Quantum Games with Strategies Induced by Basis Change Rules

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Abstract The aim of the paper is to draw attention to a special class of two parameter unitary strategies in the Eisert-Wilkens-Lewenstein scheme for quantum games. We identify the players' strategies with basis change matrices. Then we prove that the resulting quantum game is invariant with respect to isomorphic transformations of the input game. Moreover, it is shown that the game so obtained may not be trivial with respect to pure Nash equilibria, compared with the model with strategies being the special unitary group $SU(2)$.

Keywords Quantum game · Strategic-form game · Game isomorphism

1 Introduction

The EWL scheme introduced in [1] has become commonly used quantum scheme for games in strategic form. It has found application in many branches of game theory: from simple 2×2 games [2] to evolutionary games [3] and games with imperfect recall [4]. The players' strategies in this scheme are identified with unitary operators whose range is crucial for the result of a game. If the players are allowed to use general unitary operators, it is highly probable that there are no pure Nash equilibria. More precisely, given the EWL scheme for 2×2 games and the players' strategies being the special unitary group of degree 2, a pure Nash equilibrium exists only if there is an outcome in the 2×2 game with maximal payoffs for both players. The majority of games studied in quantum game theory do not satisfy this

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requirement. Hence more sophisticated techniques are required to find optimal solutions in the game (see, for example, [5]). This difficulty may disappear if the players' strategy sets are restricted to suitably chosen subsets of $SU(2)$. As shown, for example, in [1, 3, 6, 7], interesting results (compared to playing a game classically) can be obtained by using some types of two-parameter unitary operators. However, these two parameter operators are questionable. First, it seems unlikely that the restriction to the two-parameter set of unitary operators reflects any reasonable physical constraint [8]. Second, they are not reasonable from a game theory point of view - the EWL approach defined in this way may yield different optimal strategy profiles depending on the order of players' strategies in the classical game. In other words, this scheme is not invariant under isomorphic transformations of the input game [9].

Fortunately, sets of unitary strategies can be defined in many ways, in particular, the operators depending on two parameters [10]. Our goal is to determine unitary operators such that the resulting EWL protocol does not depend on the numbering of players' strategies in a classical game. At the same time we find a scheme that justifies this type of unitary strategies.

2 Preliminaries

In order to make our paper self-contained we give the necessary preliminaries from game theory and quantum game theory.

2.1 Strategic Form Game

First we recall the definition of strategic form game and Nash equilibrium [11].

Definition 1 A game in strategic form is a triple $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in which

- $N = \{1, 2, \dots, n\}$ is a finite set of players.
- S_i is the set of strategies of player i , for each player $i \in N$.
- $u_i: S_1 \times S_2 \times \dots \times S_n \rightarrow \mathbb{R}$ is a function associating each vector of strategies $s = (s_i)_{i \in N}$ with the payoff $u_i(s)$ to the player i , for every player $i \in N$.

Nash equilibrium is a solution concept commonly used in game theory. It is a strategy vector at which no player has a profitable deviation when all remaining players do not change their strategies.

Definition 2 A strategy vector $s^* = (s_1^*, \dots, s_n^*)$ is a Nash equilibrium if for each player $i \in N$ and each strategy $s_i \in S_i$ the following is satisfied:

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*), \text{ where } s_{-i}^* = (s_1^*, s_2^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*). \quad (1)$$

2.2 Strong Isomorphism

The notion of strong isomorphism defines classes of games that are the same up to numbering the players and order of players' strategies. The following definitions are taken from [12] (see also [13, 14] and [15]). The first one defines a mapping that associates players and their strategies in one game with players and their strategies in the other game.

Definition 3 Given $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and $\Gamma' = (N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$, a game mapping f from Γ to Γ' is a tuple $f = (\eta, (\varphi_i)_{i \in N})$, where η is a bijection from N to N and for any $i \in N$, φ_i is a bijection from S_i to $S'_{\eta(i)}$.

In general case, mapping f from $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ to $(N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$ identifies player $i \in N$ with player $\eta(i)$ and maps S_i to $S'_{\eta(i)}$. This means that strategy vector $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$ is mapped onto vector $(s'_1, \dots, s'_n)$ that satisfies equation $s'_{\eta(i)} = \varphi_i(s_i)$ for $i \in N$.

The notion of game mapping is a basis for a definition of game isomorphism. Depending on how rich structure of a game is to be preserved we can distinguish various types of game isomorphism. One that preserves the players' payoff functions is called strong isomorphism. The formal definition is as follows:

Definition 4 Given two strategic games $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and $\Gamma' = (N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$, a game mapping $f = (\eta, (\varphi_i)_{i \in N})$ is called a strong isomorphism if the relation $u_i(s) = u'_{\eta(i)}(f(s))$ holds for each $i \in N$ and each strategy vector $s \in S_1 \times \dots \times S_n$.

From the above definition it may be concluded that if there is a strong isomorphism between games Γ and Γ' , they may differ merely by the numbering of players and the order of their strategies.

The following lemma (see also [9]) shows that relabeling players or their strategies does not affect the game with respect to Nash equilibria. If f is a strong isomorphism between games Γ and Γ' , one may expect that Nash equilibria in Γ are mapped by f to Nash equilibria in Γ' .

Lemma 1 Let f be a strong isomorphism between games Γ and Γ' . Strategy vector $s^* = (s_1^*, \dots, s_n^*) \in S_1 \times \dots \times S_n$ is a Nash equilibrium in the game Γ if and only if $f(s^*) \in S'_1 \times \dots \times S'_n$ is a Nash equilibrium in Γ' .

2.3 Eisert-Wilkens-Lewenstein Scheme

Let us consider a strategic game $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ with $S_i = \{s_0^i, s_1^i\}$ for each $i \in N$. The generalized Eisert-Wilkens-Lewenstein approach [16, 17] to game Γ is defined by a triple $\Gamma_{EWL} = (N, (D_i)_{i \in N}, (M_i)_{i \in N})$, where

- D_i is a set of unitary operators from $SU(2)$. The commonly used parametrization for $U \in SU(2)$ is given by

$$U(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha} \cos \frac{\theta}{2} & ie^{i\beta} \sin \frac{\theta}{2} \\ ie^{-i\beta} \sin \frac{\theta}{2} & e^{-i\alpha} \cos \frac{\theta}{2} \end{pmatrix}, \quad \theta \in [0, \pi], \quad \alpha, \beta \in [0, 2\pi). \quad (2)$$

D_i is assumed to include set $\{U(\theta, 0, 0) : \theta \in [0, \pi]\}$. Elements $U_i \in D_i$ play the role of player i 's strategies. The players, by choosing $U_i \in D_i$, determine the final state $|\Psi\rangle$ according to the following formula:

$$|\Psi\rangle = J^\dagger \left(\bigotimes_{i=1}^n U_i(\theta_i, \alpha_i, \beta_i) \right) J |0\rangle^{\otimes n} \quad \text{where } J = \frac{1}{\sqrt{2}} (1^{\otimes n} + i\sigma_x^{\otimes n}) \quad (3)$$

(1 is the identity matrix of size 2 and σ_x is the Pauli matrix X).

- M_i is an observable defined by the formula

$$M_i = \sum_{j_1, \dots, j_n \in \{0,1\}} a_{j_1 \dots j_n}^i |j_1 \dots j_n\rangle \langle j_1 \dots j_n|. \quad (4)$$

The numbers $a_{j_1 \dots j_n}^i$ are the player i 's payoffs in Γ such that $a_{j_1 \dots j_n}^i = u_i(s_{j_1}^i, \dots, s_{j_n}^i)$. The player i 's payoff v_i in Γ_{EWL} is defined as the average value of measurement M_i , i.e.,

$$v_i \left(\bigotimes_{i=1}^n U_i(\theta_i, \alpha_i, \beta_i) \right) = \langle \Psi | M_i | \Psi \rangle. \quad (5)$$

As an example, let us consider a 2×2 game

$$\begin{array}{cc} & \begin{matrix} s_0^2 & s_1^2 \end{matrix} \\ \begin{matrix} s_0^1 \\ s_1^1 \end{matrix} & \begin{pmatrix} (a_{00}^1, a_{00}^2) & (a_{01}^1, a_{01}^2) \\ (a_{10}^1, a_{10}^2) & (a_{11}^1, a_{11}^2) \end{pmatrix} \end{array} \quad (6)$$

Then, (5) becomes

$$\begin{aligned} & v_i(U_1(\theta_1, \alpha_1, \beta_1) \otimes U_2(\theta_2, \alpha_2, \beta_2)) \\ &= a_{00}^i \left(\cos(\alpha_1 + \alpha_2) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin(\beta_1 + \beta_2) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \\ &+ a_{01}^i \left(\cos(\alpha_1 - \beta_2) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} + \sin(\alpha_2 - \beta_1) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \\ &+ a_{10}^i \left(\sin(\alpha_1 - \beta_2) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} + \cos(\alpha_2 - \beta_1) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \\ &+ a_{11}^i \left(\sin(\alpha_1 + \alpha_2) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} - \cos(\beta_1 + \beta_2) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \end{aligned} \quad (7)$$

for $i = 1, 2$.

3 Strategies as Basis Change Matrices

Let us consider a slightly different approach to the EWL scheme. Suppose that instead of choosing operators (2) directly, player $i \in \{1, 2\}$ is to choose an orthonormal basis $\{|+\rangle_i, |-\rangle_i\}$ for the vector space $\mathbb{C}^2$. Hence, the players create the initial state $|+\rangle_1|+\rangle_2$ which is then entangled by a unitary operator J' on $\mathbb{C}^2 \otimes \mathbb{C}^2$ of the form

$$J' |+\rangle_1 |+\rangle_2 = \frac{1}{\sqrt{2}} (|+\rangle_1 |+\rangle_2 + i |-\rangle_1 |-\rangle_2). \quad (8)$$

Next, operator $J^\dagger$ is applied to obtain the final state $|\Psi_f\rangle$ (see the setup shown in the left-hand side of Fig. 1).

Clearly, both models in Fig. 1 are equivalent. Note that the choice of the orthonormal basis $\{|+\rangle_i, |-\rangle_i\}$ can be identified with the unitary transformation U_i from $\{|0\rangle_i, |1\rangle_i\}$ to $\{|+\rangle_i, |-\rangle_i\}$. Hence, we have

$$J' |+\rangle_1 |+\rangle_2 = (U_1 \otimes U_2) J (U_1 \otimes U_2)^\dagger |+\rangle_1 |+\rangle_2. \quad (9)$$

As a result,

$$J^\dagger J' |+\rangle_1 |+\rangle_2 = J^\dagger (U_1 \otimes U_2) J |0\rangle_1 |0\rangle_2 = |\Psi_f\rangle. \quad (10)$$

Fig. 1 The equivalent quantum circuits that implement the EWL protocol

While the two ways of describing the quantum game protocol are virtually identical, choosing $\{|+\rangle_i, |-\rangle_i\}$ may legitimate two-parameter quantum strategies. Indeed, the generic state of a qubit may be written as

$$|\psi\rangle = e^{-i\phi/2} \cos \frac{\theta}{2} |0\rangle + e^{i\phi/2} \sin \frac{\theta}{2} |1\rangle, \quad (11)$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Such a state can be obtained with the use of the Stern-Gerlach apparatus directed along the axis singled out by the unit vector

$$\mathbf{v} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (12)$$

(see, for example, [18]). Mathematically, the apparatus is described by an observable

$$\sigma \cdot \mathbf{v} \equiv \sigma_x \sin \theta \cos \phi + \sigma_y \sin \theta \sin \phi + \sigma_z \cos \theta. \quad (13)$$

An easy computation shows that the eigenvectors of $\sigma \cdot \mathbf{v}$ are

$$\begin{aligned} |+\rangle_{\mathbf{v}} &= e^{-i\phi/2} \cos \frac{\theta}{2} |0\rangle + e^{i\phi/2} \sin \frac{\theta}{2} |1\rangle, \\ |-\rangle_{\mathbf{v}} &= -e^{-i\phi/2} \sin \frac{\theta}{2} |0\rangle + e^{i\phi/2} \cos \frac{\theta}{2} |1\rangle, \end{aligned} \quad (14)$$

with the eigenvalues +1 and -1, respectively. Now, we can assume that a beam of atoms of spin 1/2 enters the Stern-Gerlach apparatus. Then, a player is able to prepare the basis (14) by orienting the apparatus along the $\mathbf{v}$ axis. A particular state of (14) can then be obtained if the other component is blocked (see Fig. 2). We see that a choice of the basis (14) depends

Fig. 2 Preparation of states $|+\rangle$ and $|-\rangle$ with the use of the Stern-Gerlach apparatus

on the unit vector v and it reduces to a choice of two parameters θ and ϕ . It follows from (14) that we can define unitary transformation V such that $V|0\rangle = |+\rangle_v$, $V|1\rangle = |-\rangle_v$. Its matrix representation in the computational basis is

$$V(\theta, \phi) = \begin{pmatrix} e^{-i\phi/2} \cos \frac{\theta}{2} & -e^{-i\phi/2} \sin \frac{\theta}{2} \\ e^{i\phi/2} \sin \frac{\theta}{2} & e^{i\phi/2} \cos \frac{\theta}{2} \end{pmatrix}. \quad (15)$$

Thus, we can treat operators (15) in the EWL scheme as particular type of players' unitary strategies that are identified with observables (13).

Comparison of SU(2) and $\{V(\theta, \phi)\}$ in Terms of the EWL Scheme As we mentioned in the introduction, studying the EWL approach to 2×2 games with respect to pure Nash equilibria is trivial when the players have access to the full parameter unitary strategies. For every strategy of one of the players, the other player has a counter strategy that yields her the maximum payoff. For example, by (7) if player 2's strategy is $U_2(\theta_2, \alpha_2, \beta_2)$, then the result of the game is (a_{00}^1, a_{00}^2) if player 1 chooses $U_1(\theta_2, -\alpha_2, \pi/2 - \beta_2)$, (a_{01}^1, a_{01}^2) if she chooses $U_1(\pi - \theta_2, \beta_2, \alpha_2 - \pi/2)$, and so on. Thus player 1 can obtain the maximal payoff for any fixed strategy of her opponent.

We now check that the set of two-parameter unitary operators $\{V(\theta, \phi)\}$ does not provide a player with counter strategies in general. For this purpose, we first compute the players' payoff functions associated with operators (15) in the EWL scheme. We set the entangling operator J so that the quantum game scheme generalizes the classical game (6),

$$J = \frac{1}{\sqrt{2}} \left(\mathbb{1}^{\otimes 2} + iV(\pi, 0)^{\otimes 2} \right). \quad (16)$$

Then the payoff functions can be derived similarly to that of the model (7),

$$\begin{aligned} & v_i(V_1(\theta_1, \phi_1) \otimes V_2(\theta_2, \phi_2)) \\ &= a_{00}^i \left(\cos \left(\frac{\phi_1 + \phi_2}{2} \right) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \left(\frac{\phi_1 + \phi_2}{2} \right) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \\ &+ a_{01}^i \left(\cos \left(\frac{\phi_1 - \phi_2}{2} \right) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \left(\frac{\phi_1 - \phi_2}{2} \right) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \\ &+ a_{10}^i \left(\cos \left(\frac{\phi_1 - \phi_2}{2} \right) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \left(\frac{\phi_1 - \phi_2}{2} \right) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \\ &+ a_{11}^i \left(\cos \left(\frac{\phi_1 + \phi_2}{2} \right) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \left(\frac{\phi_1 + \phi_2}{2} \right) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2. \quad (17) \end{aligned}$$

Having specified the players' payoff functions associated with the general 2×2 game (6), we fix payoff profiles (a_{ij}^1, a_{ij}^2) , $i, j = 0, 1$ according to the following example:

Example 1 Let us consider a 2×2 game

$$\begin{matrix} & s_0^2 & s_1^2 \\ \begin{matrix} s_0^1 \\ s_1^1 \end{matrix} & \begin{pmatrix} (6, 7) & (7, 6) \\ (7, 6) & (0, 0) \end{pmatrix} \end{matrix} \quad (18)$$

In this case there is no strategy vector that yields the maximum payoff of 7 for both players. In other words, a possible result of the game where one of the players obtains 7 means that the other player ends with a payoff less than 7. It follows immediately that the EWL

approach to (18) with players' strategy sets equal to $SU(2)$ has no pure Nash equilibria. It turns out that a pure Nash equilibrium exists if we restrict strategies to (15).

Let us choose the following strategy vector:

$$V_1\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \otimes V_2\left(\frac{\pi}{2}, \frac{\pi}{4}\right). \quad (19)$$

Now (17) becomes

$$(v_1, v_2) \left(V_1\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \otimes V_2\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right) = (6.5, 6.5). \quad (20)$$

To see that the strategy vector (19) constitutes a Nash equilibrium it is sufficient to check the following system of inequalities:

$$\begin{cases} 6.5 \geq v_1 \left(V_1(\theta_1, \phi_1) \otimes V_2\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right) \text{ for } \theta_1 \in [0, \pi], \phi_1 \in [0, 2\pi), \\ 6.5 \geq v_2 \left(V_1\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \otimes V_2(\theta_2, \phi_2) \right) \text{ for } \theta_2 \in [0, \pi], \phi_2 \in [0, 2\pi). \end{cases} \quad (21)$$

Taking $V_2(\theta_2, \phi_2) = V_2(\pi/2, \pi/4)$ we can rewrite (17) as

$$v_1 \left(V_1(\theta_1, \phi_1) \otimes V_2\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right) = 5 + \frac{3\sqrt{2}}{4} (\cos(\phi_1 - \theta_1) - \sin(\phi_1 - \theta_1)). \quad (22)$$

Then

$$\max_{\theta_1, \phi_1} \left\{ 5 + \frac{3\sqrt{2}}{4} (\cos(\phi_1 - \theta_1) - \sin(\phi_1 - \theta_1)) \right\} = 6.5. \quad (23)$$

Similarly

$$\max_{\theta_1, \phi_1} \left\{ v_2 \left(V_1\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \otimes V_2(\theta_2, \phi_2) \right) \right\} \quad (24)$$

$$= \max_{\theta_1, \phi_1} \left\{ 5 + \frac{3\sqrt{2}}{4} (\cos(\phi_1 - \theta_1) - \sin(\phi_1 - \theta_1)) \right\} = 6.5. \quad (25)$$

4 Invariance with Respect to Game Isomorphism

In [9] one of the authors showed that some types of the two-parameter strategies are questionable from a game-theoretical point of view. Among the sets of unitary strategies $\{U(\theta, \alpha, \beta)\}$, $\{U(\theta, \alpha, 0)\}$ and $\{U(\theta, 0, \beta)\}$ only the EWL scheme with the three-parameter operators is invariant with respect to strongly isomorphic transformations of input games. Interestingly, there are different ways to define two-parameter subsets of $SU(2)$. Unitary operators similar to (15) have already been investigated with respect to Nash equilibria by Flitney and Hollenberg [10]. In what follows, we prove that the EWL scheme based on (15) is invariant with respect to strong isomorphism. Thus, in particular, we show, that the scheme is invariant when the strategies in the input game are permuted.

We can see from (17) that replacing $V_1(\theta_1, \phi_1) \otimes V_2(\theta_2, \phi_2)$ by, for example, $V_1(\pi - \theta_1, \phi_1) \otimes V_2(\theta_2, -\pi - \phi_2)$ we obtain the expected payoff vector (5) that corresponds to a 2×2 game with permuted player 2' strategies, i.e.,

$$\begin{matrix} & s_0^2 & s_1^2 \\ \begin{matrix} s_0^1 \\ s_1^1 \end{matrix} & \begin{pmatrix} a_{01}^1 & a_{01}^2 \\ a_{11}^1 & a_{11}^2 \end{pmatrix} & \begin{pmatrix} a_{00}^1 & a_{00}^2 \\ a_{10}^1 & a_{10}^2 \end{pmatrix} \end{matrix} \quad (26)$$

Hence, given strongly isomorphic games (6) and (26) the resulting EWL counterparts with available strategies (15) are also strongly isomorphic. The corresponding game isomorphism is defined by a mapping

$$\tilde{f}(V_1(\theta_1, \phi_1) \otimes V_2(\theta_2, \phi_2)) = V_1(\pi - \theta_1, \phi_1) \otimes V_2(\theta_2, -\pi - \phi_2). \quad (27)$$

It turns out that the EWL approach with $\{V(\theta, \phi)\}$ is invariant with respect to strongly isomorphic transformations of input games also in the general, n -person case. In order to use the general entangling operator J defined in (3), it is convenient to consider slightly modified matrix representation of the unitary operators,

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi/2} \cos \frac{\theta}{2} & ie^{i\phi/2} \sin \frac{\theta}{2} \\ ie^{-i\phi/2} \sin \frac{\theta}{2} & e^{-i\phi/2} \cos \frac{\theta}{2} \end{pmatrix}. \quad (28)$$

We see at once that $U(\theta, \phi)$ maps the computational basis $\{|0\rangle, |1\rangle\}$ onto

$$\begin{aligned} |+\rangle_u &= e^{i\phi/2} \cos \frac{\theta}{2} |0\rangle + ie^{-i\phi/2} \sin \frac{\theta}{2} |1\rangle, \\ |-\rangle_u &= ie^{i\phi/2} \sin \frac{\theta}{2} |0\rangle + e^{-i\phi/2} \cos \frac{\theta}{2} |1\rangle. \end{aligned} \quad (29)$$

They are eigenvectors of an observable

$$\begin{pmatrix} \cos \theta & -ie^{i\phi} \sin \theta \\ ie^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix} = \sigma_x \sin \theta \sin \phi + \sigma_y \sin \theta \cos \phi + \sigma_z \cos \theta. \quad (30)$$

Hence, (30) still represents the Stern-Gerlach apparatus singled out by the unit vector

$$\mathbf{u} = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta). \quad (31)$$

We can now formulate the following result:

Proposition 1 Let $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and $\Gamma' = (N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$ be strongly isomorphic strategic form games with $|S_i| = |S'_i| = 2$ and let $\Gamma_{EWL} = (N, (U_i)_{i \in N}, (M_i)_{i \in N})$ and $\Gamma'_{EWL} = (N, (U'_i)_{i \in N}, (M'_i)_{i \in N})$ with U_i and U'_i given by (28) be the corresponding quantum games. Then Γ_{EWL} and Γ'_{EWL} are strongly isomorphic.

Proof Let us consider the generalized EWL scheme defined by (3)–(5) and (28). Without loss of generality we can assume that the map η in the strong isomorphism $f = (\eta, (\varphi_i)_{i \in N})$ between Γ and Γ' is the identity map, i.e., $\eta(i) = i$, for $i \in N$. We prove the proposition directly by determining the general players' payoff functions. As a first step we specify $(\bigotimes_{j=1}^n U_j) J|0\rangle^{\otimes n}$,

$$\begin{aligned} & \left(\bigotimes_{j=1}^n U_j \right) J|0\rangle^{\otimes n} \\ &= \sum_{i_1, \dots, i_n \in \{0,1\}} \frac{1}{\sqrt{2}} e^{i((-1)^{i_1} \phi_1/2 + \dots + (-1)^{i_n} \phi_n/2)} \\ & \quad \left(i^{\sum_{j=1}^n i_j} \prod_{j=1}^n \cos \xi_j + i^{n+1 - \sum_{j=1}^n i_j} \prod_{j=1}^n \sin \xi_j \right) |i_1 \dots i_n\rangle, \end{aligned} \quad (32)$$

where

$$\xi_j = \begin{cases} \theta_j/2 & \text{if } i_j = 0, \\ (\pi - \theta_j)/2 & \text{if } i_j = 1. \end{cases} \quad (33)$$

Since for a state $|\Psi\rangle = \sum c_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle$ we have

$$J^\dagger \sum c_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle = \sum \left(c_{i_1 i_2 \dots i_n} - i c_{\bar{i}_1 \bar{i}_2 \dots \bar{i}_n} \right) |i_1 i_2 \dots i_n\rangle, \quad (34)$$

for $\bar{i}_j = i_j \oplus_2 1$, it follows that

$$J^\dagger \left(\bigotimes_{j=1}^n U_j \right) J |0\rangle^{\otimes n} \quad (35)$$

$$= \left(i^{\sum_{j=1}^n i_j} \cos \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \cos \xi_j \right. \\ \left. + i^{n+2-\sum_{j=1}^n i_j} \sin \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \sin \xi_j \right) |i_1 \dots i_n\rangle. \quad (36)$$

Write $|\Psi_f\rangle = J^\dagger \left(\bigotimes_{j=1}^n U_j \right) J |0\rangle^{\otimes n}$. When n is an odd number, $|\langle i_1 \dots i_n | \Psi_f \rangle|^2$, we have

$$|\langle i_1 \dots i_n | \Psi_f \rangle|^2 = \cos^2 \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \cos^2 \xi_j \\ + \sin^2 \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \sin^2 \xi_j, \quad (37)$$

and when n is even,

$$|\langle i_1 \dots i_n | \Psi_f \rangle|^2 = \left(\cos \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \cos \xi_j \right. \\ \left. - (-1)^{n/2+\sum_{j=1}^n i_j} \sin \left(\sum_{j=1}^n (-1)^{i_j} \phi_j / 2 \right) \prod_{j=1}^n \sin \xi_j \right)^2. \quad (38)$$

If the games Γ and Γ' differ by the order of player 1's strategies, the expected player i 's payoff in the EWL approach to Γ' is of the form

$$v'_i \left(\bigotimes_{i=1}^n U_i(\theta_i, \phi_i) \right) = \sum_{i_1, \dots, i_n \in \{0,1\}} a_{i_1 \dots i_n}^i |\langle i_1 \oplus_2 1, i_2 \dots i_n | \Psi \rangle|^2. \quad (39)$$

Define a game mapping $\tilde{f}$,

$$\tilde{f} \left(\bigotimes_{i=1}^n U_i(\theta_i, \phi_i) \right) = U_1(\theta_1, -\pi - \phi) \otimes \left(\bigotimes_{i=2}^n U_i(\pi - \theta_i, \phi_i) \right). \quad (40)$$

In case n is even (see, formula (38)), term $|\langle i_1 \oplus_2 1, i_2 \dots i_n | \Psi \rangle|^2$ associated with

$$v'_i \left(\tilde{f} \left(\bigotimes_{i=1}^n U_i(\theta_i, \phi_i) \right) \right) \quad (41)$$

becomes

$$|\langle i_1 \oplus_2 1, i_2 \dots i_n | \Psi \rangle|^2 \quad (42)$$

$$= \left(\cos \left((-1)^{i_1 \oplus_2 1} (-\pi - \phi_1)/2 \right) + \sum_{j=2}^n (-1)^{i_j} \phi_j/2 \right) \prod_{j=1}^n \sin \xi_j \quad (43)$$

$$+ (-1)^{n/2 + \sum i_j} \sin \left((-1)^{i_1 \oplus_2 1} (-\pi - \phi_1)/2 \right) + \sum_{j=2}^n (-1)^{i_j} \phi_j/2 \prod_{j=1}^n \cos \xi_j \Big)^2$$

$$= \left(\sin \left(\sum_{j=1}^n (-1)^{i_j} \phi_j/2 \right) \prod_{j=1}^n \sin \xi_j \right.$$

$$\left. - (-1)^{n/2 + \sum i_j} \cos \left(\sum_{j=1}^n (-1)^{i_j} \phi_j/2 \right) \prod_{j=1}^n \cos \xi_j \right)^2$$

$$= |\langle i_1 \dots i_n | \Psi_f \rangle|^2. \quad (44)$$

In this way we obtain

$$v'_i \left(\tilde{f} \left(\bigotimes_{i=1}^n U_i(\theta_i, \phi_i) \right) \right) = v_i \left(\left(\bigotimes_{i=1}^n U_i(\theta_i, \phi_i) \right) \right). \quad (45)$$

Similar arguments apply to the case where n is odd. As a result, the games are strongly isomorphic. $\square$

The following example shows that the converse is not true in general.

Example 2 Let us consider two 2×2 bimatrix games that differ only in the order of payoff vectors in the anti-diagonal, i.e.,

$$\Gamma: \begin{matrix} & l & r \\ t & (a_{00}, b_{00}) & (a_{01}, b_{01}) \\ b & (a_{10}, b_{10}) & (a_{11}, b_{11}) \end{matrix} \quad \text{and} \quad \Gamma': \begin{matrix} & l' & r' \\ t' & (a_{00}, b_{00}) & (a_{10}, b_{10}) \\ b' & (a_{01}, b_{01}) & (a_{11}, b_{11}) \end{matrix} \quad (46)$$

The EWL quantum counterparts Γ_{EWL} and Γ'_{EWL} for these games are specified by $N = \{1, 2\}$, unitary strategies U_i and U'_i of the form (28) and the measurement operators

$$(M_1, M_2) = \sum_{j_1, j_2=0,1} (a_{j_1 j_2}, b_{j_1 j_2}) P_{j_1 j_2}, \quad (M'_1, M'_2) = \sum_{j_1, j_2=0,1} (a_{j_1 j_2}, b_{j_1 j_2}) P_{j_2 j_1}, \quad (47)$$

where $P_{j_1 j_2} = |j_1 j_2\rangle\langle j_1 j_2|$. Let us set a mapping $\tilde{f} = (\eta, (\tilde{\varphi}_1, \tilde{\varphi}_2))$ with $\eta(i) = i$ and

$$\tilde{\varphi}_i(V_i(\theta_i, \phi_i)) = V'_i(\pi - \theta_i, \pi/2 - \phi_i) \quad (48)$$

for $i = 1, 2$. Combining (17) with (48) gives

$$(v'_1, v'_2) \left(\tilde{f}(V_1(\theta_1, \phi_1) \otimes V_2(\theta_2, \phi_2)) \right)$$

$$= (v'_1, v'_2) (V_1(\pi - \theta_1, \pi/2 - \phi_1) \otimes V_2(\pi - \theta_2, \pi/2 - \phi_2))$$

$$= (v_1, v_2) ((V_1(\theta_1, \phi_1) \otimes V_2(\theta_2, \phi_2))). \quad (49)$$

As a result, games produced by Γ_{EWL} and Γ'_{EWL} are strongly isomorphic. This fact, however, is not sufficient to guarantee the isomorphism between Γ and Γ' . Indeed, one can check that there is no $f = (\eta, (\varphi_1, \varphi_2))$ to satisfy $u_i(s) = u'_{\eta(i)}(f(s))$ for each $s \in \{t, b\} \times \{l, r\}$ and $i = 1, 2$. Alternatively, given specific payoff profiles $(a_{00}, b_{00}) = (4, 4)$, $(a_{01}, b_{01}) = (1, 3)$, $(a_{10}, b_{10}) = (3, 1)$, $(a_{11}, b_{11}) = (2, 2)$, we can find three Nash equilibria in game Γ and just one in game Γ' . Hence, by Lemma 1 games (46) are not isomorphic.

5 Conclusions

In the theory of quantum games there are yet no formal axioms and definitions that would give explicit hints how a quantum game should look like. One condition requires the quantum game scheme to generalize the classical (input) game. This allows us to define the scheme in many different ways. In particular, it leaves the freedom of choice of the players' strategy sets in the EWL model. We have studied two parameter unitary operators determined by the basis change rules. We have proved that the quantum scheme so defined is invariant with respect to isomorphic transformations of an input game. Thus, the unitary strategies induced by basic change matrices appear to be reasonable from a game theory viewpoint in the same way as the strategy set being the special unitary group $SU(2)$. However, in contrast to $SU(2)$, the above-mentioned two-parameter strategies do not provide the players with the counter strategies [8]. As a result, pure Nash equilibria may be found in a wider class of games.

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