Editorial note to:
Wolfgang Kundt and Manfred Trümper, Contributions
to the theory of gravitational radiation fields.
Exact solutions of the field equations of the general
theory of relativity V

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This is the fifth and last of the famous ‘Hamburg bible’ series to appear as a Golden Oldie (for the first four see [1–4]). The series as a whole was described in Ellis’s editorial note to the translation of paper I [1] and its several authors are named on the title page of the present paper. This fifth contribution [5] summarizes key points of the earlier papers and applies them, in particular results from papers II and IV, in the context of the propagation of gravitational radiation when matter is present. As Ellis noted, the paper “is noteworthy for the systematic use of the covariant full Bianchi identities to determine exact properties of such solutions with null (type N) Weyl tensors, proving interesting non-existence theorems for the matter flows in these cases and examining properties of pure radiation fields.” The study of algebraically special solutions with perfect fluids, which this paper began, now fills a chapter (Chapter 33) of [6], and many algebraically special solutions with electromagnetic or pure radiation fields appear elsewhere in Part III of [6].

The paper opens by giving the field equations and Bianchi identities. In the latter the decomposition of curvature (1.1.3) is used, leading to the equivalent set of equations (1.1.6) and (1.1.7), and the identity (1.1.10). As the authors remark, “(1.1.9) are the equations of motion for the sources, (1.1.10) are differential equations for the free part

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of the field, and (1.1.8) give the interaction between the sources and the free part of the field.”

The next section gives some of the formulae for matter discussed by Ehlers [4], in the special case of a perfect fluid, and substitutes these into the equations of the first section, slightly generalizing Ehlers’ version of (1.2.8) by allowing non-barotropic fluids. Section 1.3 then opens the issue of finding solutions containing both perfect fluid and a gravitational null field (a field of Petrov type N). Type N fields had been identified in earlier work as the “far fields of spatially bounded sources of radiation”, and the objective here was to begin to study how such radiation linked to the fields’ sources. The formalism and equations used, based on the tangent vector k^a to the null rays, largely follow paper II of the series [2].

Theorem 1.3.1 summarizes the resulting constraints on the properties of the fluid, and in particular shows that the only conformally flat perfect fluid spacetimes with an equation of state $\mu = \mu(p, t)$ are the Friedman universes (a converse appears as Theorem 1.4.1). In [7], Ellis ascribes this result to Trümper and notes that if the equation of state assumption is dropped, inhomogeneous models are allowed [8]. In fact, all conformally flat perfect fluids were found in the same year as [8] by Stephani [9]; see also [6, Theorem 37.17]. They are either generalized Schwarzschild or generalized Friedman solutions.

Section 1.5 ends Part 1 with a discussion of Petrov N fields with an irrotational fluid. Evaluating the previous equations under this restriction leads to the Theorem 1.5.1 that there are no such solutions if the fluid is ‘dust’ (i.e. pressureless) or if the fluid flow is geodesic. It was later shown that even if rotation is allowed there are still no solutions with geodesic flow [10]. Kundt and Trümper end by inferring that there could only be irrotational solutions if $\mu = p + A(t)$, which they discard on physical grounds: these solutions were later determined by Oleson [10, 11], cf. [6, section 33.4]. See also [12].

Part 2 of the paper considers twist-free pure radiation Petrov type N fields. Here the definition of “pure radiation” is somewhat different from that used by [6], where it means just that the energy momentum tensor takes the form

$$T_{ab} = \Phi^2 k_a k_b \quad (1)$$

for some null vector k^a . The definition used in the paper is given in section 4 of Chapter 2. It refers specifically to the Einstein–Maxwell case and defines such solutions as being pure radiation when an eigenvector of the Maxwell tensor is tangent to a shearfree null geodesic congruence (a “ray congruence” as defined in the paper).

As the authors state, the notation and results of paper II in the series [2] are used heavily: the main change is that the k^a of paper II becomes ℓ^a here. Section 2.2 reviews the properties of geodesic null congruences: the correspondence with the near contemporary, and now widely-used, Newman–Penrose notation [13], as used in [6], is that z , σ , β , Ω and ζ are respectively the NP quantities $-\rho$, $-\bar{\sigma}$, $\gamma + \bar{\gamma}$, τ and $(\bar{\alpha} + \beta)$.

Theorem¹ 2.1 states an easy consequence of the NP equation for the derivative of ρ along the rays, which complements a result in paper II. There it was shown that when $R_{ab}\ell^a\ell^b = 0$ (with the present ℓ^a) $\sigma = \theta = 0 \Rightarrow \omega = 0$; here it is shown that similarly $\omega = \theta = 0 \Rightarrow \sigma = 0$.

The next section considers Maxwell fields in Jordan's gravitational theory, in which the coefficient κ between the curvature and energy-momentum terms in Einstein's field equations becomes a variable. This scalar-tensor theory of gravity is nowadays usually referred to as Brans–Dicke or Jordan–Brans–Dicke (JBD) theory (for a history of the origins of this name and the genesis of the theory see [14]). It is still widely used as an illustrative modified theory, although its value as a possible replacement for general relativity is severely constrained by observations made later than 1962 (see e.g. [15]). The JBD theory is often discussed in the form (3.5) obtained by using a conformal transformation found by Schucking, assuming the coefficient $\zeta \neq 3/2$ (note this is not the same ζ as in (2.2)). One has to be careful about which metric, g_{ab} or $\tilde{g}_{ab}$, is the one on whose geodesics test particles move. That choice leads to physically distinct gravity theories.

As the authors point out, in the case where κ is constant, the Jordan–Maxwell theory becomes Einstein–Maxwell theory. They then prove Theorem 3.1 for the Einstein–Maxwell case. It assumes a rather generic additional matter tensor and $\sigma = \theta = 0$. The first part has the same conclusion as Theorem 2.1, and the second shows that ℓ^a is an eigenvector of the Maxwell tensor, while the third part shows the additional energy-momentum must vanish. Thus such a setup does not allow study of radiation within a source, and therefore the remaining discussion assumes there is no energy-momentum additional to the Maxwell field.

Theorems 3.3 and 3.4 are the Mariot–Robinson theorem and its corollary, and the Kundt–Thompson theorem (cf. [6, Chapter 7]). The latter generalizes the Goldberg–Sachs theorem [16]: see also [17, 18]. The proof is not given here (and part of it is also not given in [6]). The theorem's formulation here differs slightly from that of Pirani [6, 19], which says that any two of (A), (B) and (C) imply the third. Here the result is stated as (A) $\Leftrightarrow$ (B) if and only if (C). However, the implication (A) and (B) $\Rightarrow$ (C) is quite easy to prove.

This is followed by two more results. The first, Lemma 3.5, shows that (B) $\Rightarrow$ (C) if certain conditions apply to the Ricci tensor, and thus that (B) $\Rightarrow$ (A) in the Einstein–Maxwell case if an eigenvector of the Maxwell field defines a ray congruence. The second, Theorem 3.6, shows that except in a special case an aligned Maxwell field in an algebraically special spacetime will define a (shearfree) ray congruence ('alignment' here meaning that of an eigenvector of the Maxwell field and the repeated principal null direction of the Weyl tensor). While the result, which is credited to Trümper, is interesting, the proof method may have been more influential, because here the authors turn to the use of the two-component spinor formalism and use the Bianchi identities to obtain (in the NP notation)

$$(2\kappa\Phi_1\bar{\Phi}_1 + 3\psi_2)\sigma = 0, \quad (-2\kappa\Phi_1\bar{\Phi}_1 + 3\psi_2)\kappa = 0 \quad (2)$$

¹ The equations and theorems in Chapter 2, unlike those of Chapter 1, are not labelled with the Chapter number as a leading digit.

from which the result follows, the exceptional case being where

$$4(\kappa\Phi_1\overline{\Phi}_1)^2 = 9(\Psi_2)^2.$$

Section 4 proposes and discusses the paper's definition of pure radiation and its extension to the Jordan–Maxwell case, and defines various subclasses. Section 5 goes on to consider the choice of coordinates in the twist-free subclass: the diverging cases ($\theta \neq 0$) are included in the Robinson–Trautman class [20], see also [6, Section 28.2], and the non-diverging cases in the Kundt class [21], see also [6, Section 31.2]. Table 5.1 gives a detailed list of possible specializations of coordinates within the Kundt class under various assumptions and in Table 7.1 a list of all the possible “pure radiation” classes, and the arbitrary functions in them, is given.

The remainder of section 5 considers invariants in the Robinson–Trautman subclass. This is related to the use of invariants in the “equivalence problem” [22]: for a later treatment of the latter see [6, Chapter9].

Section 6 complements the studies of Robinson and Trautman [20] and reference [18] of the paper with some results for spacetimes with additional fields, starting with the generalization to the Jordan–Maxwell case. Theorem 6.1 is required for Theorem 6.2 which considers the case with a “pure radiation” field and an additional Maxwell field. For the case of a Maxwell null field, Theorem 6.3 gives the reduction of the field equations.

After Table 7.1 the plane-fronted, *pp*, and plane wave cases are distinguished, and references to the known solutions are given. The authors then note that Table 7.1 also applies with pure radiation in the sense of Eq. (1) above, i.e. as in [6], see equation (7.1) here. Theorem 7.1 states the existence of an Einstein–Maxwell solution if there is a null Maxwell field with twist-free rays, and derives the restrictions, for the expanding case, on the Weyl tensor. This prompts consideration of the relation of gravitational and electromagnetic radiation, leading to Theorem 7.2 which maps plane-fronted Maxwell fields in flat space to plane-fronted gravitational waves.

The paper ends by considering changes in Petrov type in “pure radiation” fields (Theorem 7.3) an issue further discussed by, e.g., [23], the subclass of solutions with a null Killing vector (Theorem 7.4, a result also found by Dautcourt: for a more general discussion see [6, Section 24.4]) and the geodesic completeness of plane waves (Theorem 7.5). Noting that Theorem 7.5 cannot be extended to *pp*-waves, singularities occurring instead, possible gravitational waveguides are discussed. Several authors discussed such global properties later: see e.g. [24, Section 4.4] and [25, Chapters 17 and 18].

An important aim of the paper was not entirely met, in that exact solutions showing the relation of radiation to source were not, and have not been, achieved. That issue was taken up by Szekeres [26], using similar methods to those in this paper: he found some Einstein–Maxwell fields but none for perfect fluids. The systems of equations for more general fluids and radiation have been studied in later work e.g. [27, 28], and a later paper on gravitational waves in matter by Madore, [29], obtained a dispersion relation in a perturbative treatment; see also [30].

Today the relation of source and field is treated by numerical simulations, now that computing hardware and algorithms have improved sufficiently for the rather

complicated equations to be handled. The time-reversed situation of radiation and absorber is of course fundamental to experimental searches for gravitational waves.

Despite that lacuna, the paper made first steps in, and prompted further work in, a number of areas, as mentioned above, and its methods, such as the use of Bianchi identities, and of tetrad and spinor techniques, became widely used later.

Biographies of the authors

A brief autobiography of Wolfgang Kundt appeared with Paper I of the series [1]. Biographies of the other contributors to the series have appeared with the earlier papers, except for that of Manfred Trümper.

Manfred Trümper—a brief autobiography

By Manfred Trümper

I was born in 1934 in Wernigerode, Germany, as the second child of a vocational school teacher. I graduated from a high school in Bernburg, Germany, in June 1952. In fall 1952 I enrolled as a student in physics at the Martin Luther University of Halle. As a child of bourgeois parents I did not qualify for a higher education, but it happened that in this year Walter Ulbricht, Secretary General of the “United Socialist Party = SED”, had declared a drive to economically overtake capitalist West Germany within the next 10 years. This meant increasing by about a factor 4 the number of physics students. As there were not enough applications from high school graduates with a proletarian background, I was admitted for studies in physics, and so was my elder brother Joachim.

In June of 1953, after the uprising of workers against the communist regime, there began a phase with lowered restrictions on travel to West Germany. However, a year later, in spring and summer of 1954, it became clear that the regime was tightening its grip of power and returning to repressive measures. Therefore, in the fall of 1954 I left the “workers’ paradise” and I enrolled at Hamburg University.

Leaving the DDR was not an easy task. The university administration kept under lock all my academic records, including the high school diploma. I gambled on the hypothesis that I could not be accused of subversion and secret preparation of “Republikflucht” if I told everybody about my plan to move to Hamburg. I said that in Hamburg I wanted to study General Relativity which was not represented in Halle. My fellow students thought I was nuts. Everybody knew that it was virtually impossible to change the place of studies even within the DDR. And I was talking about going to the “Klassenfeind” in the West! So I went to W. Messerschmidt, professor of physics, who I knew was quite unhappy about the excessive number of physics students which had been forced upon him and which greatly exceeded the capacity of his laboratory. He wrote a strong letter of recommendation in support of my application for an exmatriculation. The university administration was confused when I submitted my application because such a case had not occurred before. In Berlin, people were fleeing by the hundreds a day by just crossing from the soviet occupied East sector to one of the three West sectors. But in my case the university administration was faced with a request for moving to the West. Within a day I got my exmatriculation and my academic records

and then, in a final gesture of triumph, I went to the office of the youth organization FDJ (Freie Deutsche Jugend) and resigned from my membership.

My teachers in Halle had been H. Grötzsch and O.-H. Keller in mathematics, and W. Messerschmidt, M. Hieke in physics. My teachers in Hamburg were, among others, L. Collatz, H. Hasse, H. Braun in mathematics, and W. Lenz, P. Jordan, H. Raether, H. Lehmann in physics. I got my doctorate in Physics in 1962 with Pascual Jordan as my thesis advisor. The subject of my thesis dealt with the restrictions imposed on the flow of test matter in a relativistic gravitational field.

From there, I subsequently held the following positions.

Syracuse University, Postdoc with Peter Bergmann

Yeshiva University, Postdoc with Peter Bergmann

Hamburg University, Assistant to Pascual Jordan

North Texas State University, Visiting Profesor of Physics

Texas A&M, Associate Professor of Physics

Université d'Oran, Algeria, Professor of Physics

MPI for Astrophysics, Munich, Scientific Collaborator in the General Relativity Group

Université de Rennes /France, Visiting Professor

ISP Bukavu/Kivu, Zaire, Professeur Ordinaire

ENSSAT Lannion/France, Visiting Professor

From 1992 I worked as a representative for TÜV Süd (a major certification body for product safety and quality management) in Japan and in China. I retired in 1999 and moved in 2000 to the city of Uzès in southern France, where I now live.

Some of my contributions to science are

- (1) About shear-free and irrotational flows in an Einstein vacuum field of type I (1962).
- (2) Evaluation of the full Bianchi identities (for a 4-dimensional normal hyperbolic Riemannian space) in the presence of a timelike vector field (based on work done in 1963 and 1964).

The key results of this work were distributed in a preprint to various GR groups in Europe and the US.

I gave seminar talks about it

- on 15 Dec. 1964 in Cambridge at the invitation by Dennis Sciama and George Ellis,
- on 6 Feb. 1965 at the Institute Henri Poincaré in Paris, at the Invitation by Mme. M. A. Tonnelat, and in the presence of A. Lichnerowicz,
- on 8 Feb. 1965 at the Institut de Mathématique, Université Libre de Bruxelles at the invitation by R. Debever and M. Cahen.

I was preparing the publication of the results as a joint publication with P.G. Bergmann as coauthor, when the key equations were published by S. Hawking in 1966. These equations are nowadays widely used in numerical calculations. A more detailed account of the history of this work will be given elsewhere.

- (3) How to furnish the Lagrangian configuration space-time of a holonomic system with a linear connection so that the trajectory of the system is a geodesic line.
- (4) Determined the complete tree for the “33 hole central vacancy peg solitaire problem.”

- (5) Current research on the Collatz Conjecture. “Handles, Hooks, and Scenarios: A fresh Look at the Collatz Conjecture”, [arXiv:math/0612228](https://arxiv.org/abs/math/0612228)
- (6) Further results on the Collatz Conjecture, “The Collatz Problem in the Light of an Infinite Free Semigroup”, Chinese Journal of Mathematics, vol. 2014, Article ID 756917, 21 pages, 2014. doi:[10.1155/2014/756917](https://doi.org/10.1155/2014/756917)

Addendum by the Golden Oldie Editor: In addition to the works mentioned, and the present paper, Dr. Trümper published a number of other papers in physics. Among them, the following papers in relativity are cited in [6]:

“On a Special Class of Type-I Gravitational Fields”, M. Trümper, *J. Math. Phys.* **6**, 584 (1965)

“Orthogonal decomposition of axi-symmetric stationary spacetimes”, W. Kundt and M. Trümper, *Z. Phys.* **192**, 419 (1966)

“Einsteinische Feldgleichungen für das axialsymmetrische, stationäre Gravitationsfeld im Innern einer starr rotierenden idealen Flüssigkeit” M. Trümper, *Z. Naturforsch.* **22a**, 1347 (1967).

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