

# $(q, t)$ -Deformations of multivariate hook product formulae

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**Abstract** We generalize multivariate hook product formulae for  $P$ -partitions. We use Macdonald symmetric functions to prove a  $(q, t)$ -deformation of Gansner’s hook product formula for the generating functions of reverse (shifted) plane partitions. (The unshifted case has also been proved by Adachi.) For a  $d$ -complete poset, we present a conjectural  $(q, t)$ -deformation of Peterson–Proctor’s hook product formula.

**Keywords** Hook product formula · Reverse plane partition · Macdonald symmetric functions ·  $P$ -partition ·  $d$ -complete poset

## 1 Introduction

R. Stanley [14] introduced the notion of  $P$ -partitions for a poset  $P$ , and studied univariate generating functions of them. A  $P$ -partition is an order-reversing map from  $P$  to the set of non-negative integers  $\mathbb{N}$ , i.e., a map  $\sigma : P \rightarrow \mathbb{N}$  satisfying the condition:

$$\text{if } x \leq y \text{ in } P, \text{ then } \sigma(x) \geq \sigma(y).$$

Let  $\mathcal{A}(P)$  denote the set of all  $P$ -partitions.

Typical examples of  $P$ -partitions are reverse plane partitions and reverse shifted plane partitions. If  $\lambda$  is a partition, then its diagram

$$D(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\}$$

can be viewed as a poset by defining  $(i, j) \geq (k, l)$  if  $i \leq k$  and  $j \leq l$ , and the resulting poset is called a *shape*. A  $P$ -partition for this poset  $P = D(\lambda)$  is a *reverse plane*

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partition of shape  $\lambda$ , which is an array of non-negative integers

$$\begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,\lambda_1} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,\lambda_2} \\ \vdots & \vdots & & \\ \pi_{r,1} & \pi_{r,2} & \cdots & \pi_{r,\lambda_r} \end{matrix}$$

satisfying

$$\pi_{i,j} \leq \pi_{i,j+1}, \quad \pi_{i,j} \leq \pi_{i+1,j}$$

whenever both sides are defined. If  $\mu$  is a strict partition, then its shifted diagram

$$S(\mu) = \{(i, j) \in \mathbb{Z}^2 : i \leq j \leq \mu_i + i - 1\}$$

is also a poset called a *shifted shape*, and a  $S(\mu)$ -partition is a *reverse shifted plane partition* of shifted shape  $\mu$ , which is an array of non-negative integers

$$\begin{matrix} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \cdots & \sigma_{1,\mu_1} \\ & \sigma_{2,2} & \sigma_{2,3} & \cdots & \sigma_{2,\mu_2+1} \\ & & \ddots & & \\ & & & \sigma_{r,r} & \cdots & \sigma_{r,\mu_r+r-1} \end{matrix}$$

satisfying

$$\sigma_{i,j} \leq \sigma_{i,j+1}, \quad \sigma_{i,j} \leq \sigma_{i+1,j}$$

whenever both sides are defined.

E. Gansner [4] considered multivariate (trace) generating functions for reverse (shifted) plane partitions. Let  $P$  be a shape or a shifted shape. To each  $P$ -partition  $\sigma$ , we associate a monomial defined by

$$z^{\text{tr}(\sigma)} = \prod_{(i,j) \in P} z_{j-i}^{\sigma(i,j)},$$

where  $z_k$  ( $k \in \mathbb{Z}$ ) are indeterminates. This weights  $\sigma$  by the sums of its diagonals. To state Gansner’s hook product formulae, we introduce the notion of hooks for shapes and shifted shapes. For a partition  $\lambda$  and a cell  $(i, j) \in D(\lambda)$ , the *hook* at  $(i, j)$  in  $D(\lambda)$  is defined by

$$\begin{aligned} H_{D(\lambda)}(i, j) &= \{(i, j)\} \cup \{(i, l) \in D(\lambda) : l > j\} \\ &\cup \{(k, j) \in D(\lambda) : k > i\}. \end{aligned} \tag{1}$$

For a strict partition  $\mu$  and a cell  $(i, j) \in S(\mu)$ , the *shifted hook* at  $(i, j)$  in  $S(\mu)$  is given by

$$\begin{aligned}
 H_{S(\mu)}(i, j) &= \{(i, j)\} \cup \{(i, l) \in S(\mu) : l > j\} \\
 &\cup \{(k, j) \in S(\mu) : k > i\} \\
 &\cup \{(j + 1, l) \in S(\mu) : l > j\}.
 \end{aligned}
 \tag{2}$$

For a finite subset  $H \subset \mathbb{Z}^2$ , we write

$$z[H] = \prod_{(i,j) \in H} z_{j-i}.
 \tag{3}$$

Gansner [4] used the Hillman–Grassl correspondence to prove the following hook product formulae. (See also [13].)

**Theorem 1.1** ([4, Theorems 5.1 and 7.1]) *Let  $P$  be a shape  $D(\lambda)$  or a shifted shape  $S(\mu)$ . Then the multivariate generating function for  $\mathcal{A}(P)$  is given by*

$$\sum_{\sigma \in \mathcal{A}(P)} z^{\text{tr}(\sigma)} = \prod_{v \in P} \frac{1}{1 - z[H_P(v)]}.
 \tag{4}$$

The first aim of this paper is to prove a  $(q, t)$ -deformation of Gansner’s hook product formulae. Let  $q$  and  $t$  be indeterminates and put

$$f_{q,t}(n; m) = \prod_{i=0}^{n-1} \frac{1 - q^i t^{m+1}}{1 - q^{i+1} t^m},
 \tag{5}$$

for non-negative integers  $n$  and  $m$ . And we use the notation

$$F(x; q, t) = \frac{(tx; q)_\infty}{(x; q)_\infty},
 \tag{6}$$

where  $(a; q)_\infty = \prod_{i \geq 0} (1 - aq^i)$ . If we take  $q = t$ , then  $f_{q,q}(n; m) = 1$  and  $F(x; q, q) = 1/(1 - x)$ . Our main theorem is the following:

**Theorem 1.2** (a) *Let  $\lambda$  be a partition. We define a weight  $W_{D(\lambda)}(\pi; q, t)$  of a reverse plane partition  $\pi \in \mathcal{A}(D(\lambda))$  by putting*

$$\begin{aligned}
 &W_{D(\lambda)}(\pi; q, t) \\
 &= \prod_{(i,j) \in D(\lambda)} \prod_{m \geq 0} \frac{f_{q,t}(\pi_{i,j} - \pi_{i-m,j-m-1}; m) f_{q,t}(\pi_{i,j} - \pi_{i-m-1,j-m}; m)}{f_{q,t}(\pi_{i,j} - \pi_{i-m,j-m}; m) f_{q,t}(\pi_{i,j} - \pi_{i-m-1,j-m-1}; m)},
 \end{aligned}
 \tag{7}$$

where we use the convention that  $\pi_{k,l} = 0$  if  $k < 0$  or  $l < 0$ . Then we have

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} W_{D(\lambda)}(\pi; q, t) z^{\text{tr}(\pi)} = \prod_{v \in D(\lambda)} F(z[H_{D(\lambda)}(v)]; q, t).
 \tag{8}$$

(b) Let  $\mu$  be a strict partition. We define a weight  $W_{S(\mu)}(\sigma; q, t)$  of a reverse shifted plane partition  $\sigma \in \mathcal{A}(S(\mu))$  by putting

$$\begin{aligned}
 &W_{S(\mu)}(\sigma; q, t) \\
 &= \prod_{\substack{(i,j) \in S(\mu) \\ i < j}} \prod_{m \geq 0} \frac{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m-1}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m}; m)}{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m-1}, m)} \\
 &\quad \times \prod_{(i,i) \in S(\mu)} \prod_{m \geq 0} \frac{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-1,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-1}; 2m+1)}{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-2}; 2m+1)}, \tag{9}
 \end{aligned}$$

where  $\sigma_{k,l} = 0$  if  $k < 0$ . Then we have

$$\sum_{\sigma \in \mathcal{A}(S(\mu))} W_{S(\mu)}(\sigma; q, t) z^{\text{tr}(\sigma)} = \prod_{v \in S(\mu)} F(z[H_{S(\mu)}(v)]; q, t). \tag{10}$$

Note that there are only finitely many terms different from 1 in the products in (7) and (9). If we put  $q = t$ , then  $f_{q,q}(n; m) = 1$  and all weights  $W_P(\sigma; q, q)$  are equal to 1, so Theorem 1.2 reduces to Gansner’s hook product formula (Theorem 1.1).

S. Adachi [1] proves formula (8) by generalizing the argument of [15]. We give a similar, but more transparent proof. If  $\lambda$  is a rectangular partition ( $r^c$ ), then reverse plane partitions of shape ( $r^c$ ) can be viewed as plane partitions by rotating 180°, and formula (8) gives Vuletić’s generalization of MacMahon’s formula [15, Theorem A]. O. Foda et al. [2, 3] use fermion calculus to re-derive the Schur ( $q = t$ ) and Hall–Littlewood ( $q = 0$ ) case of Vuletić’s generalization.

Gansner’s hook product formulae are generalized to other posets than shapes and shifted shapes. R. Proctor [10, 11] introduced a wide class of posets, called *d-complete posets*, enjoying “hook-length property” and “jeu-de-taquin property” as a generalization of shapes and shifted shapes. And he announced [12] that, in collaboration with D. Peterson, he obtained the hook product formula for *d*-complete posets, but their formulation and proof are still unpublished. K. Nakada [7] gives a purely algebraic proof of a hook product formula equivalent to Peterson–Proctor’s formula. The second aim of this paper is to present a conjectural ( $q, t$ )-deformation of Peterson–Proctor’s multivariate hook formula.

This paper is organized as follows. In Section 2, we use the theory of Macdonald symmetric functions to give a generating function for reverse shifted plane partitions with prescribed shape and profile. Section 3 is devoted to the proof of our main theorem by using the result in Section 2. In Section 4, we consider *d*-complete posets and give a conjectural ( $q, t$ )-deformation of Peterson–Proctor’s hook formula.

## 2 Reverse shifted plane partitions of given shape and profile

In this section, we give a generating function for reverse shifted plane partitions of given shape and profile, from which our main theorem (Theorem 1.2) follows. Our approach is based on an observation made in A. Okounkov et al. [8, 9].

Let  $\mu$  be a strict partition of length  $r$  and  $\sigma$  a reverse shifted plane partition of shape  $\mu$ . We put

$$\sigma[0] = (\sigma_{r,r}, \sigma_{r-1,r-1}, \dots, \sigma_{2,2}, \sigma_{1,1})$$

and call it the *profile* of  $\sigma$ . And we associate to  $\sigma$  a weight  $V_{S(\mu)}(\sigma; q, t)$  defined by

$$\begin{aligned} &V_{S(\mu)}(\sigma; q, t) \\ &= \prod_{\substack{(i,j) \in S(\mu) \\ i < j}} \prod_{m \geq 0} \frac{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m-1}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m}; m)}{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m-1}, m)} \\ &\times \prod_{(i,i) \in S(\mu)} \prod_{m \geq 0} \frac{f_{q,t}(\sigma_{i,i} - \sigma_{i-m-1,i-m}; m)}{f_{q,t}(\sigma_{i,i} - \sigma_{i-m,i-m}; m)}. \end{aligned} \tag{11}$$

Here we use the convention that  $\sigma_{k,l} = 0$  if  $k \leq 0$  or  $k > l$ , so only finitely many terms in the product over  $m$  are different from 1. For a partition  $\tau$ , we denote by  $\mathcal{A}(S(\mu), \tau)$  the set of reverse shifted plane partitions of shape  $\mu$  and profile  $\tau$ .

**Theorem 2.1** *Given strict partition  $\mu$  of length  $r$ , let  $N$  be such that  $N \geq \mu_1$ . Let  $\mu^c$  be the complement of  $\mu$  in  $[N] = \{1, 2, \dots, N\}$ , i.e.,  $\{\mu_1, \dots, \mu_r\} \cup \{\mu_1^c, \mu_2^c, \dots, \mu_{N-r}^c\} = [N]$ . Let  $\tau$  be a partition of length  $\leq r$ . Then we have*

$$\sum_{\sigma \in \mathcal{A}(S(\mu), \tau)} V_{S(\mu)}(\sigma; q, t) z^{\text{tr}(\sigma)} = \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \cdot Q_\tau(\tilde{z}_{\mu_1}, \dots, \tilde{z}_{\mu_r}; q, t),$$

where the product is taken over all pairs  $(k, l)$  satisfying  $\mu_k^c < \mu_l$ , and

$$\tilde{z}_0 = 1, \quad \tilde{z}_k = z_0 z_1 \cdots z_{k-1} \quad (k \geq 1).$$

And  $Q_\tau(x; q, t)$  is the Macdonald symmetric function.

Here we recall the definition of Macdonald symmetric functions. (See [6, Chap. VI] for details.) Let  $\Lambda$  be the ring of symmetric functions in  $x = (x_1, x_2, \dots)$  with coefficients in a field of characteristic 0, which contains  $q, t$  and formal power series in  $z_k$ 's. Define a bilinear form on  $\Lambda$  by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$

where  $p_\lambda$  is the power-sum symmetric function. Then [6, Chap. VI, (4.7)] there exists a unique family of symmetric functions  $P_\lambda(x; q, t) \in \Lambda$ , indexed by partitions, satisfying the following two conditions:

- (i)  $P_\lambda$  is a linear combination of monomial symmetric functions of the form

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda, \mu} m_\mu,$$

where  $<$  denotes the dominance order on partitions.

- (ii) If  $\lambda \neq \mu$ , then  $\langle P_\lambda, P_\mu \rangle = 0$ .

Then the  $P_\lambda$ 's form a basis of  $\Lambda$ . The Macdonald  $Q$  functions  $Q_\lambda(x; q, t)$  are the dual basis of  $P_\lambda$ , i.e.,

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda, \mu}.$$

We use operator calculus on the ring of symmetric functions  $\Lambda$  to show Theorem 2.1. We denote by  $g_n$  the Macdonald symmetric function  $Q_{(n)}$  corresponding to a one-row partition  $(n)$ . Let  $g_n^+$  and  $g_n^-$  be the multiplication and skewing operators on  $\Lambda$  associated to  $g_n$ , respectively. They satisfy

$$g_n^+(h) = g_n h, \quad \langle g_n^-(h), f \rangle = \langle h, g_n f \rangle$$

for any  $f, h \in \Lambda$ . We consider the generating functions

$$G^+(u) = \sum_{n=0}^\infty g_n^+ u^n, \quad G^-(u) = \sum_{n=0}^\infty g_n^- u^n.$$

Also we introduce the degree operator  $D(y)$  defined by

$$D(y)P_\lambda = y^{|\lambda|} P_\lambda.$$

For a strict partition  $\mu$  and a partition  $\tau$ , we put

$$R_{S(\mu), \tau}(z; q, t) = \sum_{\sigma \in \mathcal{A}(S(\mu), \tau)} V_{S(\mu)}(\sigma; q, t) z^{\text{tr}(\sigma)}.$$

Fix a positive integer  $N$  satisfying  $N \geq \mu_1$  and define a sequence  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$  of  $+$  and  $-$  by putting

$$\varepsilon_k = \begin{cases} + & \text{if } k \text{ is a part of } \mu, \\ - & \text{if } k \text{ is not a part of } \mu. \end{cases}$$

The first step of the proof of Theorem 2.1 is the following lemma.

**Lemma 2.2** *In the ring of symmetric functions  $\Lambda$ , we have*

$$\begin{aligned} & \sum_{\tau} R_{S(\mu), \tau}(z; q, t) P_{\tau}(x; q, t) \\ &= D(z_0)G^{\varepsilon_1}(1)D(z_1)G^{\varepsilon_2}(1)D(z_2)G^{\varepsilon_3}(1) \cdots G^{\varepsilon_{N-1}}(1)D(z_{N-1})G^{\varepsilon_N}(1)1, \end{aligned}$$

where  $\tau$  runs over all partitions of length at most the length of  $\mu$ .

For example, if  $\mu = (6, 5, 2)$  and  $N = 6$ , then  $\varepsilon = (-, +, -, -, +, +)$  and

$$\sum_{\tau} R_{S(\mu),\tau}(z; q, t) P_{\tau}(x; q, t) = D(z_0)G^-(1)D(z_1)G^+(1)D(z_2)G^-(1)D(z_3)G^-(1)D(z_4)G^+(1)D(z_5)G^+(1)1.$$

*Proof* For a map  $\sigma : S(\mu) \rightarrow \mathbb{N}$  (or an array of non-negative integers of shape  $\mu$ ) and an integer  $k$  ( $0 \leq k \leq N$ ), we define the  $k$ th trace  $\sigma[k]$  to be the sequence  $(\dots, \sigma_{2,k+2}, \sigma_{1,k+1})$  obtained by reading the  $k$ th diagonal from SE to NW. For example, a reverse shifted plane partition

$$\sigma = \begin{array}{cccccc} & & & 0 & 0 & 1 & 2 & 3 & 3 \\ & & & & & 1 & 2 & 3 & 3 & 3 \\ & & & & & & & & & 2 & 4 \end{array}$$

has traces

$$\sigma[0] = (2, 1, 0), \quad \sigma[1] = (4, 2, 0), \quad \sigma[2] = (3, 1), \quad \sigma[3] = (3, 2), \\ \sigma[4] = (3, 3), \quad \sigma[5] = (3), \quad \sigma[6] = \emptyset.$$

For two partitions  $\alpha$  and  $\beta$ , we write  $\alpha \succ \beta$  if  $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$ , i.e., the skew diagram  $\alpha/\beta$  is a horizontal strip. Then it is clear from definition that a map  $\sigma : S(\mu) \rightarrow \mathbb{N}$  is a shifted reverse plane partition if and only if each  $\sigma[k]$  ( $0 \leq k \leq N$ ) is a partition and

$$\begin{cases} \sigma[k-1] \succ \sigma[k] & \text{if } \varepsilon_k = +, \\ \sigma[k-1] \prec \sigma[k] & \text{if } \varepsilon_k = -. \end{cases}$$

In the above example, we have

$$\sigma[0] \prec \sigma[1] \succ \sigma[2] \prec \sigma[3] \prec \sigma[4] \succ \sigma[5] \succ \sigma[6].$$

A key ingredient is the Pieri rule for Macdonald symmetric functions. For two partitions  $\alpha$  and  $\beta$  satisfying  $\alpha \succ \beta$ , we put

$$\varphi_{\alpha,\beta}^+(q, t) = \prod_{i \leq j} \frac{f_{q,t}(\alpha_i - \beta_j; j - i) f_{q,t}(\beta_i - \alpha_{j+1}; j - i)}{f_{q,t}(\alpha_i - \alpha_j; j - i) f_{q,t}(\beta_i - \beta_{j+1}; j - i)}, \\ \varphi_{\beta,\alpha}^-(q, t) = \prod_{i \leq j} \frac{f_{q,t}(\alpha_i - \beta_j; j - i) f_{q,t}(\beta_i - \alpha_{j+1}; j - i)}{f_{q,t}(\alpha_i - \alpha_{j+1}; j - i) f_{q,t}(\beta_i - \beta_j; j - i)}. \tag{12}$$

Then the Pieri rule [6, Chap. VI, (6.24)] can be stated as follows:

$$G^+(u)P_{\beta} = \sum_{\alpha \succ \beta} \varphi_{\alpha,\beta}^+(q, t) u^{|\alpha|-|\beta|} P_{\alpha}, \\ G^-(u)P_{\alpha} = \sum_{\beta \prec \alpha} \varphi_{\beta,\alpha}^-(q, t) u^{|\alpha|-|\beta|} P_{\beta}, \tag{13}$$

where  $\alpha$  in the first summation (resp.,  $\beta$  in the second summation) runs over all partition satisfying  $\alpha \succ \beta$  (resp.,  $\beta \prec \alpha$ ). (In [6, Chap. VI, (6.24)], the coefficients  $\varphi_{\alpha,\beta}^+ = \varphi_{\alpha,\beta}$  and  $\varphi_{\beta,\alpha}^- = \psi_{\alpha,\beta}$  are given in terms of arm and leg lengths, but it is not hard to rewrite them in the form (13) by using  $f_{q,t}(m, n)$  defined by (5). See also [6, Chap. VI, 6, Ex. 2].) And the weight function  $V_{S(\mu)}(\sigma; q, t)$  is expressed in terms of  $\varphi_{\alpha,\beta}^\pm(q, t)$  as

$$V_{S(\mu)}(\sigma; q, t) = \prod_{k=1}^N \varphi_{\sigma[k-1], \sigma[k]}^{\varepsilon_k}(q, t).$$

Now the claim of lemma easily follows by induction on  $\mu_1$ . □

The second step is to rewrite the composite operators on the right-hand side of Lemma 2.2 by using some commutation relations.

**Lemma 2.3** *Let  $\mu^c$  be the strict partition formed by the complement of  $\mu$  in  $[N]$ , i.e.,*

$$\{\mu_1, \dots, \mu_r\} \sqcup \{\mu_1^c, \dots, \mu_{N-r}^c\} = [N].$$

*Then we have*

$$\begin{aligned} & D(z_0)G^{\varepsilon_1}(1)D(z_1)G^{\varepsilon_2}(1)D(z_2)G^{\varepsilon_2}(1) \cdots G^{\varepsilon_{N-1}}(1)D(z_{N-1})G^{\varepsilon_N}(1) \\ &= \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r G^+(\tilde{z}_{\mu_k}) \prod_{l=1}^{N-r} G^-(\tilde{z}_{\mu_l^c}) D(\tilde{z}_N), \end{aligned}$$

where the first product is taken over all pairs  $(k, l)$  satisfying  $\mu_k^c < \mu_l$ .

*Proof* By the Pieri rule (14), we have

$$\begin{aligned} D(z) \circ G^+(u) &= G^+(zu) \circ D(z), \\ D(z) \circ G^-(u) &= G^-(z^{-1}u) \circ D(z). \end{aligned}$$

Also we have

$$D(z) \circ D(z') = D(zz').$$

By using these relations, we move  $D(z_i)$ 's to the right and see that

$$\begin{aligned} & D(z_0)G^{\varepsilon_1}(1)D(z_1)G^{\varepsilon_2}(1)D(z_2)G^{\varepsilon_2}(1) \cdots G^{\varepsilon_{N-1}}(1)D(z_{N-1})G^{\varepsilon_N}(1) \\ &= G^{\varepsilon_1}(\tilde{z}_1^{\varepsilon_1})G^{\varepsilon_2}(\tilde{z}_2^{\varepsilon_2}) \cdots G^{\varepsilon_N}(\tilde{z}_N^{\varepsilon_N})D(\tilde{z}_N), \end{aligned}$$

where  $z^+ = z$  and  $z^- = z^{-1}$ .

By the same argument as in [6, Chap. III, 5, Ex. 8] for Hall–Littlewood functions, we can show that

$$G^-(u) \circ G^+(v) = F(uv; q, t)G^+(v) \circ G^-(u),$$



where  $F(x; q, t)$  is defined by (6). (Details are left to the reader.) It follows from this commutation relation that

$$G^{\varepsilon_1}(\tilde{z}_1^{\varepsilon_1})G^{\varepsilon_2}(\tilde{z}_2^{\varepsilon_2}) \cdots G^{\varepsilon_N}(\tilde{z}_N^{\varepsilon_N}) = \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r G^+(\tilde{z}_{\mu_k}) \prod_{l=1}^{N-r} G^-(\tilde{z}_{\mu_l^c}). \quad \square$$

Now we are in position to finish the proof of Theorem 2.1.

*Proof of Theorem 2.1* It follows from Lemmas 2.2 and 2.3 that

$$\begin{aligned} & \sum_{\tau} R_{S(\mu), \tau}(\mathbf{z}; q, t) P_{\tau}(x; q, t) \\ &= \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r G^+(\tilde{z}_{\mu_k}) \prod_{l=1}^{N-r} G^-(\tilde{z}_{\mu_l^c}) D(\tilde{z}_N) 1. \end{aligned}$$

By definition, we have  $D(z)1 = 1$  and  $G^-(u)1 = 1$ , so we see that

$$\sum_{\tau} R_{S(\mu), \tau}(\mathbf{z}; q, t) P_{\tau}(x; q, t) = \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r G^+(\tilde{z}_{\mu_k}) 1.$$

Since the generating function of  $g_n(x; q, t)$  where  $x = (x_1, x_2, \dots)$  is given by (see [6, Chap. VI, (2.8)])

$$\sum_{n=0}^{\infty} g_n(x; q, t) u^n = \prod_i F(x_i u; q, t),$$

we have

$$\sum_{\tau} R_{S(\mu), \tau}(\mathbf{z}; q, t) P_{\tau}(x; q, t) = \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r \prod_i F(x_i \tilde{z}_{\mu_k}; q, t).$$

It follows from the Cauchy identity [6, Chap. VI, (4.13)] that

$$\prod_{k=1}^r \prod_i F(x_i \tilde{z}_{\mu_k}; q, t) = \sum_{\tau} Q_{\tau}(\tilde{z}_{\mu_1}, \dots, \tilde{z}_{\mu_r}; q, t) P_{\tau}(x; q, t).$$

Hence we see that

$$\begin{aligned} & \sum_{\tau} R_{S(\mu), \tau}(\mathbf{z}; q, t) P_{\tau}(x; q, t) \\ &= \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}; q, t) \sum_{\tau} Q_{\tau}(\tilde{z}_{\mu_1}, \dots, \tilde{z}_{\mu_r}; q, t) P_{\tau}(x; q, t). \end{aligned}$$

Equating the coefficients of  $P_{\tau}(x; q, t)$  completes the proof. □

### 3 Proof of Theorem 1.2

In this section, we derive Theorem 1.2 from Theorem 2.1. If we put

$$b_\tau(q, t) = \prod_{i \leq j} \frac{f_{q,t}(\tau_i - \tau_{j+1}; j - i)}{f_{q,t}(\tau_i - \tau_j; j - i)},$$

then we have (see [6, Chap. VI, (4.12) and (6.19)])

$$Q_\tau(x; q, t) = b_\tau(q, t) P_\tau(x; q, t).$$

*Proof of Theorem 1.2(b)* First, we prove our  $(q, t)$ -deformation for shifted shapes. By comparing (9) and (11), for a reverse shifted plane partition  $\sigma$  of shifted shape  $\mu$ , we have

$$W_{S(\mu)}(\sigma; q, t) = \frac{b_\tau^{\text{el}}(q, t)}{b_\tau(q, t)} V_{S(\mu)}(\sigma; q, t),$$

where  $\tau = \sigma[0]$  is the profile of  $\sigma$  and

$$b_\tau^{\text{el}}(q, t) = \prod_{\substack{i \leq j \\ j - i \text{ is even}}} \frac{f_{q,t}(\tau_i - \tau_{j+1}; j - i)}{f_{q,t}(\tau_i - \tau_j; j - i)},$$

with the product taken over all  $i$  and  $j$  such that  $i \leq j$  and  $j - i$  is even. Hence it follows from Theorem 2.1 that

$$\begin{aligned} & \sum_{\sigma \in \mathcal{A}(S(\mu))} W_{S(\mu)}(\sigma; q, t) \mathbf{z}^{\text{tr}(\sigma)} \\ &= \sum_{\tau} \sum_{\sigma \in \mathcal{A}(S(\mu), \tau)} \frac{b_\tau^{\text{el}}(q, t)}{b_\tau(q, t)} V_{S(\mu)}(\sigma; q, t) \mathbf{z}^{\text{tr}(\sigma)} \\ &= \sum_{\tau} \frac{b_\tau^{\text{el}}(q, t)}{b_\tau(q, t)} \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k}^{-1} \tilde{z}_{\mu_l}; q, t) Q_\tau(\tilde{z}_{\mu_1}, \dots, \tilde{z}_{\mu_r}; q, t). \end{aligned}$$

By applying the Schur–Littlewood type formula [6, Chap. VI, 7, Ex. 4 (ii)]

$$\sum_{\tau} \frac{b_\tau^{\text{el}}(q, t)}{b_\tau(q, t)} Q_\tau(x; q, t) = \prod_i F(x_i; q, t) \prod_{i < j} F(x_i x_j; q, t),$$

we see that

$$\begin{aligned} & \sum_{\sigma \in \mathcal{A}(S(\mu))} W_{S(\mu)}(\sigma; q, t) \mathbf{z}^{\text{tr}(\sigma)} \\ &= \prod_{\mu_k^c < \mu_l} F(\tilde{z}_{\mu_k}^{-1} \tilde{z}_{\mu_l}; q, t) \prod_{k=1}^r F(\tilde{z}_{\mu_k}; q, t) \prod_{1 \leq k < l \leq r} F(\tilde{z}_{\mu_k} \tilde{z}_{\mu_l}; q, t). \end{aligned}$$

On the other hand, we can compute the monomial  $z[H_{S(\mu)}(i, j)]$  associated to the shifted hook  $H_{S(\mu)}(i, j)$  at  $(i, j) \in S(\mu)$ . By using the fact that the length of column  $j$  ( $j > r$ ) of the shifted diagram  $S(\mu)$  is equal to  $j - \mu_{N-j+1}^c$ , we have

$$z[H_{S(\mu)}(i, j)] = \begin{cases} \tilde{z}_{\mu_i} \tilde{z}_{\mu_{j+1}} & \text{if } i \leq j < r, \\ \tilde{z}_{\mu_i} & \text{if } i \leq j = r, \\ \tilde{z}_{\mu_{N-j+1}^c}^{-1} \tilde{z}_{\mu_i} & \text{if } i \leq r < j. \end{cases}$$

Noticing that  $(i, j) \in S(\mu)$  if and only if  $\mu_i > \mu_{N-j+1}^c$  for  $1 \leq i \leq r < j \leq N$ , the desired product has been obtained. □

*Proof of Theorem 1.2(a)* Next we show the  $(q, t)$ -deformation for shapes. For a given partition  $\lambda$ , let  $r = \#\{i : \lambda_i \geq i\}$  be the number of cells on the main diagonal of  $D(\lambda)$  and define two strict partitions  $\mu = (\mu_1, \dots, \mu_r)$  and  $\nu = (\nu_1, \dots, \nu_r)$  by

$$\mu_i = \lambda_i - i + 1, \quad \nu_i = {}^t\lambda_i - i + 1 \quad (1 \leq i \leq r),$$

where  ${}^t\lambda$  is the conjugate partition of  $\lambda$ . Also we put

$$\begin{aligned} x_0 &= z_0^{1/2}, & x_k &= z_k \quad (k \geq 1), \\ y_0 &= z_0^{1/2}, & y_k &= z_{-k} \quad (k \geq 1). \end{aligned}$$

Then a reverse plane partition  $\pi \in \mathcal{A}(D(\lambda))$  is obtained by gluing two reverse shifted plane partitions  $\sigma \in \mathcal{A}(S(\mu))$  and  $\rho \in \mathcal{A}(S(\nu))$  with the same profile  $\tau = \sigma[0] = \rho[0]$ , and

$$\begin{aligned} z^{\text{tr}(\pi)} &= \mathbf{x}^{\text{tr}(\sigma)} \mathbf{y}^{\text{tr}(\rho)}, \\ W_{D(\lambda)}(\pi; q, t) &= \frac{1}{b_\tau(q, t)} V_{S(\mu)}(\sigma; q, t) V_{S(\nu)}(\rho; q, t). \end{aligned}$$

Hence it follows from Theorem 2.1 that

$$\begin{aligned} &\sum_{\pi \in \mathcal{A}(D(\lambda))} W_{D(\lambda)}(\pi; q, t) z^{\text{tr}(\pi)} \\ &= \sum_{\tau} \sum_{\substack{\sigma \in \mathcal{A}(\mu, \tau) \\ \rho \in \mathcal{A}(\nu, \tau)}} \frac{1}{b_\tau(q, t)} V_{S(\mu)}(\sigma; q, t) V_{S(\nu)}(\rho; q, t) \mathbf{x}^{\text{tr}(\sigma)} \mathbf{y}^{\text{tr}(\rho)} \\ &= \sum_{\tau} \frac{1}{b_\tau(q, t)} \prod_{\mu_k^c < \mu_l} F(\tilde{x}_{\mu_k^c}^{-1} \tilde{x}_{\mu_l}; q, t) Q_\tau(\tilde{x}_{\mu_1}, \dots, \tilde{x}_{\mu_r}; q, t) \\ &\quad \times \prod_{\nu_k^c < \nu_l} F(\tilde{y}_{\nu_k^c}^{-1} \tilde{y}_{\nu_l}; q, t) Q_\tau(\tilde{y}_{\nu_1}, \dots, \tilde{y}_{\nu_r}; q, t). \end{aligned}$$

Now applying the Cauchy identity [6, Chap. VI, (4.13)], we obtain

$$\begin{aligned} & \sum_{\pi \in \mathcal{A}(D(\lambda))} W_{D(\lambda)}(\pi; q, t) \mathbf{z}^{\text{tr}(\pi)} \\ &= \prod_{\mu_k^c < \mu_l} F(\tilde{x}_{\mu_k^c}^{-1} \tilde{x}_{\mu_l}; q, t) \prod_{\nu_k^c < \nu_l} F(\tilde{y}_{\nu_k^c}^{-1} \tilde{y}_{\nu_l}; q, t) \prod_{i,j=1}^r F(\tilde{x}_{\mu_i} \tilde{y}_{\nu_j}; q, t). \end{aligned}$$

On the other hand, the monomial  $\mathbf{z}[H_{D(\lambda)}(i, j)]$  associated to the hook  $H_{D(\lambda)}(i, j)$  at  $(i, j) \in D(\lambda)$  is given by

$$\mathbf{z}[H_{D(\lambda)}(i, j)] = \begin{cases} \tilde{x}_{\mu_i} \tilde{y}_{\nu_j} & \text{if } 1 \leq i, j \leq r, \\ \tilde{x}_{\mu_{N-j+1}^c}^{-1} \tilde{x}_{\mu_i} & \text{if } 1 \leq i \leq r < j, \\ \tilde{y}_{\nu_{N-i+1}^c}^{-1} \tilde{y}_{\nu_j} & \text{if } 1 \leq j \leq r < i. \end{cases}$$

This completes the proof of Theorem 1.2. □

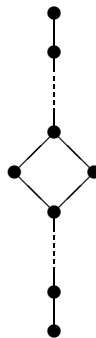
Theorem 1.2(a) can be also obtained from Theorem 2.1 by specializing  $\tau = \emptyset$ .

### 4 (q, t)-Deformation of Peterson–Proctor’s hook formula

In this section, we give a conjectural  $(q, t)$ -deformation of Peterson–Proctor’s hook formulae for  $d$ -complete posets.

First, we review the definition and some properties of  $d$ -complete posets. (See [10, 11].) For  $k \geq 3$ , we denote by  $d_k(1)$  the poset consisting of  $2k - 2$  elements with the Hasse diagram shown in Fig. 1. For example,  $d_3(1)$  is isomorphic to the shape  $D((2, 2))$ , while  $d_4(1)$  is isomorphic to the shifted shape  $S((3, 2, 1))$ . The poset  $d_k(1)$  is called the *double-tailed diamond* poset.

Let  $P$  be a poset. An interval  $[w, v] = \{x \in P : w \leq x \leq v\}$  is called a  $d_k$ -interval if it is isomorphic to  $d_k(1)$ . Then  $w$  and  $v$  are called the *bottom* and *top* of  $[w, v]$ , respectively, and the two incomparable elements of  $[w, v]$  are called the *sides*. If



**Fig. 1** Double-tailed diamond  $d_k(1)$

$k \geq 4$ , then an interval  $[w, v]$  is a  $d_k^-$ -interval if it is isomorphic to the poset obtained by removing the maximum element from  $d_k(1)$ . A  $d_3^-$ -interval  $[w; x, y]$  consists of three elements  $x, y$  and  $w$  such that  $w$  is covered by  $x$  and  $y$ . (Precisely speaking, a  $d_3^-$ -interval is not an interval, but we use this terminology after Proctor [10, 11].)

A poset  $P$  is  $d$ -complete if it satisfies the following three conditions for every  $k \geq 3$ :

- (D1) If  $I$  is a  $d_k^-$ -interval, then there exists an element  $v$  such that  $v$  covers the maximal elements of  $I$  and  $I \cup \{v\}$  is a  $d_k$ -interval.
- (D2) If  $I = [w, v]$  is a  $d_k$ -interval and the top  $v$  covers  $u$  in  $P$ , then  $u \in I$ .
- (D3) There are no  $d_k^-$ -intervals which differ only in the minimal elements.

It is clear that rooted trees, viewed as posets with their roots being the maximum elements, are  $d$ -complete posets. And it can be shown that shapes and shifted shapes are  $d$ -complete posets.

**Proposition 4.1** ([10, §3]) *Let  $P$  be a  $d$ -complete poset. Suppose that  $P$  is connected, i.e., the Hasse digram of  $P$  is connected. Then we have*

- (a)  $P$  has a unique maximal element  $v_0$ .
- (b) For each  $v \in P$ , every saturated chain from  $v$  to the maximum element  $v_0$  has the same length. Hence  $P$  admits a rank function  $r : P \rightarrow \mathbb{N}$  such that  $r(x) = r(y) + 1$  if  $x$  covers  $y$ .

Let  $P$  be a poset with a unique maximal element. The top tree  $T$  of  $P$  is the subgraph of the Hasse diagram of  $P$ , whose vertex set consists of all elements  $x \in P$  such that every  $y \geq x$  is covered by at most one other element.

**Proposition 4.2** ([11, Proposition 8.6]) *Let  $P$  be a connected  $d$ -complete poset and  $T$  its top tree. Let  $I$  be a set of colors whose cardinality is the same as  $T$ . Then a bijection  $c : T \rightarrow I$  can be uniquely extended to a map  $c : P \rightarrow I$  satisfying the following four conditions:*

- (C1) If  $x$  and  $y$  are incomparable, then  $c(x) \neq c(y)$ .
- (C2) If  $x$  covers  $y$ , then  $c(x) \neq c(y)$ .
- (C3) If an interval  $[w, v]$  is a chain, then the colors  $c(x)$  ( $x \in [w, v]$ ) are distinct.
- (C4) If  $[w, v]$  is a  $d_k$ -interval then  $c(w) = c(v)$ .

Such a map  $c : P \rightarrow I$  is called a  $d$ -complete coloring.

**Example 4.3** In the case of shapes and shifted shapes,  $d$ -complete colorings are given by contents.

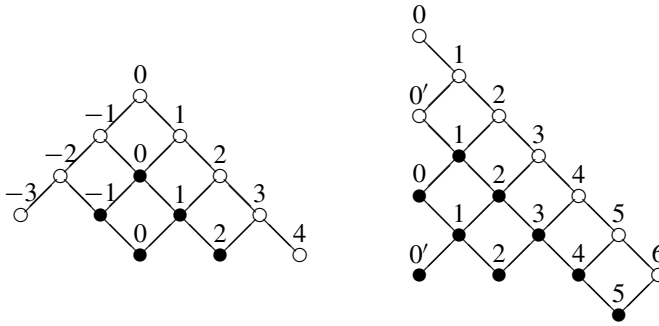
- (a) Let  $\lambda$  be a partition. Then the top tree of the shape  $D(\lambda)$  is given by

$$T = \{(i, 1) : 1 \leq i \leq \lambda_1\} \cup \{(1, j) : 1 \leq j \leq \lambda_1\}$$

and the content function  $c : D(\lambda) \rightarrow \{-\lambda_1 + 1, \dots, -1, 0, 1, \dots, \lambda_1 - 1\}$  defined by

$$c(i, j) = j - i \quad ((i, j) \in D(\lambda))$$

is a  $d$ -complete coloring.



**Fig. 2** Top trees and  $d$ -complete colorings

(b) If  $\mu$  is a strict partition with length  $\geq 2$ , then the top tree of the shifted shape  $S(\mu)$  is given by

$$T = \{(1, j) : 1 \leq j \leq \mu_1\} \cup \{(2, 2)\},$$

and a  $d$ -complete coloring  $c : S(\mu) \rightarrow \{0, 0', 1, 2, \dots, \mu_1 - 1\}$  is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd,} \\ 0' & \text{if } i = j \text{ and } i \text{ is even.} \end{cases}$$

Figure 2 illustrates the top trees and  $d$ -complete colorings of the shape  $D((5, 4, 3, 1))$  and the shifted shape  $S((7, 6, 3, 1))$ . The top tree consists of the nodes denoted by  $\circ$ .

Let  $P$  be a connected  $d$ -complete poset and  $c : P \rightarrow I$  a  $d$ -complete coloring. Let  $z_i$  ( $i \in I$ ) be indeterminates. For a  $P$ -partition  $\sigma \in \mathcal{A}(P)$ , we put

$$z^\sigma = \prod_{v \in P} z_{c(v)}^{\sigma(v)}.$$

For example, if we use the  $d$ -complete colorings given in Example 4.3, the monomial  $z^{\text{tr}(\sigma)}$  is the same as  $z^\sigma$  for a reverse plane partition  $\sigma \in \mathcal{A}(D(\lambda))$ , while for a reverse shifted plane partition  $\sigma \in \mathcal{A}(S(\mu))$ ,  $z^{\text{tr}(\sigma)}$  is obtained from  $z^\sigma$  by putting  $z_0 = z_{0'}$ . Instead of giving a definition of hooks  $H_P(v)$  for a general  $d$ -complete poset  $P$ , we define associated monomials  $z[H_P(v)]$  directly by induction as follows:

(a) If  $v$  is not the top of any  $d_k$ -interval, then we define

$$z[H_P(v)] = \prod_{w \leq v} z_{c(w)}.$$

(b) If  $v$  is the top of a  $d_k$ -interval  $[w, v]$ , then we define

$$z[H_P(v)] = \frac{z[H_P(x)] \cdot z[H_P(y)]}{z[H_P(w)]},$$

where  $x$  and  $y$  are the sides of  $[w, v]$ .

It is easy to see that for shapes this definition of  $z[H_{D(\lambda)}(v)]$  is consistent with the definition (1) and (3) given in Introduction. And, for shifted shapes, the monomial  $z[H_{S(\mu)}(v)]$  defined above reduces to the monomial  $z[H_{S(\mu)}(v)]$  defined by (2) and (3), if we put  $z_0 = z_{0'}$ .

Now we are ready to state our conjectural  $(q, t)$ -deformation of Peterson–Proctor’s hook formula. Let  $P$  be a connected  $d$ -complete poset with the maximum element  $v_0$ , and the rank function  $r : P \rightarrow \mathbb{N}$ . Let  $T$  be the top tree of  $P$ . Take  $T$  as a set of colors and let  $c : P \rightarrow T$  be the  $d$ -complete coloring such that  $c(v) = v$  for all  $v \in T$ . Given a  $P$ -partition  $\sigma \in \mathcal{A}(P)$ , we define a weight  $W_P(\sigma; q, t)$  by putting

$$\begin{aligned}
 &W_P(\sigma; q, t) \\
 &= \frac{\prod_{\substack{x, y \in P \\ \text{s.t. } x < y, c(x) \sim c(y)}} f_{q,t}(\sigma(x) - \sigma(y); d(x, y)) \prod_{\substack{x \in P \\ \text{s.t. } c(x) = v_0}} f_{q,t}(\sigma(x); e(x, v_0))}{\prod_{\substack{x, y \in P \\ \text{s.t. } x < y, c(x) = c(y)}} f_{q,t}(\sigma(x) - \sigma(y); e(x, y)) f_{q,t}(\sigma(x) - \sigma(y); e(x, y) - 1)},
 \end{aligned}
 \tag{14}$$

where  $c(x) \sim c(y)$  means that  $c(x)$  and  $c(y)$  are adjacent in  $T$ , and

$$d(x, y) = \frac{r(y) - r(x) - 1}{2}, \quad e(x, y) = \frac{r(y) - r(x)}{2}.$$

Note that if  $c(x) \sim c(y)$  then  $r(y) - r(x)$  is odd, and if  $c(x) = c(y)$  then  $r(y) - r(x)$  is even, hence  $d(x, y)$  and  $e(x, y)$  are integers.

If we consider the extended poset  $\hat{P} = P \sqcup \{\hat{1}\}$ , and its top tree  $\hat{T} = T \sqcup \{\hat{1}\}$ , where  $\hat{1}$  is the new maximum element of  $\hat{P}$  and  $\hat{1}$  is adjacent to  $v_0$  in  $\hat{T}$ , then the weight  $W_P(\sigma; q, t)$  can be expressed in the following form:

$$\begin{aligned}
 &W_P(\sigma; q, t) \\
 &= \frac{\prod_{\substack{x, y \in \hat{P} \\ \text{s.t. } x < y, \hat{c}(x) \sim \hat{c}(y)}} f_{q,t}(\hat{\sigma}(x) - \hat{\sigma}(y); d(x, y))}{\prod_{\substack{x, y \in P \\ \text{s.t. } x < y, c(x) = c(y)}} f_{q,t}(\sigma(x) - \sigma(y); e(x, y)) f_{q,t}(\sigma(x) - \sigma(y); e(x, y) - 1)},
 \end{aligned}
 \tag{15}$$

where  $\hat{c} : \hat{P} \rightarrow \hat{T}$  and  $\hat{\sigma} : \hat{P} \rightarrow \mathbb{N}$  are the extensions of  $c$  and  $\sigma$  defined by  $\hat{c}(\hat{1}) = \hat{1}$  and  $\hat{\sigma}(\hat{1}) = 0$ , respectively.

**Conjecture 4.4** *Let  $P$  be a connected  $d$ -complete poset. Using the notations defined above, we have*

$$\sum_{\sigma \in \mathcal{A}(P)} W_P(\sigma; q, t) z^\sigma = \prod_{v \in P} F(z[H_P(v)]; q, t).
 \tag{16}$$

If we put  $q = t$ , then  $W_P(\sigma; q, q) = 1$  and  $F(x; q, q) = 1/(1 - x)$ , hence Conjecture 4.4 reduces to Peterson–Proctor’s hook formula (see [7]).

Our main theorem (Theorem 1.2) supports Conjecture 4.4.

- Proposition 4.5** (a) Conjecture 4.4 is true for shapes.  
 (b) Conjecture 4.4 is true for shifted shapes.

*Proof* First, we check that expressions (7) and (9) coincide with (14). Let  $\sigma$  be a reverse shifted plane partition of shape  $\mu$ . Then, by noting the convention that  $\sigma_{k,l} = 0$  for  $k < 0$ , expression (9) can be rewritten as

$$\begin{aligned} &W_{S(\mu)}(\sigma; q, t) \\ &= \prod_{\substack{(i,j) \in S(\mu) \\ i < j}} \prod_{m=0}^{i-2} \frac{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m-1}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m}; m)}{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m-1}, m)} \\ &\quad \times \frac{f_{q,t}(\sigma_{i,j} - \sigma_{1,j-i}; i-1)}{f_{q,t}(\sigma_{i,j} - \sigma_{1,j-i+1}; i-1)} \\ &\quad \times \prod_{\substack{(i,i) \in S(\mu) \\ i \text{ is odd}}} \prod_{m=0}^{(i-3)/2} \frac{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-1,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-1}; 2m+1)}{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-2}; 2m+1)} \\ &\quad \times \frac{f_{q,t}(\sigma_{i,i}; i-1)}{f_{q,t}(\sigma_{i,i} - \sigma_{1,1}; i-1)} \\ &\quad \times \prod_{\substack{(i,i) \in S(\mu) \\ i \text{ is even}}} \prod_{m=0}^{i/2-2} \frac{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-1,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-1}; 2m+1)}{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-2}; 2m+1)} \\ &\quad \times \frac{f_{q,t}(\sigma_{i,i} - \sigma_{1,2}; i-2)}{f_{q,t}(\sigma_{i,i} - \sigma_{2,2}; i-2)}. \end{aligned}$$

This expression can be transformed into (14). Similarly, we can show that Expression (7) is the same as (14) for shapes.

Now (a) follows from Theorem 1.2(a). To prove (b), we use the  $d$ -complete coloring given in Example 4.3(b) and modify the proof of Theorem 1.2(b), which corresponds to the case of  $z_0 = z_{0'}$ . (This modification is due to M. Ishikawa [5].) For a partition  $\tau$ , we denote by  $o(\tau)$  the number of columns of odd length in the diagram of  $\tau$ . Since we have

$$\tau_1 + \tau_3 + \dots = \frac{1}{2}(|\tau| + o(\tau)), \quad \tau_2 + \tau_4 + \dots = \frac{1}{2}(|\tau| - o(\tau)),$$

the monomial  $z^\sigma$  associated to  $\sigma \in \mathcal{A}(S(\mu))$  is given by

$$z^\sigma = \begin{cases} \left(\frac{z_{0'}}{z_0}\right)^{(|\tau|+o(\tau))/2} z^{\text{tr}(\sigma)} & \text{if } l(\mu) \text{ is even,} \\ \left(\frac{z_0'}{z_0}\right)^{(|\tau|-o(\tau))/2} z^{\text{tr}(\sigma)} & \text{if } l(\mu) \text{ is odd,} \end{cases}$$



where  $\tau = \sigma[0]$  is the profile of  $\sigma$ . We appeal to Warnaar’s generalization of the Schur–Littlewood type identity [16, (1.18)]

$$\sum_{\tau} a^{o(\tau)} \frac{b_{\tau}^{\text{el}}(q, t)}{b_{\tau}(q, t)} Q_{\tau}(x; q, t) = \prod_i F(ax_i; q, t) \prod_{i < j} F(x_i x_j; q, t).$$

Replacing  $a, x_i$  by  $a^{1/2}, a^{1/2}x_i^{1/2}$ , or  $a^{-1/2}, a^{1/2}x_i^{1/2}$ , respectively, we obtain

$$\sum_{\tau} a^{(|\tau|+o(\tau))/2} \frac{b_{\lambda}^{\text{el}}(q, t)}{b_{\lambda}(q, t)} Q_{\tau}(x; q, t) = \prod_i F(ax_i; q, t) \prod_{i < j} F(ax_i x_j; q, t),$$

$$\sum_{\tau} a^{(|\tau|-o(\tau))/2} \frac{b_{\lambda}^{\text{el}}(q, t)}{b_{\lambda}(q, t)} Q_{\tau}(x; q, t) = \prod_i F(x_i; q, t) \prod_{i < j} F(ax_i x_j; q, t).$$

We use these identities with  $a = z_0'/z_0$  together with Theorem 2.1 to derive (16) for  $P = S(\mu)$ , by the argument similar to the proof of Theorem 1.2(b). □

**Proposition 4.6** *Conjecture 4.4 is true for rooted trees.*

*Proof* Let  $T$  be a rooted tree with root  $v_0$ . Then  $z[H_T(v)] = \prod_{w \leq v} z_w$ , and the weight function (14) reduces to

$$W_T(\sigma; q, t) = \prod_{\substack{x, y \in P \\ \text{s.t. } y \text{ covers } x}} f_{q,t}(\sigma(x) - \sigma(y); 0) \cdot f_{q,t}(\sigma(v_0); 0)$$

because  $T$  itself is the top tree.

We proceed by induction on the number of vertices in the tree  $T$ . Since it is obvious for  $\#T = 1$ , we may assume  $\#T > 1$ . Let  $v_1, \dots, v_r$  be the children of  $v_0$ , and  $T_1, \dots, T_r$  be the rooted subtrees of  $T$  with roots  $v_1, \dots, v_r$ , respectively. Then there is a natural bijection

$$S : \mathcal{A}(T_1) \times \dots \times \mathcal{A}(T_r) \times \mathbb{N} \longrightarrow \mathcal{A}(T)$$

which associates to  $(\pi_1, \dots, \pi_r, k) \in \mathcal{A}(T_1) \times \dots \times \mathcal{A}(T_r) \times \mathbb{N}$  the  $T$ -partition  $\pi \in \mathcal{A}(T)$  defined by

$$\pi(v) = \begin{cases} \pi_i(v) + k & \text{if } v \in T_i, \\ k & \text{if } v = v_0. \end{cases}$$

Under this bijection  $S$ , we have

$$W_T(\pi; q, t) = f_{q,t}(k; 0) \prod_{i=1}^r W_{T_i}(\pi_i; q, t), \quad z^{\pi} = z[T]^k \prod_{i=1}^r z^{\pi_i}.$$

Hence we have

$$\sum_{\pi \in \mathcal{A}(T)} W_T(\pi; q, t) z^\pi = \left( \sum_{k \geq 0} f_{q,t}(k; 0) z[T]^k \right) \cdot \prod_{i=1}^r \left( \sum_{\pi_i \in \mathcal{A}(T_i)} W_{T_i}(\pi_i; q, t) z^{\pi_i} \right).$$

By using the binomial theorem

$$\sum_{k \geq 0} f_{q,t}(k; 0) z[T]^k = F(z[T]; q, t) = F(z[H_T(v_0)]; q, t)$$

and the induction hypothesis

$$\sum_{\pi_i \in \mathcal{A}(T_i)} W_{T_i}(\pi_i; q, t) z^{\pi_i} = \prod_{v \in T_i} F(z[H_{T_i}(v)]; q, t),$$

we can complete the proof.  $\square$

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