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Special Issue "Pseudo-Hermitian Hamiltonians in Quantum Physics in 2014"

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To some physicists the title "Pseudo-Hermitian Hamiltonians in quantum physics" (PHHQP) of the present special issue referring to a rather specific mathematical concept might sound enigmatic, so the name probably deserves a brief introductory comment first of all. For the guest-editor's making such an introductory clarification there exist several options. Besides a purely formal reference to the series of international conferences on this particular subject (carrying the same PHHQP name – see their common webpage [1]) or to the related proceedings and/or special issues of various international scientific Journals [2] one could accept an alternative strategy of recalling at least a few most characteristic publications in the field (in this setting, the Daniel Hook's dedicated webpage [3] seems to be the best "bookkeeping" of the current related publication activities). Thirdly (and let me adopt this, less formal approach here) one could merely remind the readers about a few, randomly selected historical roots of the ideas which seem shared by the participants of the above-mentioned traditional conferences and/or by the authors of the related recent publications.

For the purpose let me try to proceed in a parallel to the introductory phrase of ref. [4] and classify the roots as three – not entirely independent – research territories and evolutionary branches. Out of them, the oldest one (let me call it a "historical tendency A") originates in quantum many-body physics. Its basic ideas could be attributed to Freeman Dyson [5]. The mathematical essence of the Dyson's approach to the problem of the construction of bound states may be briefly characterized as an isospectral-mapping transition from a known, "realistic", Hermitian but, alas, prohibitively user-unfriendly Hamiltonian $\mathfrak{h}$ of the system in question to its auxiliary, non-Hermitian but much friendlier representation

$$H = \Omega^{-1}\mathfrak{h}\Omega \qquad (1)$$

where the invertible "intertwiner" Ω should be assumed, for the sake of non-triviality of the analysis, non-unitary. In opposite direction, precisely this assumption broadened the scope of the theory. In other words, the broadened possibilities of the judicious choice of the

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non-unitary Dyson's maps Ω offer also an explanation of the practical computational success of the "operator preconditioning" (1) (cf., e.g., its "interacting boson" applications in nuclear physics as reviewed by Scholtz et al. [6]).

In contrast to recipe "A", the other, much younger historical tendency "B" proceeds in an opposite direction reconstructing, whenever asked for, the complicated (and initially unknown) physical Hamiltonian $\mathfrak{h}$ from an input ansatz (or rather a trial-and-error choice) of a sufficiently elementary form of some elementary non-Hermitian right-hand-side operator $H \neq H^\dagger$ of (1). In this alternative approach, extremely successfully introduced and advocated by Carl Bender with multiple co-authors [7], one must first translate the obligatory, physical Hermiticity condition $\mathfrak{h} = \mathfrak{h}^\dagger$ into the language of Hs. This yields the Dieudonné's [8] relation

$$H^\dagger \Theta = \Theta H \tag{2}$$

Let us abbreviate $\Omega^\dagger\Omega = \Theta$ calling this product a "new Hilbert-space metric" [6, 9]. Now, approach "B" requires that one re-factorizes the metric back into its Dyson-mapping factors, completing the picture and reconstructing the "missing" textbook Hamiltonian $\mathfrak{h}$ whenever necessary.

In 1993, a very nice and mathematically rigorous illustration of the latter, two-representation relationship (1) between alternative Hamiltonian-operator representations has been provided by Buslaev and Grecchi. Indeed, today, their originally half-forgotten paper [10] may be read as one of the first successful derivations of a closed-form Hermitian $\mathfrak{h}$ from a given non-Hermitian input H. Unfortunately, this analytic and exact, non-numerical model proved so exceptional that it took more than ten years before a similar realistic physical sample of the correspondence (1) has been found and described, by Ali Mostafazadeh, in his remarkable innovative application of tendency "B" to first-quantized Klein-Gordon fields [11].

The climax and main success of the Dyson's philosophy of simplification $\mathfrak{h} \to H$ emerged between the years 1998 - 2004 during which Bender et al proposed and defended their ideas of further simplification of the formalism which we might call here "the youngest historical tendency C". Incidentally, the authors themselves already made these ideas well known under the nickname of $\mathcal{PT}-$symmetric quantum mechanics. The essence of their contribution may be briefly characterized as based on the most intuitive perception of the breakdown of spatial parity $\mathcal{P}$ of quantum systems and of an entirely new and enormously inspiring requirement of restoration of the conservation of symmetry after the multiplication of $\mathcal{P}$ by an antilinear time-reversal operator $\mathcal{T}$.

It is a certain paradox that although the motivation of "tendency C" was deeply rooted in physics (typically, its authors decided to require an obligatory observability of a "charge" $\mathcal{C}$), their main achievement may be seen now, unexpectedly, in an unexpectedly efficient simplification of mathematics. Indeed, in their language (see the Bender's own comprehensive review paper [7]), the necessary physical Hilbert-space metric Θ becomes simply equal to product $\Theta^{(Bender)} = \mathcal{PC}$. In parallel, another compact presentation of the latter key merit of the third approach "C" to non-Hermitian quantum Hamiltonians with real bound-state spectra was given, from a perceivably more general perspective (which could be traced back to the nuclear physics context [6]), in the review paper "Pseudo-Hdermitian representation of quantum mechanics" by Ali Mostafazadeh [9]. This author emphasized (cf. also [12] for an origin of this observation) that the Bender's main technical requirement $H\mathcal{PT} = \mathcal{PT}H$ of the *additional, simplifying* $\mathcal{PT}-$symmetry property of the Hamiltonians H should be re-written in the form

$$H^\dagger\mathcal{P} = \mathcal{P}H \tag{3}$$

re-read, in the traditional language of mathematics, as the mere Krein-space Hermiticity condition *alias* **pseudo-Hermiticity** property. In 2003 these terminology-related considerations gave, ultimately, the name to the above-mentioned series of conferences as well as to present special issue.

Naturally, the whole field is developing quickly. As a consequence, the names like $\mathcal{PT}-$symmetry or pseudo-Hermiticity became over-schematic and do not cover the full range of the theory anymore. Thus, also the contents of our present special issue reflect these new evolution tendencies which are not restricted to the reports on developments of quantum theory along the above-outlined traditional lines. In the context of this preface one could easily update the pattern and speak about a "scope-broadening tendency D". Indeed, already a quick glimpse at the list of contributions reveals that besides the multiple "expected" detailed studies of various versions and aspects of $\mathcal{PT}-$symmetric quantum oscillators the readers will find here the papers also about their relativistic extensions and/or about their perturbation and/or path-integral and/or discrete (i.e., e.g., quantum-lattice) simplifications. New exactly solvable quantum models emerge, often together with an enrichment of underlying mathematics. Particular emphasis is being put on the theory of analytic functions and/or on the theory of representations of Jordan or C^* or Lie algebras and of their deformations, etc.

One of the most productive recent innovations of the concept of $\mathcal{PT}-$symmetry may be seen in its transfer to the dynamical regime of scattering. In this issue the readers may find detailed studies of exactly solvable benchmark models. This led, naturally, to the interest in systems in which the $\mathcal{PT}-$symmetry becomes spontaneously broken and/or in which the unitarity is inadvertently lost. In the former context the readers find here several papers on the quantum systems near their exceptional-point degeneracies. In the latter, non-unitary-evolution perception of the models a very natural "next move" is made towards the transfer of the role of $\mathcal{PT}-$symmetry to non-linear Schrödinger equations and, as a consequence, to the semiclassical and even purely non-quantum domains of classical physics and, most typically, optics.

Multiple fresh ideas emerge presented in the new mathematical as well as physical context. First of all, the newly allowed natural emergence of non-linearities gives rise to a number of new questions to be answered. New theoretical initiatives then quickly enter the scene, e.g., via the ideas of possible experimental realizations of $\mathcal{PT}-$symmetry-related mathematics or, vice versa, via a new-wisdom-inspired return to some traditional theoretical concepts involving, typically, supersymmetry and/or non-commutative geometries. New tractability also emerges in connection with the traditional phenomenological challenges involving, e.g., the problems in the quantum information theory and quantum computers as well as the questions concerning the pions and Bose-Einstein condensates, or tachyons, or magnetohydrodynamical systems - just to name a few.

Last but not least, a special attention is being paid here to the studies of open quantum systems. But at this point this preface has to be stopped because at this moment the above-outlined pilgrim's tour of applied non-Hermitian mathematics closes a full circle and returns to the point of view of multiple traditional textbooks on quantum mechanics in which one finds illustrative examples of manifestly non-Hermitian Hamiltonians generating the resonances at complex energies. Naturally, these days even the world of traditional quantum models of resonances did not remain the same as before. Its present mathematical tools as well as phenomenological language were perceivably influenced by the shift of emphasis from the traditional domains like complex resonances to the narrower area in which the strictly real bound states energies appear generated by pseudo-Hermitian Hamiltonians.

There is no final message provided by the contents of this Special Issue. The PHHQP field of research is by far not at the end of its current developments. Open questions abound in both the applications and in mathematics. In 2014 the newest continuation of the series of the PHHQP-related conferences (in Setif, Algeria [13]) was still well attended (see the attached photo of participants) and the series is also going to be continued, in Palermo, in 2015 [14].

References

1. http://gemma.ujf.cas.cz/~znojil/conf/index.html
2. http://gemma.ujf.cas.cz/~znojil/conf/proceedphhqp.html
3. http://ptsymmetry.net
4. Gallia est omnis divisa in partes tres, quarum unam incolunt Belgae, aliam Aquitani, tertiam qui ipsorum lingua Celtae, nostra Galli appellantur. in IVLI CAESARIS COMMENTARIORVM DE BELLO GALLICO LIBER PRIMVS" (58-49 BC)(http://www.thelatinlibrary.com/caesar/gall1.shtml(2008))
5. Dyson, F.J.: Phys. Rev. **102**, 1217 (1956)
6. Scholtz, F.G., Geyer, H.B., Hahne, F.J.W.: Ann. Phys. **213**, 74 (1992)
7. Bender, C.M.: Rep. Prog. Phys. **70**, 947 (2007)
8. Dieudonne, J.: Proc. Int. Symp. Lin. Spaces, Pergamon, pp. 115–122. Oxford (1961)
9. Mostafazadeh, A.: Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010)
10. Buslaev, V., Grechi, V.: J. Phys. A: Math. Gen. **26**, 5541 (1993)
11. Mostafazadeh, A.: Class. Quantum Grav. **20**, 155–171 (2003)
12. Langer, H., Tretter, Ch.: Czech. J. Phys. **54**, 1113 (2004)
13. http://phhqp14.univ-setif.dz/
14. http://www.math.unipa.it/~phhqp15/