Editorial note to:

Pascual Jordan, Jürgen Ehlers and Rainer K. Sachs, Contributions to the theory of pure gravitational radiation.

Exact solutions of the field equations of the general theory of relativity II

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Part I

By Malcolm A. H. MacCallum

The overall arc of the ‘Hamburg bible’ has already been described in Ellis’ editorial note to paper I in the series [1]. This second paper discusses the geometry of affinely-parametrized geodesic null congruences, reviews and develops the classification of conformal curvature using spinor techniques, and then puts the two together in studying the propagation of gravitational and electromagnetic radiation. In doing so, the authors say that they intended to address the question ‘whether null fields and the other “special” fields in the sense of the Petrov classification have analytical properties, which justify their interpretation as more or less “pure” radiation fields’. Here “null fields” means what we would now refer to as fields of Petrov type N.

Paper II appeared at a time of rapid evolution in the understanding and use of null congruences and of the Petrov types. Petrov’s classification of conformal curvature, which had first been given (in Russian) in 1954 [2], naturally defines congruences

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of null curves given by the principal null directions (PNDs) of the Weyl tensor, as discussed here in Chapter 2. The present paper provided impetus to, and the basic equations for, linking the geometry of these congruences with properties of curvature. It is this aspect of the content that has been the most influential.

In the immediately following years there were several remarkable related developments. Using the ideas here, Sachs soon produced two papers [3,4] which in particular address the other question posed in the present paper's introduction "whether and in which approximation the fields of bounded sources approach such fields asymptotically"; those papers give the famous peeling theorem and generalize the results of [5]. With Goldberg, Sachs published their well-known theorem that the Weyl tensor in a vacuum is algebraically special if and only if there is a geodesic shearfree null congruence [6]. Newman and Penrose introduced their famous formalism [7] and gave a proof of the Goldberg-Sachs result using it. Very soon, Kundt and Thompson [8], and Robinson and Schild [9], generalized the Goldberg-Sachs result to the non-vacuum case. The Kundt-Thompson form, with some further remarks quoted in [10], is also given in paper V of the present series.

These advances led to much activity on algebraically special exact solutions, work which still continues. Results up to 1999 are summarized in chapters 26–32 of [10]. Basic papers on certain major classes of such solutions appeared very soon after the present paper: for solutions with geodesic, shearfree and non-twisting rays, Robinson and Trautman [11]; for geodesic twistfree rays, Newman and Tamburino [12]; and for twist and divergence-free rays, Kundt [13] and paper V of this series. Particularly important cases included the NUT solutions [14] and perhaps the most astrophysically important of all exact solutions which can be considered to describe an isolated body, the Kerr solution for a rotating black hole [15].

Chapter 1 here provides the definition and interpretation of the optical scalars. These are so much part of the foundation of the study of exact solutions, of gravitational radiation, and of the geometry of spacetime, that it is hard to trace the influence of the present treatment in detail, because the results are inextricably embedded in the work cited above that followed almost immediately. (For example, the subsequent papers of Sachs are much more widely cited than the present paper, on which they depend.) Moreover, the exposition is so clear that there is little need, in this note, to paraphrase it.

The Newman-Penrose (NP) technique has been the most widely used of the many techniques based on null vectors or related geometric structures, so I shall adopt it, in the form described in chapter 7 of [10], as a standard for reference (note that [10] uses the same signature as the present paper, and that the NP Ricci equations, (NPa)–(NP_r), are (7.21a)–(7.21r) there). Among the other related techniques are those of Cahen et al. [16] based on bivectors (some later workers used this privately even though they translated the outcomes to NP for the published versions of their work!) and Geroch et al. [17], who gave a modified calculus exploiting the discrete symmetries inherent in the structure. Ernst [18] provided a dictionary between several notations and discussed the choices of sign convention. More recently still, a method based on a single null direction has been introduced [19].

It may be worth noting that the sign convention adopted here for the Riemann tensor, which is the opposite of that used in [10], was not stated directly but given by

reference to other works which were no more easily obtained than the present one. However, reference (12) of this paper is the first of the Mainz series [1] and is therefore now readily available.

The authors assume a null congruence with geodesic tangent vector field $\mathbf{k}$, affinely parametrized by v , so that the NP quantities κ and $\epsilon + \bar{\epsilon}$ are zero. The tetrad vectors $\{t^a, k^a, m^a\}$ introduced in section 1 [of Chapter 1] would now more commonly be notated as $\{\bar{m}^a, k^a, -\ell^a\}$ respectively, with the consequence that the B of (1.1.6) is $-B$ in the notation of [10], equation (7.7a). In this editorial note we will use the notation of the paper itself, to avoid having two different meanings of m^a ; note that the definition in the paper implies that $\mathbf{m}$ is past-pointing.

The optical scalars are given in (1.3.4). A more symmetrical relation to a future-pointing timelike unit vector $\mathbf{u}$, differing from that given before (1.3.5) by a boost with a factor $1/\sqrt{2}$, would be $\sqrt{2}\mathbf{u} = \mathbf{k} - \mathbf{m}$. The z of (1.3.6) is the NP quantity $-\rho$, and the σ is the NP quantity $-\bar{\sigma}$, using the conventions of [10]. The interpretations that follow (1.3.9) give the key properties of the optical scalars: this is perhaps the most important part of the paper.

In the remaining parts of the covariant derivative of $\mathbf{k}$, defined in (1.4.1), β , Ω and ζ can be identified as the NP quantities $\gamma + \bar{\gamma}$, τ and $(\bar{\alpha} + \beta)$ respectively. Deducing some of the consequences in section 5 requires a little algebra, but is fairly straightforward. The relation (1.4.4) is important in calculating redshifts. The equation (1.5.1) is the NP equation NP_a, (7.21a) in [10], giving the evolution of the expansion and twist along the rays, and the rest of the section explores its main consequences in the shear-free case.

The set of equations (1.5.1), (1.5.8)–(1.5.11) are the NP equations a , b , c , a combination of d and e , f plus its complex conjugate, and g . Equation (1.5.10) is not the most general form arising from the assumptions of the paper. That should read

$$\dot{\beta} + \Omega \bar{\zeta} + \bar{\Omega} \zeta = R_{abcd} m^a k^b m^c k^d, \quad (1.5.10)$$

so the given form requires $\zeta = \Omega$: this is necessarily true if k_a is a gradient, so it appears that this property has been assumed although earlier (remark 3 of section 3) it was only assumed for the twistfree case. The quantity $\bar{\sigma}$ is $-\lambda$ in the NP notation and is therefore the counterpart of σ under the prime operation of the GHP formalism [see [10], (7.38)].

Theorem 1.5.1 is a first step towards the Goldberg – Sachs theorem: the condition (1.5.13) is $\Psi_0 = 0$. Similarly (1.5.15) and (1.5.16) are $\Phi_{01} = 0$ and $\Psi_1 = 0$. In the Lemmas 1.5.1 and 1.5.2 (and later in Chapter 3, section 5) the authors do not detail the relations between the various constants along the rays. When these results are presented their importance is not immediately obvious, but they play a key role in Chapter 3.

Chapter 2 is the shortest of the three. Although the authors say its contents are essentially known, it nevertheless presents some very useful formulations and formulae in spinors, which were “implicitly but not in this form contained in earlier works”. The spinor technique follows [20], and the appendix to the present paper establishes the basic algebra and analysis. (Note that dotted rather than primed indices

corollary of the Mariot-Robinson theorem stating that the Weyl tensor of Einstein-Maxwell fields with an electromagnetic null field is algebraically special.

Section 4 deals with the non-null electromagnetic case. The λ here is Φ_1 . Known results for this case prompt a re-derivation of the peeling behaviour, assuming a geodesic shearfree congruence so that $\Phi_0 = \Psi_0 = \Psi_1 = 0$. This makes essential use of Lemmas 1.5.1 and 1.5.2 and the forms in Theorem 3.4.1 use a choice of null rotation (3.4.2). Remark 1 of Chapter 1, section 4, showed that a null rotation can set $\Omega = 0$ at a point, so the result of Theorem 1.5.2 then implies that at all points on the ray $\Omega = 0$. The null rotations f given in (3.4.18) are those between the tangent to the geodesic shearfree rays and the basis for the canonical form (3.4.1).

Section 5 similarly uses a null rotation from the basis in which the algebraically special Weyl tensors take their canonical forms to a parallelly-propagated one. The resulting Weyl tensor forms are given. The motivation is that, as in the electromagnetic case, this will facilitate the expansion along the rays in powers of v , z or r_A . This is then applied in sections 6-9 to each of the algebraically special types in turn. In NP notation, the leading terms ν , ρ and λ are Ψ_4 , $-\Psi_3$ and Ψ_2 respectively.

The type N case is quickly shown to be closely analogous to the electromagnetic null case. The type III case is not much more difficult and somewhat analogous. One should be aware from the ‘Notations and conventions’ section that the tilde relating the two N terms in the schematic form given in (3.7.4), and the similar subsequent forms in (3.8.4) and (3.9.2), implies proportionality.

Type D is treated in a different way, by using the results for non-null electromagnetic fields on the eigenbispinor (eigenbivector), giving a very neat proof of the forms in Theorem 3.8.2. The integral invariant (3.8.5) is interpreted for the Schwarzschild solutions as related to the mass m . The same arguments applied to type II give the behaviour of Ψ_2 and the null rotation f , leaving only the behaviour of ν (i.e. Ψ_4) to be determined. This readily follows from the Bianchi identity (3.9.1) and earlier results.

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Part II

By Wolfgang Kundt

We were all very young then: Rainer Sachs, Jürgen Ehlers, and myself, in 1961, when “Mainz II” was released, under the “mastership” of Pascual Jordan, then president of the “Academy of Sciences and Literature at Mainz”, and one of the three authors of the 1927 “Dreimännerarbeit” on the foundations of quantum mechanics, three men, two of whom had just acquired maturity (21 years of age) when they first published the commutation relation $PQ - QP = \hbar/(2\pi i)$, in 1925.

In 1961, Jordan ranked like our father: always friendly and helpful, always on time for our joint seminars, and always concerned that we received his monthly purses on time, a support which he provided for us via his contract with the American airforce. In return, every now and then, he expected us to provide written reports of our findings, for publication in the proceedings of the Academy of Mainz.

Jordan's weekly Hamburg seminars were devoted to a deeper understanding of Einstein's General Theory of Relativity (GRT). For some 20 years, Engelbert Schücking, Jürgen Ehlers, and myself, plus a handful of younger researchers who joined us later, worked hard to convert the young GRT from an elegant mathematical scheme into a useful physical tool, into a transparent generalisation of Newton's and Maxwell's theories towards higher speeds and higher densities of matter. How many relationships survive? How many one- or two-line proofs can be generalised to the new theory? Can they be obtained by hand within finite amounts of time, without the help of a computer?

The present "Mainz II", or "JES" paper is an early version of papers by the Hamburg group, though with the replacement of me by Ray Sachs. Ray arrived at Hamburg as a fresh post-doc, from Peter Bergmann's group at Syracuse N.Y., when I was sent to Syracuse as a fresh post-doc, to join Bergmann's active theoretical group in the U.S. Ray and I worked largely in parallel during that year of exchange: he on outgoing waves, me on expansion-free waves. Our work had been triggered by A. Z. Petrov's algebraic classification of the Weyl tensor followed by Felix Pirani's recognition of its physical significance, again followed by skillful calculations of Ivor Robinson and Andrzej Trautman. A common goal of all these calculations was to explore the analogies between electromagnetic radiation and gravitational radiation, not only within the linearised regime of GRT. Both Ray and I have profited from the care and hindsight provided by Jürgen Ehlers during those Hamburg seminars, and beyond.

How much has been achieved, in Mainz II? This lengthy paper is a concise report on exact relationships obtained in part by using Roger Penrose's spinor calculus, whose equations tend to be shorter than the corresponding tensor equations. They are aesthetic, and they convey the clear impression that GRT is the correct spacetime theory, without any further modifications needed. The richness of its classes of exact solutions does not fall short of what one expects it to offer.

And the physics behind it? It keeps us waiting. No single gravitational wave has been recorded yet, despite the ever improving detectors. Perhaps because scientists have been rash in believing in black-hole formations, or of naked singularities. The original hope was unjustified that from the shapes of detected gravitational signals, we should be able to conclude at the properties of their sources. Those signals have simply been too faint so far. Theory has been ahead of its verifications. Even more likely, humankind will never get into a position to measure gravitational waves beyond the linear regime, so that Mainz II will transcend testable human insight. It was a big challenge, though, to those who worked on it.

Comment by the Golden Oldies Editor: Short biographies of the authors of the paper were published in our journal on other occasions, namely:

1. Pascual Jordan: *Gen. Relativ. Gravit.* **41**(9), 2182 (2009),
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2. Jürgen Ehlers: *Gen. Relativ. Gravit.* **41**(9), 1899 (2009),
doi:[10.1007/s10714-009-0841-7](https://doi.org/10.1007/s10714-009-0841-7),
3. Rainer K. Sachs: *Gen. Relativ. Gravit.* **39**(11), 1941 (2007),
doi:[10.1007/s10714-007-0448-9](https://doi.org/10.1007/s10714-007-0448-9).

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