place of i_j , so we could define τ with i'_j in place of i_j , and obtain an alternative formula for the action of c_j on $\mathcal{X}_\phi^+(P)$ to that in (5.3).

5.6 The classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules

Now that we have described the component group action we can state classification of finite dimensional irreducible $U(\mathfrak{g}, e)$ -modules with integral central character from [2].

Theorem 5.5 [2, Theorem 5.13] *Let $A \in \text{sTab}_\phi^{\leq}(P)$. Then the $U(\mathfrak{g}, e)$ -module $L(A)$ is finite dimensional if and only if A is $\tilde{C}(e)$ -conjugate to a table that is justified row equivalent to column strict.*

Remark 5.6 In case all parts of $\mathbf{p}$ have the same parity, then P is justified, thus there is a natural notion of $A \in \text{sTab}_\phi^{\leq}(P)$ being row equivalent to column strict as an s-table. By [2, Lemma 4.13] this is equivalent to being row equivalent to column strict in the not skew-symmetric sense.

5.7 The restricted Weyl group

In this subsection we explain how to change highest weight theories using elements of the restricted Weyl group $\tilde{W}^e$ as in Sect. 3.2.

First we recall the structure of the restricted Weyl group $\tilde{W}^e$ from [3, Sects. 4, 6 and 7]. For $i = 1, \dots, m$, we let $\bar{m}_i$ be the multiplicity of i in $\mathbf{p}$. Then $\tilde{W}^e$ is the subgroup of W_m consisting of permutations of $\{\pm 1, \dots, \pm m\}$ that permute numbers labelling rows in P of equal length; so $W^e \cong W_{\bar{m}_1} \times \dots \times W_{\bar{m}_m}$. We note that in [3] only the group W^e is considered, but it is straightforward to deduce our assertions for $\tilde{W}^e$. For $k = 1, \dots, m$ we let $r_k = (k, -k) \in \tilde{W}^e$.

Recall the subgroups Z^e and $(\tilde{W}^e)^\circ$ of $\tilde{W}^e$ from Sect. 3.2. As explained in [3, Sect. 4], we have Z^e is isomorphic to $\mathbb{Z}_2^d$, where as before d is the number of ϕ -odd parts of $\mathbf{p}$. Let $i_1 < \dots < i_d$ be as in Sect. 5.5, then one can easily calculate that Z^e is generated by the elements $r_{i_j} \in \tilde{W}^e$. Also we see that $(\tilde{W}^e)^\circ$ is the subgroup $W'_{\bar{m}_1} \times \dots \times W'_{\bar{m}_m}$ where $W'_{\bar{m}_i}$ is the Weyl group of type $D_{\bar{m}_i}$ if i is ϕ -odd and $W'_{\bar{m}_i} = W_{\bar{m}_i}$ if i is ϕ -even.

Let $k \in \{1, \dots, m\}$. We can write $r_k = z_k v_k$, where $z_k \in Z^e$ and $v_k \in (\tilde{W}^e)^\circ$ as in Sect. 3.2. If p_k is ϕ -odd, then we let j be such that $p_k = p_{i_j}$ and we see that $z_k = r_{i_j}$.

Further, recalling the maps ι and κ from Sect. 3.2, we find that $\iota(z_k) = \kappa(c_j)$. If p_k is ϕ -even, then we see that $z_k = 1$.

All the assertions above can be verified with the explicit descriptions of centralizers given in [8, Sect. 3].

Let $\sigma \in W_m$ and $B \in \text{sTab}_\phi^+(\sigma \cdot P)$. By applying Proposition 3.3, where we consider $r_k \in \tilde{W}^e$ acting on $L_\sigma(B)$, we obtain

$$[L_\sigma(B)] = \begin{cases} [L_{r_{\sigma^{-1}(k)}\sigma}(c_j \cdot B)] & \text{if } p_k \text{ is } \phi\text{-odd, where } j \text{ is such that } p_k = p_{i_j}; \\ [L_{r_{\sigma^{-1}(k)}\sigma}(B)] & \text{if } p_k \text{ is } \phi\text{-even.} \end{cases} \quad (5.4)$$

5.8 Changing highest weight theories for $U(\mathfrak{g}, e)$

We are now in a position to explain how to change highest weight theories in general. To do this we extend the action of $\bar{S}_m$ on $\mathcal{X}_\phi^+(P)$ to an action of W_m . The important step in doing this is to define the action of $r = (m, -m) \in W_m$. This can be done in terms of the restricted Weyl group as in the previous subsection.

Let $\sigma \in W_m$ and let $B \in \text{sTab}_\phi^+(\sigma \cdot P)$. By (5.4) for $k = \sigma(m)$ we have

$$[L_\sigma(B)] = \begin{cases} L_{r\sigma}(c_j \cdot B) & \text{if } p_k \text{ is } \phi\text{-odd, where } j \text{ is such that } p_k = p_{i_j}; \\ L_{r\sigma}(B) & \text{if } p_k \text{ is } \phi\text{-even.} \end{cases}$$

Using Remark 5.4 we see that in both cases this says that

$$[L_\sigma(B)] = [L_{r\sigma}(c \cdot B)],$$

where the operation of c is defined in Sect. 5.5. Thus we define the star action of r on $\mathcal{X}_\phi^+$ by

$$r \star B = c \cdot B.$$

We can now extend the $\star$ -actions of $\bar{S}_m$ and r on $\mathcal{X}^+$ to a $\star$ -action of W_m similarly to the type A case as in (4.3). Then we get the following analogue of Theorem 4.3.

Theorem 5.7

- (i) *The $\star$ -action of W_m on $\mathcal{X}_\phi^+(P)$ is well defined.*
- (ii) *Let $\sigma, \tau \in W_m$, and $\bar{B} \in \text{sTab}_\phi^+(\sigma \cdot F)$, $\bar{B}' \in \text{sTab}_\phi^+(\tau \sigma \cdot P)$. Then $L_\sigma(\bar{B}) \cong L_{\tau\sigma}(\bar{B}')$ if and only if $B' = \tau \star \bar{B}$.*

We demonstrate Theorem 5.7 with an example for $\mathfrak{g} = \mathfrak{sp}_{10}$. We take A as in (5.2) and $\tau = (1, -2)(2, -1) = r\bar{s}_1r$. To calculate $\tau \star A$ we first calculate $r \star A$. Since the length of row 1 of A is ϕ -even, we see that $r \star A = A$. Next we calculate $\bar{s}_1 \star A$ using

the row swapping operation and we get

$$\bar{s}_1 \star A = \begin{array}{|c|c|c|} \hline -3 & -2 & 7 \\ \hline 1 & 4 & \\ \hline -4 & -1 & \\ \hline -7 & 2 & 3 \\ \hline \end{array}$$

To finish off we have to apply $r \star$ to $\bar{s}_1 \star A$, which means applying the operation of c and gives

$$\tau \star A = \begin{array}{|c|c|c|} \hline -3 & -2 & 7 \\ \hline -4 & 1 & \\ \hline -1 & 4 & \\ \hline -7 & 2 & 3 \\ \hline \end{array}$$

Remark 5.8 In case all parts of $\mathbf{p}$ are equal we say that e is a rectangular nilpotent element. In this case the classification of finite dimensional $U(\mathfrak{g}, e)$ -modules given in [1] does not have the restriction to integral central characters. If $\mathbf{p}$ has an even number of parts it is easy to see that Theorem 5.7 holds in this case without the restriction to integral central characters. We note that in this case the action of each $\bar{s}_i$ is trivial, as explained in Remark 4.6, and the action of r is given by c . This is explained by the fact that all possible choices of parabolic subalgebras can be attained using the action of $\tilde{W}^e$.

When $\mathbf{p}$ has an odd number of parts, it is still the case that all possible choices of parabolic subalgebras can be attained using the action of $\tilde{W}^e$. Therefore, as the component group is trivial in this case, we see that all changes of highest weight theory are given by a trivial action on s-tables.

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