Editorial notes to:

3. On the theory of gravitation.

By Hermann Weyl

and to:

**5. Comment on the axially symmetric solutions
to Einstein's equations of gravitation.**

By Hermann Weyl

and to:

New solutions to Einstein's equations of gravitation.

B. Explicit determination of static, axially symmetric fields.

By Rudolf Bach.

With a supplement on the static two-body problem.

By H. Weyl

Gernot Neugebauer · David Petroff · Bahram Mashhoon

Published online: 14 February 2012
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Keywords Hermann Weyl · Weyl solution · Golden Oldie

Part I: The Weyl—Bach papers and their influence on later work

By Gernot Neugebauer and David Petroff

One could say that the General Theory of Relativity was born under a lucky star. In addition to Einstein himself, early research on the theoretical side of the subject was led

The two parts of the editorial note were written independently at the invitation of the editor of the Golden Oldies, and not in collaboration.

The republications of the original papers can be found in this issue following the editorial note and online via doi:[10.1007/s10714-011-1310-7](https://doi.org/10.1007/s10714-011-1310-7) (Weyl 1917), doi:[10.1007/s10714-011-1311-6](https://doi.org/10.1007/s10714-011-1311-6) (Weyl 1919), and via doi:[10.1007/s10714-011-1312-5](https://doi.org/10.1007/s10714-011-1312-5) (Bach and Weyl 1922).

G. Neugebauer · D. Petroff (✉)
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität, Max-Wien-Platz 1, Jena, Germany
e-mail: D.Petroff@tpi.uni-jena.de

B. Mashhoon
Department of Physics and Astronomy, University of Missouri, Columbia, MO 65211, USA
e-mail: MashhoonB@missouri.edu

by such great thinkers as Hilbert, Schwarzschild, Droste, Weyl, Bach and Levi-Civita, and on the experimental side by the likes of Eddington.

In his 1917 paper “On the Theory of Gravitation” Weyl examined a Hamiltonian formulation of the field equations, conservation laws for energy and momentum, the role of the observer in the theory and introduced important concepts and methods for dealing with static, axially symmetric spacetimes. This last topic was examined in more detail in 1919 in response to criticisms published by Levi-Civita and in particular in a joint paper from 1922 with Rudolf Bach,¹ which also served to clarify issues regarding regularity on the axis of symmetry.

We present these three papers here in English translation. This short introductory note is intended to point out some of the papers’ most notable contributions to the subject and indicate how these influenced later work. We cannot hope to do so with any degree of completeness and mean merely to convey a sense of their impact. The nature of research is to proceed by trial and error and even if our understanding of the Schwarzschild metric today relies heavily on the concept of an event horizon, Weyl’s attempt at understanding it in terms of the inside and outside of a point particle remains fruitful: he was correct in believing that the introduction of an appropriate coordinate system could be helpful when trying to overcome the “singularity” at the Schwarzschild radius.

The paper “On the Theory of Gravitation” begins with the derivation of a variational principle leading to the field equations for electrically charged, incoherent matter. Weyl emphasised the importance of this approach, taking into account the difficulties inherent in treating continuous media that are best described phenomenologically. In this formulation, the energy-momentum balance arises as a consequence of the invariance of the Lagrangian with respect to an infinitesimal, arbitrary coordinate transformation, along the lines of the (later published) Noether theorem [1].

In the paper’s third section he turns his attention to the very important question of how measurements (e.g. of light signals from stars) are to be understood within relativity, given that an observer moves arbitrarily in spacetime. With this in mind, the tangent vector to the observer’s world line is used to define a local time direction, thus providing a means of splitting up the four dimensional spacetime locally into $3 + 1$ dimensions. In the philosophical language of Weyl, the observer is reduced to a bare minimum and assumes the form of a “monad” or “point eye”. The work in this section laid the foundations for Weyl’s general frequency shift formula for sources and observers moving arbitrarily in a general spacetime, see Ref. [2]. An important application is Fermat’s principle of fastest arrivals for light rays, which he derived for a static spacetime from the general principle that the world line of a light signal is a null geodesic.

The fourth section is devoted to a derivation and interpretation of the spherically symmetric solutions to the Einstein vacuum and the Einstein–Maxwell case, i.e. what are usually referred to as the Schwarzschild metric [3,4] and the Reissner–Nordström metric [5,6]. Weyl attempts to make sense of the Schwarzschild metric by concentrating on its spatial part, $dt = 0$. He is misled by the coordinate singularity at the

¹ This name was a pseudonym, see “Bach’s” biography at the end of this note.

Schwarzschild radius $r = 2M$ into believing that the mass is distributed on the sphere $r = 2M$, which then induces a true singularity and marks an absolute boundary in spacetime. Both by embedding a slice through the origin in a Euclidean three-space and by introducing isotropic coordinates, Weyl concludes that the spacetime with $r > 2M$ is naturally divided into two regions, which he refers to as the inside and outside of the “point mass”, whereas the portion with $r < 2M$ is inaccessible. His two regions are nothing more than a double covering of the part of the Schwarzschild metric with $r > 2M$. The idea of introducing appropriate coordinates is good, but it took a long while before appropriate spacetime coordinates such as those found by Kruskal [7] allowed one to overcome Weyl’s difficulties and arrive at the concept of a Black Hole.

The fifth and sixth sections deal with axisymmetric and static spacetimes and mark an important contribution toward a systematic approach to solving Einstein’s equations. Weyl’s hope was that the non-linearity of gravitational phenomena could help clarify the nature of quantum theory and thus provide an understanding of the inner structure of the atom, an interesting idea that did not prove successful however.

On the other hand, the introduction of the canonical coordinates named after him was a milestone along the path toward studying the axisymmetric and stationary (and not only static) Einstein–Maxwell equations. In such coordinates, the field equations for the static and uncharged case are nothing more than Poisson equations, where the source terms cannot be prescribed in general however. Moreover, the electrostatic Einstein–Maxwell equations in these coordinates reduce to two coupled, semi-linear partial differential equations, which further simplify to one such Poisson equation for a constant ratio of charge to mass density.

The two further articles presented here in translation mark important further developments in the construction and interpretation of exact solutions to Einstein’s equations. In the first of these, reacting to a critical comment made by Levi-Civita [8], Weyl analysed the necessity for the existence of stresses in order to compensate for gravitational attraction in the static setting. In the second, a paper by Bach with an addendum by Weyl, this topic was discussed in greater detail by considering concrete examples for which gravitational potentials had been constructed by Bach.

Bach begins the article by presenting Einstein’s vacuum field equations in Weyl’s canonical (cylindrical) coordinates. Weyl had already shown in the 1917 paper that, assuming $T_r^r + T_z^z = 0$, coordinates can be found in which the line element reads

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi} \left[r^2 d\vartheta^2 + e^{2\gamma} (dr^2 + dz^2) \right],$$

where r, z, ϑ are cylindrical coordinates. In particular, Einstein’s equations reduce to Laplace’s equation for ψ and a line integral for γ . The importance of these papers to later work lies in the fact that even in this simple setting (a static and axially symmetric spacetime), one is confronted with a number of central issues, which are typical and unique in Einstein’s theory: formulating an appropriate problem from the standpoint of the physics, the problematic choice of suitable coordinates and new types of singularities.

Each of these plays a role in the Bach/Weyl paper. A good example is provided by the treatment of two “sphere-like” bodies. After having presented the Schwarzschild solution in Weyl’s canonical coordinates, which turned out to be a solution of Laplace’s equation with a rod-like source on the axis of symmetry, Bach attempts to model two sphere-like sources by considering a superposition of two such sources. This approach is quite natural from a mathematical standpoint, since Laplace’s equation is, of course, linear, but disregards that one would have to select an appropriate matter model *a priori* and then formulate a free-boundary value problem (at least for two *fluid* balls). The next question is then whether or not the two bodies can remain at rest. A necessary condition is that the metric function γ vanish on the matter-free parts of the axis for them to be regular. Bach notes that this condition cannot be satisfied on the portion of the axis between the two bodies as expected. Weyl’s closing remarks (published as a supplement to Bach’s paper) focus on the interpretation of this type of singularity. He uses stress components of the energy-momentum tensor (which do not vanish in the matter-free region defined by $\rho = 0$ (see (48)) on the portion of the axis joining the bodies!) to define a force of attraction between these bodies. This force turns out to be proportional to the constant value of γ on this portion of the axis. The critical point here is Weyl’s rather artificial introduction of matter-free regions *with stresses*. These considerations were of the utmost importance for the study of axisymmetric and stationary vacuum spacetimes later, since they led to the Weyl–Papapetrou [9] coordinates needed to write the field equations in the form of completely integrable systems. The symmetries that become apparent in these coordinates are what finally facilitated the analytic generation techniques for large classes of solutions to Einstein’s equations. For a description of modern work in this direction, including many references, see Sect. 20, Sect. 34 in Ref. [10] and for an excellent historical overview, see [11].

Weyl’s work covered a broad range of topics and had a substantial influence on the development of the field. It deals with field-theoretical aspects (application of the Hamiltonian formalism), contributed fundamentally via the concept of the observer to an understanding of the physics of gravity and constitutes the beginning of a systematic approach toward solving classes of Einstein’s field equations and interpreting them as far as possible. On the whole, the articles presented here provide insight into Weyl’s particular genius that is based on a combination of great mathematical ability coupled with a deep understanding of physics.

Acknowledgements G.N. & D.P. want to express their thanks to Andrzej Kasiński for his perseverance and patience in seeing this project through and, in particular, for his meticulous proof-reading.

Part II: Weyl’s metric and the exterior gravitational field of a rotating star

By Bahram Mashhoon

In 1917, Hermann Weyl introduced what has now become one of the classic line elements of general relativity. This spacetime metric describes the general static exterior gravitational field of a non-rotating axisymmetric body. An excellent discussion of the vacuum spacetimes for the exterior of one or a system of such bodies is contained in chapter VIII of Synge’s book [12]. Weyl’s metric has served as the starting point

for various developments in general relativity [13]; this brief commentary reflects the present author's own experience.

Consider, for simplicity, the general static axisymmetric line element for the Ricci-flat exterior field of a body. It may be written in terms of Weyl's canonical coordinates (t, ρ, ϕ, z) as

$$ds^2 = -e^{2\psi} dt^2 + e^{-2\psi} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where ψ and γ are functions only of ρ and z , and

$$\psi_{,\rho\rho} + \frac{1}{\rho} \psi_{,\rho} + \psi_{,zz} = 0, \quad (2)$$

$$\gamma_{,\rho} = \rho(\psi_{,\rho}^2 - \psi_{,z}^2), \quad (3)$$

$$\gamma_{,z} = 2\rho \psi_{,\rho} \psi_{,z}. \quad (4)$$

In these formulas a comma represents partial differentiation. Equation (2) is the integrability condition for Eqs. (3) and (4). The symmetry axis is regular (i.e., elementary flat) if γ vanishes for $\rho \rightarrow 0$. Moreover, from a given solution (ψ, γ) , one can generate a set of solutions $(\delta\psi, \delta^2\gamma)$, where δ is a real parameter. Remarkably, Eq. (2) is equivalent to $\nabla^2\psi = 0$, so that ψ is a harmonic function. It is therefore natural to associate ψ with the exterior Newtonian gravitational potential of the body.

It is useful to transform the Weyl metric to prolate spheroidal coordinates (t, x, y, ϕ) , where

$$x = \frac{1}{2\sigma}(r_+ + r_-), \quad y = \frac{1}{2\sigma}(r_+ - r_-). \quad (5)$$

Here $x \geq 1$, $-1 \leq y \leq 1$, σ is a constant length and

$$r_{\pm}^2 = \rho^2 + (z \pm \sigma)^2. \quad (6)$$

The general solution of Laplace's equation corresponding to an asymptotically flat axisymmetric spacetime with a regular axis is given by [14]

$$\psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n Q_n(x) P_n(y), \quad (7)$$

where Q_n and P_n are respectively the associated Legendre functions of the second kind and the Legendre polynomials. The corresponding expression for γ has been explicitly given in [14]. For $\sigma = m \neq 0$, $q_0 = 1$ and $q_n = 0$ for $n > 0$ together with $x = -1 + r/m$ and $y = \cos \theta$, the solution reduces to the Schwarzschild solution with mass m in spherical polar coordinates (r, θ, ϕ) . The quantities q_n are proportional to the Newtonian multipole moments; in fact, Eq. (7) corresponds to the expansion of the exterior Newtonian potential in terms of the multipole moments of the gravitational

source. For instance, if the constants q_n vanish except for $q_0 = 1$ and q_2 , one recovers the Erez–Rosen solution [15] for a mass with quadrupole moment.

The exterior vacuum field of a spherically symmetric distribution of matter is uniquely described by the Schwarzschild solution. Therefore, the static Schwarzschild solution with an infinite set of parameters that pertain to axisymmetric deformations of a non-rotating source is interesting for astrophysical applications. However, astronomical sources generally rotate. In the 1980s, powerful methods became widely available for generating solutions of Einstein’s field equations—see [13] for a discussion of these generation techniques. In particular, the approach developed by Kinnersley, Hoenselaers, and Xanthopoulos, known as the HKX transformations, could be applied in this case to generate stationary axisymmetric vacuum solutions. The task of generating the Kerr solution with a full set of gravitoelectric (due to mass) and gravitomagnetic (due to spin) multipole moments from the static solution corresponding to Eq. (7) via HKX transformations was first accomplished by Quevedo in his doctoral work [16, 17]. It is possible to extend the solution to include electric charge and Taub-NUT parameter [18–20]. The Quevedo solutions, also known as the generalized Kerr spacetimes [21], are of current interest in relativistic astrophysics [22].

“Rudolf Bach”—a brief biography

By Hubert Goenner

Rudolf Förster (1885–1941) studied mathematics in Göttingen and received his PhD with David Hilbert in 1908. He then worked as an engineer with the manufacturer Krupp in Essen. From Nov. 1917 to March 1918 he corresponded with A. Einstein on an extension of general relativity to include the electromagnetic field via a skew-symmetric metric. By his employer, he was forbidden to publish. Therefore he assumed the pen-name R. Bach [23].

Hubert Goenner
Institute for Theoretical Physics
University of Göttingen
Friedrich-Hund-Platz 1
37077 Göttingen
Germany

A brief biography of Hermann Weyl was printed together with another Golden Oldie in *Gen. Relativ. Gravit.* **41**, p. 1657 (2009), doi:[10.1007/s10714-009-0825-7](https://doi.org/10.1007/s10714-009-0825-7).

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