



# Quantum Teleportation via Entangled State of Light in Schwarzschild Black Hole

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## Abstract

In this paper, we describe the properties of quantum entanglement and teleportation between Alice and Bob who is freely falling toward the Schwarzschild black hole. To this aim, in the flat Minkowski spacetime before a black hole is formed, Alice and Bob share a two-mode entangled coherent state (ECS) generated by Kerr medium and the beam splitter. We will show that the degree of entanglement decreases with increasing the radius of black hole  $r$ . Moreover, the fidelity of quantum teleportation via ECS, as a quantum channel, is also reduced because of the Hawking-Unruh effect.

**Keywords** Quantum teleportation · Entangled State · Schwarzschild black hole

## 1 Introduction

Quantum entanglement as an important resource, plays an essential role in quantum information theory as well as for quantum teleportation [1], communication and quantum computing [2, 3]. A pair of quantum systems in an entangled state can be used as a quantum teleportation channel to transfer the quantum state of a particle onto another particle. So the generation of the entangled state attracts tremendous interest in the last decades. In the original work by Bennett et al. [1] the system is considered to be isolated from external forces, and the maximally entangled qubit pair (Bell state) is unitarily evolved.

The importance of understanding entanglement in a relativistic setting has received considerable attention recently [4–6]. The properties of quantum entanglement and teleportation in the background of stationary and rotating curved spacetimes via a maximally entangled Bell state were studied in [7], and it was found that a maximally entangled Bell

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state in an inertial frame becomes less entangled in curved spacetime due to the well-known Hawking–Unruh effect.

In 2003, in [8], teleportation was investigated between Alice and Bob so that Bob was in a uniform acceleration frame relative to Alice. It has also been shown that fidelity decreases due to the Unruh radiation within the framework of Bob. Almost a year after the paper, Schuller and his colleague Mann in the [9] examined two observers that determine the entanglement of two free bosonic modes. They concluded that the entanglement of the state which is maximally entangled, decreases in the inertial frame, if the frames are accelerated relative to each other. This phenomenon, which is a consequence of the Unruh effect, shows that the entanglement in the non-inertial frames, is an observer-dependent quantity. In [10], the entanglement between two modes of free Dirac fields is studied by two accelerated observers. In this paper, it is shown that the entanglement is suppressed due to the Unruh effect and asymptotically reaches a minimum value at very large acceleration. In [11, 12], the entanglement between the two scalar and Dirac-free field modes have been examined from the point of view of accelerated observers, which yielded similar results to previous works. In confirmation of previous results in [13, 14], the entanglement of fermionic fields as well as arbitrary spin fields has been studied and it has been concluded that at infinite acceleration there is no significant difference between the different spin fields. In [15], the suppression of entanglement in the tripartite GHZ (Greenberger-Horne-Zielinger) and W modes were studied from the perspective of accelerated moving observers. As another step along the lines of previous works, Ling et al in [16], examined quantum entanglement of the electromagnetic field specifically the entangled state of the photon helicity in the non-inertial framework and showed that unlike previous works, the logarithmic negativity in non-inertial and inertial framework is similar. In [17], the behavior of quantum and classical correlations in space-like regions with an event horizon between fermionic and bosonic fields were studied. This paper concluded that the emergence of stability laws in quantum and classical entanglement suggests that the statistical distribution function plays a fundamental role in the teleportation of information in the horizon. In [18], an interesting method is proposed for storing quantum information in the field modes of cavities which are moving in spacetime. In contrast to previous results, they found the quantum information in such systems is screened. In [19], the entanglement between particle and antiparticle modes of a Dirac field was investigated from the perspective of inertial and accelerated observers. They have shown that at infinite acceleration, redistribution of entanglement between the particle and antiparticle states plays a key role in the survival of the fermion entanglement. Whereas, for charged boson field modes in [20], different behaviors were observed, so that redistribution did not prevent the entanglement from disappearing in the infinite acceleration limit.

Due to the extensive use of coherent states in quantum optics, they are of particular importance. Coherent states of the simple harmonic oscillator are well known since the foundational work of Schrodinger [21]. Generation of multipartite ECS's and entanglement of multipartite states have been investigated in [22, 23]. In [24], the entanglement properties of the generalized balanced N-mode coherent states generated by the beam splitter have been stated which is a general form of the two-mode ECS

$$|\psi\rangle = \frac{1}{\sqrt{M}}(|\alpha\rangle|\alpha\rangle + \mu|\beta\rangle|\beta\rangle). \quad (1)$$

In this paper, we assume that Alice and Bob share a two-mode entangled coherent state. In Ref. [24], an experimental set up is used for generating N-mode entangled coherent state, using the parity, displacement, and beam splitting operator. In order to produce a two-mode

ECS, one must first obtain the superposition of two coherent states [25, 26].  $\mu$  is complex parameter which depends on the amplitude of coherent states. In this work, we will use two-mode ECS,  $|\psi\rangle = (1/\sqrt{M})(|\alpha\rangle|\alpha\rangle + \mu|-\alpha\rangle|-\alpha\rangle)$  in which two coherent states  $|\alpha\rangle$  and  $|-\alpha\rangle$  are in general nonorthogonal and span a two-dimensional qubit like Hilbert space  $\{|0\rangle, |1\rangle\}$ . By definition the orthonormal basis as

$$|0\rangle = |\alpha\rangle, |1\rangle = \frac{|-\alpha\rangle - p|\alpha\rangle}{\sqrt{1 - p^2}}, \tag{2}$$

the two-mode ECS,  $|\psi\rangle$  can be recast in a two-qubit form so we can use it as a quantum channel to transfer the quantum state from sender (Alice) to receiver (Bob). The main purpose of this work is to investigate the entanglement of two-mode ECS and teleportation in Schwarzschild black hole spacetime [27]. After some mass collapses to form a black hole, Alice stays stationary at the flat region of a black hole while Bob moves from Alice’s place toward the black hole. We study the influences of this effect on entanglement using negativity. Moreover, the measurements of teleportation can be performed between Alice and Bob in curved spacetime, so we will find that the fidelity of teleportation decreases with increasing the radius of black hole  $r$ .

The outline of this paper is as follows: In Section 2, we will discuss the normal mode solutions for a Schwarzschild black hole. In Section 3, we will study entanglement of entangled coherent states in noninertial frames. Moreover, we recall the usual flat space teleportation protocol and explore the degradation of the fidelity of the teleported state when one of the participants undergoes uniform radius of black hole. Finally, conclusions are discussed in Section 4.

## 2 Normal Mode Solutions for Schwarzschild Black Hole

The stationary Schwarzschild black hole is represented by the metric [28]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where the metric is written in spherical coordinates so  $r$  represents the radial coordinate. When  $r = 2M$ , the singularity appears, in other words, this radius is called the black hole radius indicated by  $r_s$ . In (3),  $M$  is the mass of the black hole and at  $r_s = 2M$ , the Schwarzschild spacetime has an event horizon. In the Schwarzschild metric, the scalar field  $\Phi$  which corresponds to a massless particle with zero spin, applies in the following equation [29]:

$$\sqrt{-g}\partial_\mu[g^{\mu\nu}\sqrt{-g}\partial_\nu]\Phi(x) = 0, \tag{4}$$

where the Schwarzschild metric is defined as follows:

$$g_{\mu\nu} = \begin{bmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}. \tag{5}$$

Perhaps the simplest way of treating a partial differential equation such as (3) is to split it into a set of ordinary differential equations [30]. This may be done as follows, let

$$\Phi(x) = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega t} F_{\omega l}(r) Y_{lm}(\theta, \phi), \tag{6}$$

here we have chosen the positive frequency normal mode solution. By substituting into (3), we obtain a differential equation for  $F_{\omega l}$ :

$$\frac{\partial^2 F_{\omega l}(r)}{\partial r^{*2}} + \omega^2 F_{\omega l}(r) - \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) F_{\omega l}(r) = 0, \tag{7}$$

where  $r^* = r + 2M \ln(\frac{r}{2M} - 1)$  which is called ‘‘tortoise coordinate’’ [31],  $r$  ranges from  $2M$  to  $\infty$  and  $r^*$  goes from  $-\infty$  to  $+\infty$ . As you can see, the Schwarzschild metric is singular at  $r = 2M$ , but it should be noted that this one is only a coordinate singularity. This kind of singularity can be eliminated by selecting a proper coordinate system. We introduce the null coordinates  $u, v$  which have the direction of null geodesics by [32]

$$u = t - r^* \quad \text{and} \quad v = t + r^*. \tag{8}$$

Inserting  $u, v$  in (3), we obtain:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dudv + r^2(d\theta^2 + \text{Sin}^2\theta d\phi^2). \tag{9}$$

By rewriting the metric via new coordinates

$$U = -4Me^{-\frac{u}{4M}} \quad \text{and} \quad V = 4Me^{\frac{v}{4M}}, \tag{10}$$

we find:

$$ds^2 = -2M \frac{e^{-r/2M}}{r} dUdV + r^2(d\theta^2 + \text{Sin}^2\theta d\phi^2). \tag{11}$$

Now with solving (6), one can obtain the positive frequency solutions in Kruskal coordinates [33]

$$\Phi^{(I)} = e^{-i\omega t} F_{\omega l} = e^{-i\omega v}, \tag{12}$$

$$\Phi^{(II)} = e^{i\omega t} F_{\omega l} = e^{-i\omega u}. \tag{13}$$

By using the above equations, we can quantize the scalar field

$$\Phi = \sum_p (b_p^{(I)} \Phi_p^{(I)} + b_p^{(II)} \Phi_p^{(II)} + h.c.), \tag{14}$$

where  $p$  stands for  $(\omega, l, m)$  and the operators  $b_p^{(I)}$  and  $b_p^{(II)}$  are annihilation operators which correspond to the different regions in Penrose diagram. Those operators define the Fulling-Rindler vacuum:

$$b_p^{(I)} |0\rangle^{(I)} \otimes |0\rangle^{(II)} = b_p^{(II)} |0\rangle^{(I)} \otimes |0\rangle^{(II)} = 0. \tag{15}$$

Let us define the normalized modes  $\phi_1$  and  $\phi_2$  in terms of Kruskal coordinates:

$$\phi_1 = e^{-\pi\omega a/2} \Theta(-u) (-u/a)^{i\omega a} + e^{\pi\omega a/2} \Theta(u) (u/a)^{i\omega a}, \tag{16}$$

$$\phi_2 = e^{\pi\omega a/2} \Theta(u) (-u/a)^{-i\omega a} + e^{-\pi\omega a/2} \Theta(-u) (u/a)^{-i\omega a}, \tag{17}$$

where  $\Theta(\pm u)$  is the Heaviside step function. We can quantize the scalar field as

$$\phi = \sum_p \frac{1}{\sqrt{2 \sinh(\pi\omega a)}} [d_p^{(I)} \phi_1 + d_p^{(II)} \phi_2 + h.c.], \tag{18}$$

where  $b_p$  can be written in  $d_p$  as follows

$$b_p^{(I)} = \frac{1}{\sqrt{2 \sinh(\pi \omega a)}} [e^{\pi \omega a/2} d_p^{(I)} + e^{-\pi \omega a/2} d_p^{(II)\dagger}], \tag{19}$$

$$b_p^{(II)} = \frac{1}{\sqrt{2 \sinh(\pi \omega a)}} [e^{\pi \omega a/2} d_p^{(II)} + e^{-\pi \omega a/2} d_p^{(I)\dagger}]. \tag{20}$$

The Minkowski vacuum can be defined as

$$d_p^{(I)} |0\rangle_M = d_p^{(II)} |0\rangle_M = 0. \tag{21}$$

The following relation shows the simplicity of Minkowski’s vacuum

$$|0\rangle_M = \sum_{n_p=0}^{\infty} \prod_P \cosh^{-1} r \tanh^{n_p} r |n_p\rangle_I \otimes |n_p\rangle_{II}, \tag{22}$$

where  $r$  represents the radial coordinate and  $|n_p\rangle_I = (b_p^{(I)\dagger})^n |0\rangle_I / \sqrt{n!}$  and  $|n_p\rangle_{II} = (b_p^{(II)\dagger})^n |0\rangle_{II} / \sqrt{n!}$ , are orthonormal bases for Hilbert spaces.

### 3 Entanglement of Entangled Coherent States in Noninertial Frames

Let us assume prior to forming a black hole, Alice and Bob share an entangled coherent state in the flat Minkowski spacetime such as follows

$$|\psi\rangle = \frac{1}{\sqrt{M}} \{ |\alpha\rangle_A |\alpha\rangle_B + \mu |-\alpha\rangle_A |-\alpha\rangle_B \}, \tag{23}$$

where the states  $|\alpha\rangle$  and  $|-\alpha\rangle$  are coherent states and subscripts  $A$  and  $B$  indicate the modes associated with the observers Alice and Bob, respectively. For simplicity we assume that  $\mu$  and  $\alpha$  are real numbers and  $M$  is a normalization factor,  $M = 1 + \mu^2 + 2\mu p^2$  in which  $p = \langle \alpha | -\alpha \rangle = e^{-2\alpha^2}$ . It should be noted that after the formation of the black hole, Alice is in the flat region while Bob is inside the black hole. If Bob undergoes a uniform acceleration or stays in curved spacetime, Bob’s state must be specified in Rindler or Schwarzschild coordinates. Then the state (23) can be rewritten in terms of the Minkowski modes for Alice and Schwarzschild modes for Bob. Bob is causally disconnected from region  $II$ , so we take the trace over the mode  $II$  [7]. It leads to a mixed density matrix between Alice and Bob which is given by

$$\rho_{AB} = \frac{1}{M \cosh^2 r} \sum_n \tanh^{2n} r \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{pmatrix}, \tag{24}$$

where the elements of this matrix are arranged in Table 1.

**Table 1** Elements of matrix  $\rho_{AB}$  with  $R = \sqrt{1 - p^2}$

$\rho_{11} = (1 + \mu p^2)^2$	$\rho_{12} = \frac{\mu p R (1 + \mu p^2) \sqrt{n+1}}{\cosh r}$
$\rho_{13} = \mu p R (1 + \mu p^2)$	$\rho_{14} = \frac{\mu R^2 (1 + \mu p^2) \sqrt{n+1}}{\cosh r}$
$\rho_{22} = \frac{\mu^2 p^2 R^2 (n+1)}{\cosh^2 r}$	$\rho_{23} = \frac{\mu^2 p^2 R^2 \sqrt{n+1}}{\cosh r}$
$\rho_{24} = \frac{\mu^2 p R^3 (n+1)}{\cosh^2 r}$	$\rho_{33} = \mu^2 p^2 R^2$
$\rho_{34} = \frac{\mu^2 p R^3 \sqrt{n+1}}{\cosh r}$	$\rho_{44} = \frac{\mu^2 R^4 (n+1)}{\cosh^2 r}$

The amount of entanglement between two qubits  $A$  and  $B$  can be measured by logarithmic negativity which is defined by [34]

$$E_{\mathcal{N}}(\rho) = \log_2(2\mathcal{N}(\rho) + 1), \tag{25}$$

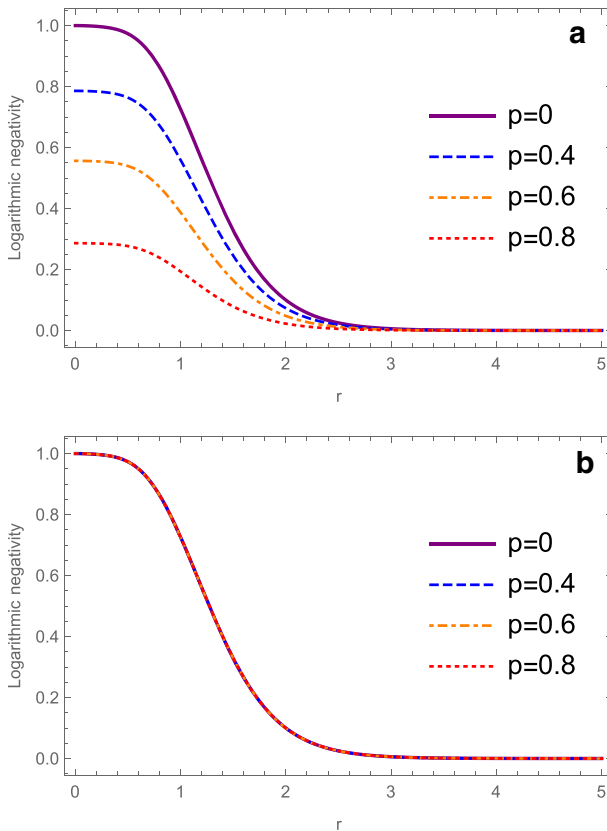
where  $\mathcal{N}(\rho)$  is the negativity of the state which is specified as

$$\mathcal{N}(\rho) = \left| \sum_i \mu_i \right|, \tag{26}$$

in that  $\mu_i$ 's are the negative eigenvalues of partial transpose with respect to  $A$ . So we can obtain

$$\mathcal{N}(\rho) = \frac{|\mu| \sqrt{2(1+n)} (1-p^2) \tanh^{2n} r}{M \cosh^2 r \sqrt{1 + \cosh 2r}}. \tag{27}$$

In Fig. 1, the logarithmic negativity is plotted as a function of black hole radius  $r$ .



**Fig. 1** (Color online) Logarithmic negativity as a function of  $r$  for given  $p$ , **a**  $\mu = 1$  and **b**  $\mu = -1$

Figure 1a and b show that the logarithmic negativity is a monotone function of radius  $r$ . When  $r \rightarrow 0$ , logarithmic negativity  $\mathcal{N}(\rho)$  is equal to one, which corresponds to the situation that Alice and Bob are Minkowski observers, i.e., no black hole is formed. In the limit  $r \rightarrow \infty$ , the logarithmic negativity tends to zero. Moreover, for  $\mu = 1$ , by increasing  $p$  the logarithmic negativity is reduced, whereas for  $\mu = -1$  the logarithmic negativity is independent of  $p$ .

Consider an arbitrary one-qubit state  $|\varphi\rangle = a|0\rangle + b|1\rangle$  with  $|a|^2 + |b|^2 = 1$ , which Alice wishes to teleport to Bob [4]. For this purpose, Alice and Bob have initially shared the bipartite state  $|\psi\rangle$ . The total initial state is  $|\Psi\rangle = |\psi\rangle \otimes |\varphi\rangle$ , where two first qubits belong to Alice, and the third one belongs to Bob. Alice makes a local measurement in the Bell basis  $|\psi^\pm\rangle\langle\psi^\pm|$  and  $|\varphi^\pm\rangle\langle\varphi^\pm|$  on the two particles in her possession. Then Alice sends the result of her measurement to Bob via a classical channel. Bob is causally disconnected from region  $II$ , so by tracing out over region  $II$  Bob gets

$$\rho^I = \frac{2}{M \cosh^2 r} \sum_n \tanh^{2n} r \{ \xi^2 |n\rangle\langle n| + \eta_n^2 |n+1\rangle\langle n+1| + \xi \eta_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|) \}, \tag{28}$$

in which

$$\xi = a(1 + \mu p^2) + b\mu R p, \quad \eta_n = \frac{\sqrt{n+1}\mu R}{\cosh r} (ap + bR). \tag{29}$$

The density operator is in a  $2 \times \infty$  dimensional space. We can project the total state into  $2 \times 2$  dimensional space [35]. In a two dimensional space, the reduced density operator is

$$\rho^I = \frac{2}{M \cosh^2 r} \{ \xi^2 |0\rangle\langle 0| + \xi \eta_0 (|0\rangle\langle 1| + |1\rangle\langle 0|) + (\eta_0^2 + \xi^2 \tanh^2 r) |1\rangle\langle 1| \}. \tag{30}$$

Finally, the fidelity corresponding to the teleportation,  $F^I = \langle \varphi | \rho^I | \varphi \rangle$  is obtained as

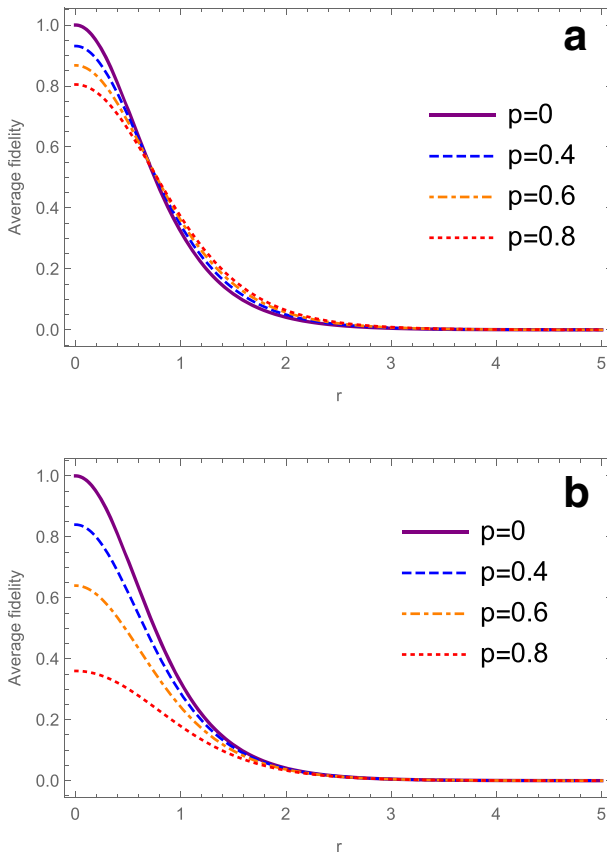
$$F^I = \frac{2}{M \cosh^2 r} \{ a^2 \xi^2 + 2ab\xi \eta_0 + b^2 (\eta_0^2 + \xi^2 \tanh^2 r) \}. \tag{31}$$

For the pure input state, it is useful to calculate average fidelity which is defined by average over all possible pure states in the Bloch sphere with  $a = \cos \theta$  and  $b = \sin \theta$  as follows [5]

$$\bar{F} = \frac{1}{\pi} \int_0^\pi d\theta F^I(r, \theta). \tag{32}$$

Where  $r$  represents the radial coordinate. In Fig. 2, average fidelity is plotted as a function of  $r$  for given  $\mu$  and  $p$ .

We concluded from Fig. 2a and b, by increasing  $r$  which scales directly with Bob’s acceleration, the fidelity decreases. Moreover, when  $r = 0$  the fidelity is unity corresponding to the case both Alice and Bob are in Minkowski space. On the other hand, for both cases  $\mu = 1$  and  $\mu = -1$  by intensifying coherent field or equivalently decreasing  $p$ , the average fidelity enhances.



**Fig. 2** (Color online) Average fidelity as a function of  $r$  for given  $\mu$  and  $p$ , **a**  $\mu = 1$  and **b**  $\mu = -1$

## 4 Conclusion

In this paper, we examined the properties of quantum entanglement and teleportation in the background of curvature spaces. We use the two-mode ECS as a quantum channel for transmitting quantum states from sender to receiver. We assumed that Alice was standing in the area described by the Minkowski metric and sending the two-mode ECS to Bob, which is described in an area with Schwarzschild metric. The main purpose of this work is to investigate the entanglement of two-mode ECS and teleportation in Schwarzschild black hole. We show that the entanglement of a two-mode ECS in an inertial frame falls due to the well-known influence of Hawking-Unruh in the curvature space. In Fig. 1, the diagram of logarithmic negativity is plotted as a function of black hole radius  $r$ . We concluded that when  $r \rightarrow 0$ , the logarithmic negativity approaches to one and there is not a black hole. In the limit  $r \rightarrow \infty$ , the logarithmic negativity tends to zero. We also concluded that for  $\mu = 1$  with increasing  $p$  the logarithmic negativity is reduced whereas for  $\mu = -1$  the logarithmic negativity is independent of  $p$ . Finally, we study the fidelity corresponding to the teleportation. In Fig. 2, we plot the average fidelity in terms of  $r$  that shows by increasing  $r$  the fidelity decreases and when  $r = 0$  the fidelity is unity corresponding to the Minkowski



space. We also deduced that for both cases  $\mu = 1$  and  $\mu = -1$  by the enhancement of coherent field or equivalently decreasing  $p$ , the average fidelity increases.

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