# The Dual Action of Fractional Multi Time Hamilton Equations

**Dumitru Baleanu · Ali Khalili Golmankhaneh ·
Alireza Khalili Golmankhaneh**

**Abstract** The fractional multi time Lagrangian equations has been derived for dynamical systems within Riemann-Liouville derivatives. The fractional multi time Hamiltonian is introduced as Legendre transformation of multi time Lagrangian. The corresponding fractional Euler-Lagrange and the Hamilton equations are obtained and the fractional multi time constant of motion are discussed.

## 1 Introduction

Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. For example, in chaotic dynamics, quantum mechanics, plasma physics, anomalous diffusion, and so many fields of physics [8, 10, 11, 15, 17, 18, 24, 29, 31]. Especially in mechanics Riewe has shown that Lagrangian involving fractional time derivatives leads to equation of motion with non conservative classical derivatives such

D. Baleanu (✉)
Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
e-mail: dumitru@cankaya.edu.tr

D. Baleanu
Institute of Space Sciences, PO Box, MG-23, 76900, Magurele, Bucharest, Romania

Alireza K. Golmankhaneh
Department of Physics, University of Pune, Pune 411007, India
e-mail: alireza@physics.unipune.ernet.in

Alireza K. Golmankhaneh · Ali K. Golmankhaneh
Department of Physics, Islamic Azad University-Uromia Branch, PO Box 969 Uromia, Iran

Ali K. Golmankhaneh
e-mail: ali.khalili.gol@gmail.com

as friction [22, 23]. Motivated by this approach many researchers have explored this area giving new insight into this problem [4, 5, 12–14, 16, 19–21, 25–27, 30]. Agrawal has presented fractional Euler-Lagrange equation in Riemann-Liouville derivatives [1]. Further fractional single time Hamiltonian formulation has been developed by Baleanu and coworkers [2, 3]. Classical multi time Hamilton equations were firstly introduced by de Donder in 1936 [7]. In recent years multi-time Hamiltonian and its applications become an interesting area for authors [6, 28]. In this paper we generalize classical multi time Lagrangian and Hamiltonian to fractional multi time motion. We show that the classical dual action can be defined in the fractional case but the connection between the periodic solution of multi-time Hamiltonian and the periodical critical points of the dual action collapses [6]. The paper is organized as follows: Sect. 2 has three subsections. The first subsection presents briefly some basic definitions of Riemann-Liouville and Caputo derivatives, the remaining ones contain the formalism of single time fractional Lagrangian and Hamiltonian. In Sect. 3 we introduce multi time fractional Lagrange equations. Section 4 work out the multi time fractional Hamilton equations. In Sect. 5 we have studied fractional conservation laws. Fractional Noether theorem is offered in the multi time fractional mechanics and we arrive at the standard cases. Further in Sect. 6 suitable Lagrangian for fractional dual action has been derived. In Sect. 7 the Euler-Lagrange equations of the fractional dual action and corresponding multi time Hamilton equations is presented and the connection between the periodic solution of multi-time Hamiltonian and the periodical critical points of the dual action in the fractional mechanics is discussed. Section 8 is devoted to our conclusions.

## 2 Basic Tools

### 2.1 Fractional Derivatives

In this section we briefly present the definition of the left and the right fractional derivatives of Riemann-Liouville as well as Caputo [28]. The left Riemann-Liouville fractional derivative is defined as

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{1}$$

and the right Riemann-Liouville fractional derivative,

$$_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha+1-n}}d\tau, \tag{2}$$

where the order $\alpha$ fulfills $n-1 \le \alpha < n$ and $\Gamma$ represent the gamma function. An alternative definition of Riemann-Liouville fractional derivative called Caputo derivative that introduced by Caputo in 1967. The left Caputo derivative defined as

$$_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{3}$$

and the right Caputo Fractional Derivative

$$_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{4}$$

where the order $\alpha$ satisfies $n - 1 \leq \alpha < n$. The Riemann-Liouville derivative of constant isn't zero, although Caputo derivative of a constant is zero.

## 2.2 Single Time Fractional Lagrangian

From the mathematical point of view both left and right derivatives could appear within a fractional Lagrangian of a system, $L(t, q^i(t), {}_aD_t^\alpha q^i, {}_tD_b^\beta q^i)$, $i = 1, \ldots, n$ where $(q^i(t)) = (q^1(t), q^2(t), \ldots, q^n(t))$, but in some physics problems, if $t$ denotes the time variable, the right fractional derivatives of $q^i(t)$ is interpreted as future state of the process $q^i(t)$. For this reason in those problems the right derivative is usually neglected in application when the present state of the process does not depend on the result of the future or at least we consider this sort of problems. Let us consider a Lagrangian as $L(t, q^i, {}_aD_t^\alpha q^i)$. The Lagrangian $L : R \times R^{2n} \to R$, $(t, q^i, {}_aD_t^\alpha q^i) \to L(t, q^i, {}_aD_t^\alpha q^i)$ and $({}_aD_t^\alpha q^i) = ({}_aD_t^\alpha q^1(t), {}_aD_t^\alpha q^2(t), \ldots, {}_aD_t^\alpha q^n(t))$ produces the single time fractional Euler-Lagrange equation given below

$$\frac{\partial L}{\partial q^i} + {}_tD_b^\alpha \frac{\partial L}{\partial {}_aD_t^\alpha q^i} = 0, \quad i = 1, 2, \ldots, n. \tag{5}$$

## 2.3 Single Time Fractional Hamilton Equations

The Lagrangian $L$ defines the diffeomorphism, ${}_aD_t^\alpha q^i(t) \to p_i = \frac{\partial L}{\partial {}_aD_t^\alpha q^i}$ called the Legendre transformation. The Legendre transformation of the Lagrangian $L$ is the Hamiltonian $H$. If $p_i = \frac{\partial L}{\partial {}_aD_t^\alpha q^i}$ be fractional $i$-momentum we obtain the fractional multi time Hamiltonian as follows

$$H(t, q^i, p_i) = {}_aD_t^\alpha q^i \frac{\partial L}{\partial {}_aD_t^\alpha q^i} - L. \tag{6}$$

Calculating the total differential of this Hamiltonian we obtain

$$dH = {}_aD_t^\alpha q^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial t} dt. \tag{7}$$

By making use of

$$dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i, \tag{8}$$

and by using the fractional Euler-Lagrange, we get the fractional Hamilton equations as

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t},$$
$$\frac{\partial H}{\partial p_i} = {}_aD_t^\alpha q^i, \tag{9}$$
$$\frac{\partial H}{\partial q^i} = {}_tD_b^\alpha p_i.$$

It is worth to mention that for $\alpha \to 1$ equations reduce to classical Hamilton equations. The fractional Hamiltonian isn't a constant of motion even though the Lagrangian doesn't depend on the time explicitly. This can be observed by making use of

$$\frac{dH}{dt} = \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} + \frac{\partial H}{\partial q^i} \frac{dq^i}{dt}, \tag{10}$$

and by substituting the values of partial derivatives of Hamiltonian from (9). As a result we obtain that

$$\frac{dH}{dt} = {_aD_t^\alpha q^i} \frac{dp_i}{dt} + {_tD_b^\alpha p_i} \frac{dq^i}{dt} \neq 0. \tag{11}$$

Using the symplectic structure

$$J = \begin{pmatrix} 0 & \delta_j^i \\ \delta_j^i & 0 \end{pmatrix}, \tag{12}$$

we may write the Hamilton equations in the matrix form

$$-\begin{pmatrix} 0 & \delta_j^i \\ \delta_j^i & 0 \end{pmatrix} \begin{pmatrix} {_aD_t^\alpha q^j} \\ {_tD_b^\alpha p_j} \end{pmatrix} + \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{13}$$

## 3 Multi Time Fractional Lagrange Equations

We consider the multi time variable

$$(t^\nu) = (t^1, t^2, \dots, t^p) \in R^p, \quad \nu = 1, \dots, p. \tag{14}$$

Let $a = (a^\nu)$ and $b = (b^\nu)$ be the borders of the integration and let us consider the functions given below

$$q^i : R^p \rightarrow R, \quad (t^1, t^2, \dots, t^p) \rightarrow q^i(t^1, t^2, \dots, t^p), \quad i = 1, \dots, n. \tag{15}$$

Multi time Lagrangian is considered to be function of left fractional derivatives as

$$L : R^{p+n+np} \rightarrow R, \quad (t^\nu, q^i, {_aD_{t^\nu}^\alpha q^i}) \rightarrow L(t^\nu, q^i, {_aD_{t^\nu}^\alpha q^i}). \tag{16}$$

By making use of the fractional variational principles the corresponding fractional Euler-Lagrange equations are obtained as follows

$$\frac{\partial L}{\partial q^i} + {_{t^\nu}D_b^\alpha} \frac{\partial L}{\partial {_aD_{t^\nu}^\alpha q^i}} = 0. \tag{17}$$

## 4 Multi Time Fractional Hamilton Equations

In 1935 de Donder obtained the multi time Hamilton equation in classical mechanics [7]. Our aim is to obtain the fractional generalization of this result. By using the definition of the fractional momenta as follows

$$p_i^\nu = \frac{\partial L}{\partial {_aD_{t^\nu}^\alpha q^i}}, \tag{18}$$

we define the fractional Hamiltonian as

$$H = {_aD_{t^\nu}^\alpha q^i} \frac{\partial L}{\partial {_aD_{t^\nu}^\alpha q^i}} - L(t^\nu, q^i, {_aD_{t^\nu}^\alpha q^i}). \tag{19}$$

The transformation $_aD_{t^\nu}^\alpha q^i \to p_i^\nu$ defines the fractional Legendre transformation in multi time case. Taking total differential of (19) and using (18) and canceling two terms we have

$$dH = -\frac{\partial L}{\partial t^\nu}dt^\nu - \frac{\partial L}{\partial q^i}dq^i + {}_aD_{t^\nu}^\alpha q^i \, dp_i^\nu. \tag{20}$$

Taking into account the fractional Euler-Lagrange equation (17) and (18) we obtain

$$dH = -\frac{\partial L}{\partial t^\nu}dt^\nu + {}_{t^\nu}D_b^\alpha p_i^\nu \, dq^i + {}_aD_{t^\nu}^\alpha q^i \, dp_i^\nu. \tag{21}$$

Equation (21) shows that $H$ is a function of $t^\nu, q^i, p_i^\nu$. Therefore we can write

$$dH(t^\nu, q^i, p_i^\nu) = \frac{\partial H}{\partial t^\nu}dt^\nu + \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i^\nu}dp_i^\nu. \tag{22}$$

Comparing (22) and (21) we obtain

$$\begin{aligned}
\frac{\partial H}{\partial t^\nu} &= -\frac{\partial L}{\partial t^\nu}, \quad \nu = 1, \ldots, p, \\
\frac{\partial H}{\partial q^i} &= {}_{t^\nu}D_b^\alpha p_i^\nu, \quad i = 1, \ldots, n, \nu = 1, \ldots, p, \\
\frac{\partial H}{\partial p_i^\nu} &= {}_aD_{t^\nu}^\alpha q^i, \quad i = 1, \ldots, n, \nu = 1, \ldots, p,
\end{aligned} \tag{23}$$

The number of (23) is $(n+1)p + n$ equations and they are the counterpart of the Euler-Lagrange equation on $R^n$.

## 5 Fractional Conservation Laws

In classical mechanics when Lagrangian is autonomous Hamiltonian is not a constant of motion [6]. In this section we study conservation laws from the existence of symmetries of the fractional variational principles in tow cases.

  *Case 1.* Suppose the functional as follows

$$I = \int_a^b L(t^\nu, q^i, {}_aD_{t^\nu}^\alpha q^i, {}_{t^\nu}D_b^\beta q^i) dt, \tag{24}$$

under the appropriate conditions. If (24) is invariant under many parameter group of infinitesimal transformations as

$$\begin{aligned}
\delta t^\nu &= \epsilon_r X_r^\nu, \quad r = 1, \ldots, R, \\
\delta q^i(t) &= \epsilon_r \psi_{ir}, \quad r = 1, \ldots, R,
\end{aligned} \tag{25}$$

where $R$ is the number of infinitesimal transformation parameters. Using fractional Noether-type theorem [9], the fractional conservation law is

$$C = \left[\frac{\partial L}{\partial {}_aD_{t^\nu}^\alpha q^i} - \frac{\partial L}{\partial {}_{t^\nu}D_b^\beta q^i}\right]\psi_{ir} + \left[L\delta_\nu^\mu - \alpha\frac{\partial L}{\partial {}_aD_{t^\mu}^\alpha q^i} {}_aD_{t^\nu}^\alpha q^i - \beta\frac{\partial L}{\partial {}_{t^\mu}D_b^\beta q^i} {}_{t^\nu}D_b^\beta q^i\right]X_r^\nu, \tag{26}$$

in the sense of the following definition [9]

$$D_{t^\nu}^\alpha \{(fg)\} = -f \sum_\mu {}_{t^\mu} D_b^\alpha g + g \sum_\mu {}_a D_{t^\mu}^\alpha f. \tag{27}$$

In (26) if we choose $\mu = 1$ and $\nu = 1$ we have corresponding conservation law for the case of single time fractional Lagrangian [9].

Equation (27) for multi time classical mechanics becomes divergence in time sense as follows

$$D(fg) = \sum_\mu \frac{\partial}{\partial t_\mu}(fg). \tag{28}$$

*Case 2.* Let us consider the following functional

$$I = \int_a^b L(t^\nu, q^i, q_\nu^i) dt, \tag{29}$$

where $q_\nu^i = (\frac{dq^i}{dt^\nu})$. Likewise case 1 if we suppose the functional (29) is invariant under the group of infinitesimal transformations then the conservation law is

$$C = \frac{\partial L}{\partial q_\nu^i} \psi_{ir} + \left( L \delta_\nu^\mu - \frac{\partial L}{\partial q_\mu^i} q_\nu^i \right) X_r^\nu. \tag{30}$$

In (30) let the variations are $\psi_{ir} = 0$ and $X_r^\nu = \delta_r^\nu$ then

$$C = T_{r\mu} = L \delta_r^\mu - \frac{\partial L}{\partial q_\mu^i} q_r^i,$$

energy tensor is conservation law.

Let us consider the group of infinitesimal transformations for single time Lagrangian as follows

$$\delta t = \epsilon_r X_r, \qquad \delta q^i = \epsilon_r \Psi_{ri}. \tag{31}$$

The corresponding conservation law is

$$C = \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \psi_{ir} + \left( L - \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \dot{q}^i \right) X_r. \tag{32}$$

In (32) let $\psi_{ir} = 0$ and $X_r = \delta_{r1}$, then the conservation law reduces to Hamiltonian,

$$H = L - \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \dot{q}^i. \tag{33}$$

As we have seen the fractional conservation law in special cases reduces to well known conservation law in classical mechanics.

## 6 Legendre Transform of Fractional Multi Time Hamiltonian Dual Action

We write the fractional multi time Hamilton equations in the form

$$-\delta_\mu^\nu \delta_j^i {}_{t^\nu} D_b^\alpha p_i^\mu + \frac{\partial H}{\partial q^j} = 0,$$

$$i = 1, \ldots, n, j = 1, \ldots, n, \nu = 1, \ldots, p, \mu = 1, \ldots, p, \tag{34}$$

$$-\delta_\mu^\nu \delta_{ja}^i D_{t^\nu}^\alpha q^i + \frac{\partial H}{\partial p_j^\mu} = 0,$$

$$i = 1, \ldots, n, j = 1, \ldots, n, \nu = 1, \ldots, p, \mu = 1, \ldots, p, \tag{35}$$

or in matrix form

$$-\begin{pmatrix} 0 & \delta_\mu^\nu \delta_j^i \\ \delta_\mu^\nu \delta_j^i & 0 \end{pmatrix} \begin{pmatrix} {}_a D_{t^\nu}^\alpha q^i \\ {}_{t^\nu} D_b^\alpha p_i^\mu \end{pmatrix} + \begin{pmatrix} \frac{\partial H}{\partial q^j} \\ \frac{\partial H}{\partial p_j^\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{36}$$

Using the notation

$$\begin{pmatrix} 0 & \delta_\mu^\nu \delta_j^i \\ \delta_\mu^\nu \delta_j^i & 0 \end{pmatrix} = \delta \otimes J, \tag{37}$$

we can write (34) and (35) as follows

$$-(\delta \otimes J) \begin{pmatrix} {}_a D_{t^\nu}^\alpha q^i \\ {}_{t^\nu} D_b^\alpha p_i^\mu \end{pmatrix} + \begin{pmatrix} \frac{\partial H}{\partial q^j} \\ \frac{\partial H}{\partial p_j^\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{38}$$

The action $\Psi$ whose fractional Euler-Lagrange equations are fractional Hamilton equations is

$$\Psi = \int_{T_0} \mathcal{L}(q^i, {}_a D_{t^\nu}^\alpha q^i, p_i^\nu, {}_{t^\nu} D_b^\alpha p_i^\nu) dt, \tag{39}$$

where $dt = dt^1 dt^2 \cdots dt^p$ and $T_0 = [0, T^1] \times \cdots \times [0, T^p]$,

$$\mathcal{L} = -\frac{1}{2}(q_{t^\nu}^i D_b^\alpha p_i^\nu + p_i^\nu {}_a D_{t^\nu}^\alpha q^i) + H(t^\nu, q^i, p_i^\nu) \tag{40}$$

$$= -\frac{1}{2} \begin{pmatrix} {}_{t^\nu} D_b^\alpha p_j^\nu & {}_a D_{t^\mu}^\alpha q^j \end{pmatrix} \begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta^{\mu\nu} \delta_{ij} \end{pmatrix} \begin{pmatrix} q^i \\ p_i^\nu \end{pmatrix} + H(t^\nu, q^i, p_i^\nu). \tag{41}$$

Since we can write

$$\begin{pmatrix} {}_{t^\nu} D_b^\alpha p_j^\nu & {}_a D_{t^\mu}^\alpha q^j \end{pmatrix} = (\delta \otimes J) D_{t^\nu}^\alpha u, \tag{42}$$

and the standard Riemannian metric from $R^{n+np}$ is

$$G = \begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta^{\mu\nu} \delta_{ij} \end{pmatrix}, \tag{43}$$

also if we suppose $u = (q^i \ p_i^\nu)$, $D_{t^\nu}^\alpha u = ({}_a D_{t^\nu}^\alpha q^i \ {}_{t^\nu} D_b^\alpha p_i^\nu)$ and $\langle, \rangle$ is scalar product and using (42), then we have

$$\mathcal{L} = -\frac{1}{2} \langle (\delta \otimes J) D_{t^\nu}^\alpha u, u \rangle + H(t^\nu, u). \tag{44}$$

We have fractional Lagrange equations as given below

$$\frac{\partial \mathcal{L}}{\partial q^i} + {}_{t^\nu}D_b^\alpha \frac{\partial \mathcal{L}}{\partial {}_a D_{t^\nu}^\alpha q^i} = 0, \quad i = 1, 2, \ldots, n, \tag{45}$$

$$\frac{\partial \mathcal{L}}{\partial p_i^\nu} + {}_a D_{t^\nu}^\alpha \frac{\partial \mathcal{L}}{\partial {}_{t^\nu} D_b^\alpha p_i^\nu} = 0, \quad i = 1, 2, \ldots, n. \tag{46}$$

We obtain the Legendre transformation of $H(t^\nu, .)$ as follows

$$\mathcal{H}^*\left(t^\nu, \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i^\nu}\right) = -q^i \frac{\partial H}{\partial q^i} - p_i^\nu \frac{\partial H}{\partial p_i^\nu} + H(t^\nu, q^i, p_i^\nu). \tag{47}$$

Substituting Hamilton equations (34) and (35) into (47) we get

$$\mathcal{H}^*(t^\nu, -{}_a D_{t^\nu}^\alpha q^i, -{}_{t^\nu}D_b^\alpha p_i^\nu) = -q_{t^\nu}^i D_b^\alpha p_i^\nu - p_{ia}D_{t^\nu}^\alpha q^i + H(t^\nu, q^i, p_i^\nu). \tag{48}$$

By using the definition of scalar product we obtain

$$\mathcal{H}^*(t^\nu, -{}_a D_{t^\nu}^\alpha q^i, -{}_{t^\nu}D_b^\alpha p_i^\nu) = -\langle (\delta \otimes J)D_{t^\nu}^\alpha u, u\rangle + H(t^\nu, q^i, p_i^\nu). \tag{49}$$

Since we have obtain the Legendre transform of $H(t^\nu, .)$, $\mathcal{H}^*(t^\nu, .)$, we would like to write $\mathcal{L}$ versus $\mathcal{H}$, namely

$$\mathcal{L} = \frac{1}{2}\langle (\delta \otimes J)D_{t^\nu}^\alpha u, u\rangle - \langle (\delta \otimes J)D_{t^\nu}^\alpha u, u\rangle + H(t^\nu, u)$$

$$= \frac{1}{2}\langle (\delta \otimes J)D_{t^\nu}^\alpha u, u\rangle + \mathcal{H}^*. \tag{50}$$

Let us define new variables

$$v = (Q, P) = (-q^i, -p_i^\nu), \tag{51}$$

$\mathcal{L}$ in terms of new variables will be denoted by $\mathcal{L}^*$. Therefore, we have the following equalities,

$$(\delta \otimes J)D_{t^\nu}^\alpha v = -(\delta \otimes J)D_{t^\nu}^\alpha u, \tag{52}$$

and

$$\langle (\delta \otimes J)D_{t^\nu}^\alpha v, v\rangle = \langle (\delta \otimes J)D_{t^\nu}^\alpha u, u\rangle. \tag{53}$$

As a result $\mathcal{L}^*$ becomes

$$\mathcal{L}^* = \frac{1}{2}\langle (\delta \otimes J)D_{t^\nu}^\alpha v, v\rangle + \mathcal{H}^*(t, (\delta \otimes J)D_{t^\nu}^\alpha v). \tag{54}$$

We see that $\mathcal{L}^*$ is only function of new variable $v$ and $(\delta \otimes J)D_t^\alpha v$.

## 7 Extremal of Dual Action and Multi Time Hamilton Equations

The dual action $\Phi$ that is defined on suitable functions uses the following Lagrangian

$$\mathcal{L}^*(t, v, D_t v) = \frac{1}{2}\langle (\delta \otimes J)D_t v, v(t)\rangle + \mathcal{H}^*(t, (\delta \otimes J)D_t v). \tag{55}$$

Although in classical dual action for periodic functions we have $\Phi(v + c) = \Phi(v)$, in fractional dual action this condition won't be true for periodic solution generally. Since $\int D_t c + \int D_t v \neq 0$ generally,

$$\Phi(v + c) = \int \frac{1}{2}\langle(\delta \otimes J)D_t(v + c), v(t) + c\rangle$$
$$+ H^*(t, (\delta \otimes J)D_t(v + c)) \neq \Phi(v). \tag{56}$$

In classical mechanics we have $\int \frac{d}{dt}c = 0$ because the derivative integer order of constant is zero. Also we have $\int \frac{d}{dt}v = 0$ if $v$ be a periodic function. Then we have $\Phi(v + c) = \Phi(v)$.

In classical multi time Hamiltonian there is close connection between the periodical critical points of $\Phi$ and periodical solutions of multi time Hamiltonian and but this connection fails for fractional order derivative for the reason that we mentioned above [6]. But a theorem is still true for fractional case.

**Theorem** *If $v$ be a critical point of the action $\Phi$ and*

$$w(t) = \nabla\mathcal{H}^*(t, v^*),$$
$$D_t^\alpha v = (-_a D_{t^\nu}^\alpha q^i, -_b D_{t^\nu}^\alpha p_i^\nu),$$
$$v^* = (-_b D_{t^\nu}^\alpha p_i^\nu, -_a D_{t^\nu}^\alpha q^i) = (\delta \otimes J)D_t^\alpha v = (v_i^{*\nu}, v^{*i}),$$

*then we have*

$$(\delta \otimes J)D_t^\alpha w = \nabla H(t, w). \tag{57}$$

*Proof* Suppose

$$w = (z^i, r_i^\nu), \qquad a = (a^\nu), \qquad b = (b^\nu), \quad \nu = 1, \dots, p. \tag{58}$$

From the definition of dual action we have

$$v = (-q_i, -p_i^\nu) \quad \text{and} \quad v^* = (-_b D_{t^\nu}^\alpha p_i^\nu, -_a D_{t^\nu}^\alpha q^i).$$

Then

$$\mathcal{L}^* = \frac{1}{2}(_{t^\nu}D_b^\alpha p_i^\nu q^i +_a D_{t^\nu}^\alpha q^i p_i^\nu) + \mathcal{H}^*(t, -_{t^\nu}D_b^\alpha p_i^\nu, -_a D_{t^\nu}q^i). \tag{59}$$

If $v = (-q^i, -p_i^\nu)$ is a critical point of the dual action, then $\mathcal{L}^*$ satisfies the fractional Euler-Lagrange equations, namely

$$\frac{\partial \mathcal{L}^*}{\partial q^i} + _{t^\nu}D_b^\alpha \frac{\partial \mathcal{L}^*}{\partial_a D_{t^\nu}^\alpha q^i} = 0, \tag{60}$$

$$\frac{\partial \mathcal{L}^*}{\partial p_i^\nu} + _a D_{t^\nu}^\alpha \frac{\partial \mathcal{L}^*}{\partial_{t^\nu} D_b^\alpha p_i^\nu} = 0. \tag{61}$$

Substituting (59) into (60) we have

$$-\frac{1}{2}_{t^\nu}D_b^\alpha p_i^\nu + _{t^\nu}D_b^\alpha\left(\frac{\partial \mathcal{H}^*}{\partial_a D_{t^\nu}^\alpha q^i} - \frac{1}{2}p_i^\nu\right) = 0, \tag{62}$$

$$_{t^\nu} D_b^\alpha \left( \frac{\partial \mathcal{H}^*}{\partial_a D_{t^\nu}^\alpha q^i} \right) = {}_{t^\nu} D_b^\alpha p_i^\nu. \tag{63}$$

Also, substituting (59) into (61) we have

$$-\frac{1}{2} {}_a D_{t^\nu}^\alpha q^i + {}_a D_{t^\nu}^\alpha \left( \frac{\partial \mathcal{H}^*}{\partial_{t^\nu} D_b^\alpha p_i^\nu} - \frac{1}{2} q^i \right) = 0, \tag{64}$$

$$_a D_{t^\nu}^\alpha \left( \frac{\partial \mathcal{H}^*}{\partial_{t^\nu} D_b^\alpha p_i^\nu} \right) = {}_a D_{t^\nu}^\alpha q^i. \tag{65}$$

Let define new variables as below

$$z^i = \frac{\partial \mathcal{H}^*}{\partial_a D_{t^\nu}^\alpha q^i}, \tag{66}$$

$$r_i^\nu = \frac{\partial \mathcal{H}^*}{\partial_{t^\nu} D_b^\alpha p_i^\nu}. \tag{67}$$

Applying (66) and (67) in (63) and (65) we get

$$_a D_{t^\nu}^\alpha z^i = {}_a D_{t^\nu}^\alpha q^i, \tag{68}$$
$$_{t^\nu} D_b^\alpha r_i^\nu = {}_{t^\nu} D_b^\alpha p_i^\nu.$$

Since $\omega = (z^i, r_i^\nu) = \nabla \mathcal{H}^*(t, v^{*i}, v_i^{*\nu})$ and in view of reversibility condition [6] we have

$$({}_{t^\nu} D_b^\alpha p_i^\nu, {}_a D_{t^\nu}^\alpha q^i) = \nabla H(t, \omega). \tag{69}$$

Using (68) we have $({}_{t^\nu} D_b^\alpha r_i^\nu, {}_a D_{t^\nu}^\alpha z^i) = \nabla H(t, w)$, then we obtain

$$_{t^\nu} D_b^\alpha r_i^\nu = \nabla_{z^i} H(t, z^i, p_i^\nu), \tag{70}$$

and

$$_a D_{t^\nu}^\alpha z^i = \frac{\partial H}{\partial r_i^\nu}, \tag{71}$$

respectively. If $w = (z^i, r_i^\nu)$ then

$$({}_{t^\nu} D_b^\alpha r_i^\nu, {}_a D_{t^\nu}^\alpha z^i) = \nabla H(t, w) \quad \text{or} \quad (\delta \otimes J) D_{t^\nu}^\alpha w = \nabla H(t, w), \tag{72}$$

as required.                                                                                          □

*Remark 1* If we change the derivative to Caputo derivative for the derivative of constant will be zero, ${}_a^c D_t^\alpha c = 0$ but we still have $\int_a^c D_t^\alpha v \neq 0$. May be useful to know the relation between the periodic functions and fractional integrals

$$_a I_t^\alpha {}_a^c D_t^\alpha f(t) = f(t) - f(a),$$

where $\alpha \in (0, 1]$ and $_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau$. If $f(t)$ be periodic function in $t \in [a, T]$, we have

$$_a I_t^\alpha {}_a^c D_T^\alpha f(t) = f(T) - f(a) = 0.$$

*Remark 2* We recovered the classical results for $\alpha$ integers.

## 8 Conclusion

Fractional mechanics describes both conservative and non-conservative systems. Recently, the fractional variation principles gained importance in studying the fractional mechanics and several versions are proposed. The fractional Lagrangian and Hamiltonian formulation for the constrained systems are still at the beginning of their development. Fractional mechanics is a non-local theory. The fractional derivatives were renormalized to construct local fractional differential operators. Using fractional Caputo derivative in fractional mechanics allows utilization of initial values of classical integer-order derivative with known physical interpretations. In this paper we generalized classical multi time Hamiltonian to fractional multi time Hamiltonian. Constant of motion in fractional multi time Hamiltonian has been introduced. Further we have shown that fractional multi time mechanics leads to the standard cases. Furthermore the dual action is defined for fractional mechanics. We have explained that the connection between the periodical solution of this dual action and fractional Hamiltonian fails for the formalism involving Riemann-Liouville derivatives. All classical results are recovered for $\alpha$ being an integer.

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