Erratum to: General spherically symmetric elastic stars in relativity

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Erratum to: Gen Relativ Gravit (2010) 42:2357–2382
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A mistake in a global sign renders wrong subsection 5.1 in the recently published paper “General spherically symmetric elastic stars in relativity” (Gen Relativ Gravit (2010) 42:2357–2382). Here, we correct that mistake and rewrite subsection 5.1, presenting examples similar to those in the above paper. The general developments, outline and conclusions in that reference remain valid.

1 Introduction

As pointed out right after equation (BCV-8) in [1], the *symmetric* energy-momentum tensor taking part in the Einstein field equations is the negative of the so-called *canonical* energy-momentum tensor (see also the paragraph following equation (BCV-6) in that paper).

Equation (BCV-33) in section 4 of [1] corresponds to the *canonical* energy-momentum tensor, and therefore, the paragraph right before (BCV-33) should better start: In [4], the *canonical* energy-momentum..., and therefore,

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the *symmetric* energy-momentum tensor should be the negative of that; and the two tensors, *canonical* and *symmetric*, should have been somehow distinguished.

This fact was overlooked there assuming that (BCV-33) corresponded in fact to the symmetric energy-momentum tensor, which is wrong. To see this, it suffices to notice that $\bar{G}_0^0 = -\rho$, whereas $\bar{T}_0^0 = \rho$. The expressions (BCV-34) are indeed correct if one is dealing with the *canonical* energy-momentum tensor, but when it comes to the Einstein's field equations, i.e., (BCV-41) to (BCV-44), there should be a minus sign, thus the sentence following (BCV-40) should read:

The Einstein field equations $\bar{G}_b^a = -8\pi \bar{T}_b^a$ can be written as follows: and the correct form for the field equations (BCV-41) to (BCV-44) is then:

$$\bar{G}_0^0 = -8\pi \bar{T}_0^0:$$

$$-\frac{\dot{\bar{Y}}}{\bar{Y}^2 \bar{a}} - \frac{\dot{\bar{Y}}}{\bar{Y}} \frac{\dot{\bar{b}}}{\bar{a} \bar{b}} + \frac{2\bar{Y}''}{\bar{Y} \bar{b}} + \frac{\bar{Y}'^2}{\bar{Y}^2 \bar{b}} - \frac{\bar{Y}'}{\bar{Y}} \frac{\bar{b}'}{\bar{b}^2} - \frac{1}{\bar{Y}^2} = -\bar{\epsilon} \bar{v} 8\pi \quad (1)$$

$$\bar{G}_0^1 = -8\pi \bar{T}_0^1:$$

$$2\frac{\dot{\bar{Y}}'}{\bar{a}} - \frac{\bar{a}'}{\bar{a}} \frac{\dot{\bar{Y}}}{\bar{Y}} - \frac{\dot{\bar{b}}}{\bar{b}} \bar{Y}' = 0, \quad (2)$$

$$\bar{G}_1^1 = -8\pi \bar{T}_1^1:$$

$$-\frac{\dot{\bar{Y}}^2}{\bar{Y}^2 \bar{a}} + \frac{\dot{\bar{Y}}}{\bar{Y}} \frac{\dot{\bar{a}}}{\bar{a}^2} + \frac{\bar{Y}'}{\bar{Y}} \frac{\bar{a}'}{\bar{a} \bar{b}} + \frac{\bar{Y}'^2}{\bar{Y}^2 \bar{b}} - \frac{2\ddot{\bar{Y}}}{\bar{Y} \bar{a}} - \frac{1}{\bar{Y}^2} = \bar{\epsilon} 2 \bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} 8\pi, \quad (3)$$

$$\bar{G}_2^2 = -8\pi \bar{T}_2^2:$$

$$\begin{aligned} & \frac{1}{2} \frac{\dot{\bar{Y}} \dot{\bar{a}}}{\bar{Y} \bar{a}^2} - \frac{1}{2} \frac{\dot{\bar{Y}} \dot{\bar{b}}}{\bar{Y} \bar{a} \bar{b}} - \frac{1}{4} \frac{\bar{a}'^2}{\bar{a}^2 \bar{b}} + \frac{1}{2} \frac{\bar{Y}' \bar{a}'}{\bar{Y} \bar{a} \bar{b}} - \frac{1}{4} \frac{\bar{a}' \bar{b}'}{\bar{a} \bar{b}^2} + \frac{\bar{Y}''}{\bar{Y} \bar{b}} - \frac{1}{2} \frac{\bar{Y}' \bar{b}'}{\bar{Y} \bar{b}^2} \\ & + \frac{1}{2} \frac{\bar{a}''}{\bar{a} \bar{b}} - \frac{1}{2} \frac{\ddot{\bar{b}}}{\bar{a} \bar{b}} + \frac{1}{4} \frac{\dot{\bar{a}} \dot{\bar{b}}}{\bar{a}^2 \bar{b}} + \frac{1}{4} \frac{\dot{\bar{b}}^2}{\bar{a} \bar{b}^2} - \frac{\ddot{\bar{Y}}}{\bar{Y} \bar{a}} = \bar{\epsilon} \bar{s} \frac{\partial \bar{v}}{\partial \bar{s}} 8\pi. \end{aligned} \quad (4)$$

This mistake was carried over to Sects. 5.1 and 5.2 rendering invalid the examples presented there. However, the general set up in those sections is still correct, and it is indeed very easy to produce correct examples constructed along the same lines as those in BCV.

Instead of pointing out to the mistakes in these two sections and giving the amended, we next rewrite them using the correct sign in the Einstein field equations and presenting the corresponding correct examples. If at any stage, a reference is made to some expression appearing in the BCV which is not contained in the subsections being rewritten, we shall include it as a footnote to make the present paper more readable.

5.1 Static shear-free solutions

In the static, shear-free case (metric (BCV-49)¹ with no dependence on t), the field equations yield

$$-\epsilon v 8\pi = 2 \frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad (5)$$

$$2\epsilon\eta \frac{\partial v}{\partial \eta} 8\pi = 2 \frac{a'}{a} \frac{Y'}{Y} + \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad (6)$$

$$\epsilon s \frac{\partial v}{\partial s} 8\pi = \frac{Y''}{Y^3} + \frac{a''}{aY^2} - \frac{Y'^2}{Y^4}, \quad (7)$$

solving (5) for ϵ and substituting it in (6) and (7), one gets two equations which depend only on r , and elementary considerations show that for given $a(r)$ and $Y(r)$, functions $y(r)$, $f(y)$ and v can be found so that the two equations are satisfied. It remains to be seen, though, that the DEC are satisfied, and therefore, the solution is physically acceptable.

The following simple example shows that solutions with these characteristics do indeed exist.

Example 1

Consider the line element

$$ds^2 = -Y^{-2}(r)dt^2 + Y^2(r) \left(dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (8)$$

A direct calculation yields

$$-8\pi\epsilon v = 2 \frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad 16\pi\epsilon\eta \frac{\partial v}{\partial \eta} = -\frac{Y'^2}{Y^4} - \frac{1}{Y^2}, \quad 8\pi\epsilon s \frac{\partial v}{\partial s} = \frac{Y'^2}{Y^4}. \quad (9)$$

The dominant energy condition (BCV-11)² implies:

$$8\pi\rho = -\left(2 \frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2} \right) \geq 0, \quad (10)$$

¹

$$ds^2 = -a^2(r, t)dt^2 + Y^2(r, t)(dr^2 + d\theta^2 + \sin^2 \theta d\phi^2).$$

In (BCV-49), we used $-a(r, t)$ instead of $-a^2(r, t)$. As it turns out, the field equations take a simpler form if we use $-a^2(r, t)$ for the metric coefficient g_{tt} , and therefore, we follow this convention here.

²

$$\rho \geq 0, \quad |p_A| \leq \rho, \quad A = 1, 2, 3.$$

$$8\pi (\rho + p_1) = -2\frac{Y''}{Y^3} \geq 0, \quad (11)$$

$$8\pi (\rho - p_1) = -2\left(\frac{Y''}{Y^3} - \frac{Y'^2}{Y^4} - \frac{1}{Y^2}\right) \geq 0 \quad (12)$$

$$8\pi (\rho + p_2) = -\left(2\frac{Y''}{Y^3} - 2\frac{Y'^2}{Y^4} - \frac{1}{Y^2}\right) \geq 0, \quad (13)$$

$$8\pi (\rho - p_2) = -\left(2\frac{Y''}{Y^3} - \frac{1}{Y^2}\right) \geq 0, \quad (14)$$

where we put $\rho = \epsilon v$, $p_1 = 2\epsilon\eta\frac{\partial v}{\partial\eta}$ and $p_2 = \epsilon s\frac{\partial v}{\partial s}$.

Now, it is immediate to see that the above conditions are all satisfied if and only if

$$-2\frac{Y''}{Y^3} \geq 0 \quad (15)$$

Take, for instance,

$$Y = e^{-\frac{5}{2}r^2}, \quad (16)$$

one then has

$$\begin{aligned} \rho = \epsilon v &= \frac{1}{8\pi}e^{5r^2}(11 - 25r^2), \\ p_1 = 2\epsilon\eta\frac{\partial v}{\partial\eta} &= -\frac{1}{8\pi}e^{5r^2}(25r^2 + 1), \quad p_2 = \epsilon s\frac{\partial v}{\partial s} = \frac{1}{8\pi}25r^2e^{5r^2} \end{aligned} \quad (17)$$

which is obviously well behaved: satisfies the dominant energy condition for values of r such that

$$r \in \left[0, \frac{1}{\sqrt{5}}\right) \quad (18)$$

and is non-singular at the origin. Notice that the radial pressure is negative (compressed material), and the tangential pressures are zero at the center, as one would expect. See Fig. 1 for details.

The field equations in this case read:

$$\epsilon v = \frac{1}{8\pi}e^{5r^2}(11 - 25r^2) \quad (19)$$

$$2\epsilon\eta\frac{\partial v}{\partial\eta} = -\frac{1}{8\pi}e^{5r^2}(25r^2 + 1) \quad (20)$$

$$\epsilon s\frac{\partial v}{\partial s} = \frac{1}{8\pi}25r^2e^{5r^2} \quad (21)$$

Fig. 1 The density, radial pressure and tangential pressure

and one has that

$$\eta = f^2(r)e^{5r^2}, \quad s = r^2 f^2(r)e^{5r^2}. \quad (22)$$

Now, dividing (20) and (21) through by (19), and setting $E \equiv \ln \eta$, $\Sigma \equiv \ln s$, one gets

$$\frac{\partial \ln v}{\partial E} = \frac{1}{2} \frac{25r^2 + 1}{25r^2 - 11}, \quad \frac{\partial \ln v}{\partial \Sigma} = -\frac{25r^2}{25r^2 - 11}. \quad (23)$$

From the expressions for η and s , one has that

$$E = 2 \ln f(r) + 5r^2, \quad \Sigma = E + 2 \ln r, \quad (24)$$

hence one can express r as a function of E , and Σ as a function of E as well, thus

$$\frac{\partial \ln v}{\partial E} = \frac{\partial \Sigma}{\partial E} \frac{\partial \ln v}{\partial \Sigma}$$

from where it follows that

$$\frac{\partial r}{\partial E} = -\frac{3}{4}r - \frac{1}{100r}, \quad \text{or else} \quad E = -\frac{2}{3} \ln(75r^2 + 1),$$

that is

$$r = \sqrt{\frac{1}{75} \left(e^{-\frac{3}{2}E} - 1 \right)}. \quad (25)$$

Plugging the expression of E in terms of r into (24), one gets that

$$f(r) = \frac{e^{-\frac{5}{2}r^2}}{(75r^2 + 1)^{\frac{1}{3}}}, \quad (26)$$

whence expressions for η , s and $\epsilon = \epsilon_0 s \sqrt{\eta}$ can be easily derived.

Next, from (25) and the first equation in (23), one can easily find an expression for the constitutive equation, namely:

$$v = F(\Sigma) \left(\frac{e^{-\frac{3}{2}E}}{(e^{-\frac{3}{2}E} - 34)^{18}} \right)^{\frac{1}{51}}, \quad (27)$$

where $F(\Sigma)$ must satisfy

$$\frac{\partial \ln F}{\partial \Sigma} = -\frac{25r^2}{25r^2 - 11},$$

where r in the right hand side of the equation has to be expressed in terms of Σ . From the second equation in (24), it follows that r must be the only real solution of

$$r^3 - 75e^{\frac{3}{2}\Sigma}r^2 - e^{\frac{3}{2}\Sigma} = 0,$$

which has a rather complicated form. In any case, one gets

$$F(\Sigma(r)) = (25r^2 - 11)^{-\frac{18}{51}} (75r^2 + 1)^{\frac{1}{51}},$$

and thus, we have proven that a solution exists, which is regular at the origin $r = 0$, satisfies the dominant energy condition in some neighborhood containing the origin and possesses a constitutive equation which can be given in a closed form.

1.0.1 A toy star model

We next construct a shear free, static elastic solution that matches to Schwarzschild vacuum solution across a spherically symmetric static surface.

Consider the line element

$$ds_1^2 = -a^2(r)dt^2 + Y^2(r) \left(dr^2 + d\Omega^2 \right) \quad (28)$$

According to (5)–(7), the field equations for this metric are:

$$8\pi\epsilon v = 8\pi\rho = -2\frac{Y''}{Y^3} + \frac{Y'^2}{Y^4} + \frac{1}{Y^2}, \quad (29)$$

$$16\pi\epsilon\eta\frac{\partial v}{\partial\eta}=8\pi p_1=2\frac{a'}{a}\frac{Y'}{Y^3}+\frac{Y'^2}{Y^4}-\frac{1}{Y^2}, \quad (30)$$

$$8\pi\epsilon s\eta\frac{\partial v}{\partial s}=8\pi p_2=\frac{Y''}{Y^3}+\frac{a''}{aY^2}-\frac{Y'^2}{Y^4}, \quad (31)$$

Consider next the vacuum Schwarzschild metric

$$ds_2^2=-\left(1-\frac{2m}{\rho}\right)^2 d\tau^2+\frac{d\rho^2}{1-\frac{2m}{\rho}}+\rho^2 d\Omega^2 \quad (32)$$

and let us match both spacetimes across a spherically symmetric surface Σ of constant radius, which will read in the two coordinate systems used and in an obvious notation:

$$\Sigma_1=\{r=r_\Sigma\}, \quad \Sigma_2=\{\rho=\rho_\Sigma\}. \quad (33)$$

Further, the (unit) normal to this surface will be in this notation:

$$\vec{n}_1=\frac{1}{Y}\partial_r, \quad \vec{n}_2=\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}\partial_\rho, \quad (34)$$

and the tangent space to Σ at each point can be written as

$$T\Sigma=\text{span}\{\partial_\tau,\partial_\theta,\partial_\phi\}. \quad (35)$$

The first fundamental form on Σ for the spacetimes (28) and (32) will read, respectively (see for instance [19] or [20])

$$d\sigma_1^2=-a^2\dot{t}^2d\tau^2+Y^2d\Omega^2 \quad (36)$$

$$d\sigma_2^2=-\left(1-\frac{2m}{\rho_\Sigma}\right)^2 d\tau^2+\rho_\Sigma^2 d\Omega^2, \quad (37)$$

where $\dot{t}\equiv(dt/d\tau)_\Sigma$ and all the quantities appearing are to be evaluated on Σ .

The non-zero, independent components of the second fundamental forms for these spacetimes are

$$K_{1\tau\tau}=-aa'\frac{1}{Y}\dot{t}^2, \quad K_{1\theta\theta}=Y' \quad (38)$$

$$K_{2\tau\tau}=-\frac{m}{\rho_\Sigma^2}\sqrt{1-\frac{2m}{\rho_\Sigma}}, \quad K_{2\theta\theta}=\rho_\Sigma\sqrt{1-\frac{2m}{\rho_\Sigma}} \quad (39)$$

where again, all functions must be evaluated on Σ .

Now, the matching conditions demand continuity of the first and second fundamental forms on the matching surface Σ , that is,

$$a^2\dot{t}^2\stackrel{\Sigma}{=}1-\frac{2m}{\rho} \quad (40)$$

$$Y^2\stackrel{\Sigma}{=}\rho^2 \quad (41)$$

$$aa' \frac{1}{Y} i^2 \stackrel{\Sigma}{=} \frac{m}{\rho^2} \sqrt{1 - \frac{2m}{\rho}} \quad (42)$$

$$Y' \stackrel{\Sigma}{=} \rho \sqrt{1 - \frac{2m}{\rho}}. \quad (43)$$

The conditions above can be rewritten as:

$$\frac{a'}{a} \stackrel{\Sigma}{=} \frac{m}{\rho} \sqrt{1 - \frac{2m}{\rho}}, \quad i^2 \stackrel{\Sigma}{=} a^{-2} \left(1 - \frac{2m}{\rho}\right), \quad (44)$$

$$Y \stackrel{\Sigma}{=} \rho, \quad Y' \stackrel{\Sigma}{=} \rho \sqrt{1 - \frac{2m}{\rho}}, \quad (45)$$

and it is then easy to see, from the expression for the radial pressure p_1 , see (30), that one then has

$$p_1 \stackrel{\Sigma}{=} 0.$$

Further, from the definition for the mass for the interior solution, $m_1 = \frac{1}{2}Y(1 - \chi)$ for $\chi = g^{ab}Y_{,a}Y_{,b} = Y^{-2}Y'^2$, it readily follows that the mass is continuous across Σ , that is,

$$m_1 \stackrel{\Sigma}{=} m.$$

The following choices of the metric functions $a(r)$ and $Y(r)$,

$$a = \left[\sec \left(r + \arcsin \frac{m}{2} \right) + \tan \left(r + \arcsin \frac{m}{2} \right) \right]^m, \quad (46)$$

$$Y = \sin \left(r + \arcsin \frac{m}{2} \right), \quad (47)$$

lead to

$$\rho = \frac{1 + 2 \sin^2 \left(r + \arcsin \frac{m}{2} \right)}{\sin^4 \left(r + \arcsin \frac{m}{2} \right)} \quad (48)$$

$$p_1 = - \frac{2 \sin^2 \left(r + \arcsin \frac{m}{2} \right) - 2m \sin \left(r + \arcsin \frac{m}{2} \right) - 1}{\sin^4 \left(r + \arcsin \frac{m}{2} \right)} \quad (49)$$

$$p_2 = \frac{m \sin^3 \left(r + \arcsin \frac{m}{2} \right) + m^2 \sin^2 \left(r + \arcsin \frac{m}{2} \right) - \cos^2 \left(r + \arcsin \frac{m}{2} \right)}{\sin^4 \left(r + \arcsin \frac{m}{2} \right) \cos^2 \left(r + \arcsin \frac{m}{2} \right)} \quad (50)$$

and it is easy to see that all quantities are finite at $r = 0$.

Further, the matching with the Schwarzschild solution is then possible for

$$\rho_{\Sigma} = \frac{m}{2} + \frac{1}{2}\sqrt{m^2 + 2}, \Leftrightarrow r_{\Sigma} = \arcsin\left(\frac{m}{2} + \frac{1}{2}\sqrt{m^2 + 2}\right) - \arcsin\frac{m}{2}. \quad (51)$$

Now, $\rho_{\Sigma} > 2m$ if and only if $m < \frac{1}{2}$; the dominant energy condition fails to be satisfied for $m = \frac{1}{2}$ but for values strictly less this, for example, $m = 0.4$, the matching surface lays outside the Schwarzschild horizon and the interior (elastic) solution satisfies the dominant energy condition.

As discussed previously, a constitutive equation $v = v(\eta, s)$ for the elastic solution always exists in the case of shear-free static solutions as the one discussed here, but in the present case, it cannot be found analytically; and the same comment applies also to the function $f(r)$. To find them, one should proceed in a way similar to that in example 1; thus, setting $E = \ln \eta$ and $\Sigma = \ln s$, from the field (29) to (31), one gets

$$\frac{\partial \ln v}{\partial E} = \frac{1}{2} \frac{p_1}{\rho}, \quad \frac{\partial \ln v}{\partial \Sigma} = \frac{p_2}{\rho}. \quad (52)$$

Further, from $E = 2 \ln f - 2 \ln Y$, $\Sigma = E + 2 \ln r$ and since all the functions depend only on r and, therefore,

$$\frac{\partial \ln v}{\partial E} = \frac{\partial \Sigma}{\partial E} \frac{\partial \ln v}{\partial \Sigma},$$

one gets

$$\frac{\partial r}{\partial E} = \frac{r}{2} \left(\frac{1}{2} \frac{p_1}{p_2} - 1 \right) \quad (53)$$

which, upon integration, yields $E = E(r)$ (and $r = r(E)$).

The function $f(r)$ can then be found substituting the function $E(r)$ thus obtained in $E = 2 \ln f - 2 \ln Y$ and solving for f . On the other hand, substitution of $r = r(E)$ in (52) allows one to get $v = v(E, \Sigma)$, and therefore, $v = v(\eta, s)$.

The integration involved in (53) cannot be performed analytically, and thus the problem can only be solved up to quadratures.

We do not claim this example to be of any particular physical relevance, but it provides a relatively simple example of a solution which is regular everywhere, satisfies the dominant energy condition and can be matched to the Schwarzschild vacuum solution.

1.1 Non-static shear-free solutions

Assume now that $\dot{Y} \neq 0$, so that $a(t, r)$ takes the form (BCV-50),³ substituting this into (BCV-49)⁴ and redefining the coordinate t so as to absorb the arbitrary function $L(t)$, one has

$$ds^2 = -\frac{\dot{Y}}{Y} dt^2 + Y^2(r, t) \left(dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (54)$$

From $G_r^t = 0$, it follows now that $Y(r, t) = A(t)B(r)$, which substituted above yields, after a trivial redefinition of the coordinate t :

$$ds^2 = -dt^2 + A^2(t)B^2(r) \left(dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (55)$$

A direct computation of the EFEs for the above metric gives

$$8\pi\epsilon v = \frac{B'^2 - 2B''B + 3\dot{A}^2 B^4 + B^2}{A^2 B^4}, \quad (56)$$

$$16\pi\epsilon\eta \frac{\partial v}{\partial \eta} = -\frac{2AB^4\ddot{A} - B'^2 + \dot{A}^2 B^4 + B^2}{A^2 B^4}, \quad (57)$$

$$8\pi\epsilon s \frac{\partial v}{\partial s} = -\frac{2AB^4\ddot{A} - B''B + \dot{A}^2 B^4 + B'^2}{A^2 B^4}. \quad (58)$$

where the energy density, radial and tangential pressures are

$$\rho = \epsilon v, \quad p_1 = 2\epsilon\eta \frac{\partial v}{\partial \eta}, \quad p_2 = \epsilon s \frac{\partial v}{\partial s}.$$

At this point, it is worth mentioning that general (i.e., not restricted by the shear-free condition) non-static solutions with vanishing radial pressure have been studied in detail by Magli [18] using Ori's massfbarea coordinates, although in a rather different context (the cosmic censorship conjecture), solving the problem up to a quadrature.

In our case (i.e., non-static and shear-free solutions), vanishing radial pressure implies, from (57) above, that

$$2A\ddot{A} - \frac{B'^2}{B^4} + \dot{A}^2 + \frac{1}{B^2} = 0$$

³

$a = L(t) \frac{\dot{Y}}{Y}.$

⁴

$ds^2 = -a(r, t)dt^2 + Y^2(r, t) \left(dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right).$

which is separable in the coordinates t and r , thus we have

$$B(r) = \frac{2Cqe^{-r}}{C^2e^{-2r} + \epsilon}, \quad 2A\ddot{A} + \dot{A}^2 = \epsilon q^{-2}$$

where⁵ $\epsilon = \pm 1$ and q and C are constants, and therefore, a well-characterized subset of the solutions considered here are included in those studied in [18].

Back to the general non-static shear-free case, the dominant energy condition $\rho \geq 0$, $\rho \pm p_1 \geq 0$ and $\rho \pm p_2 \geq 0$ implies

$$B'^2 - 2B''B + 3\dot{A}^2B^4 + B^2 \geq 0, \quad (59)$$

$$-(-B'^2 + B''B - \dot{A}^2B^4 + AB^4\ddot{A}) \geq 0, \quad (60)$$

$$AB^4\ddot{A} - BB'' + 2\dot{A}^2B^4 + B^2 \geq 0, \quad (61)$$

$$-(BB'' - 2\dot{A}^2B^4 - B^2 + 2AB^4\ddot{A}) \geq 0, \quad (62)$$

$$2B'^2 - 3B''B + 4\dot{A}^2B^4 + B^2 + 2AB^4\ddot{A} \geq 0. \quad (63)$$

We next address the question of the existence of a constitutive equation $v = v(\eta, s)$ where in this case, η and s are given by (see (BCV-31)⁶)

$$\eta = \frac{f^2(r)}{A^2(t)B^2(r)}, \quad s = \frac{r^2f^2(r)}{A^2(t)B^2(r)}. \quad (64)$$

Dividing (57) and (58) by (56), and defining as before $E = \ln \eta$, $\Sigma = \ln s$, we get

$$\frac{\partial \ln v}{\partial E} = -\frac{1}{2} \frac{2AB^4\ddot{A} - B'^2 + \dot{A}^2B^4 + B^2}{B'^2 - 2B''B + 3\dot{A}^2B^4 + B^2} \equiv V_E, \quad (65)$$

$$\frac{\partial \ln v}{\partial \Sigma} = -\frac{2AB^4\ddot{A} - B''B + \dot{A}^2B^4 + B^2}{B'^2 - 2B''B + 3\dot{A}^2B^4 + B^2} \equiv V_\Sigma. \quad (66)$$

In order for a constitutive equation $v = v(\eta, s)$ (or equivalently $v = v(E, \Sigma)$) to exist, it must be that

$$\frac{\partial^2 \ln v}{\partial \Sigma \partial E} = \frac{\partial^2 \ln v}{\partial E \partial \Sigma} \Leftrightarrow \frac{\partial V_E}{\partial \Sigma} = \frac{\partial V_\Sigma}{\partial E}. \quad (67)$$

Notice that

$$\partial_E = \frac{\partial t}{\partial E} \partial_t + \frac{\partial r}{\partial E} \partial_r, \quad \partial_\Sigma = \frac{\partial t}{\partial \Sigma} \partial_t + \frac{\partial r}{\partial \Sigma} \partial_r. \quad (68)$$

⁵ Note that the equation for A can be simplified to $\ddot{a} = \frac{3}{4}\epsilon q^{-2}a^{-\frac{1}{3}}$.

⁶

$$s = f^2(r) \frac{r^2}{Y^2}, \quad \eta = \frac{f^2(r)}{b^2}.$$

Now, from the expression (64) and the corresponding one for E and Σ , it follows that

$$r = e^{\frac{1}{2}(\Sigma-E)}, \quad A(t) = \frac{f}{B} e^{-\frac{1}{2}E}; \quad (69)$$

and differentiating them with respect to E and Σ and applying the chain rule, we get

$$\frac{\partial r}{\partial E} = -\frac{1}{2}r, \quad \frac{\partial r}{\partial \Sigma} = \frac{1}{2}r, \quad (70)$$

$$\frac{\partial t}{\partial E} = -\frac{1}{2\dot{A}} \left[r \left(\frac{f}{B} \right)' + \frac{f}{B} \right] \frac{AB}{f}, \quad \frac{\partial t}{\partial \Sigma} = \frac{1}{2\dot{A}} r \left(\frac{f}{B} \right)' \frac{AB}{f}. \quad (71)$$

Substituting the above expressions into (68), (67) reads, after some manipulations,

$$\frac{A}{\dot{A}} r \left(\frac{f'}{f} - \frac{B'}{B} \right) \partial_t (V_E + V_\Sigma) + r \partial_r (V_E + V_\Sigma) + \frac{A}{\dot{A}} \partial_t V_\Sigma = 0. \quad (72)$$

Solving for $\frac{f'}{f}$, we get, after some algebra,

$$3rB^2 \frac{f'}{f} = \frac{-K[(S + 3B^4\dot{A}^2)A_{ttt} - 6B^4\dot{A}\ddot{A}^2 + 3B^4A^{-1}\dot{A}^3] + MA^{-1}\dot{A}\ddot{A}}{[(S + 3B^4\dot{A}^2)A_{ttt} - 6B^4\dot{A}\ddot{A}^2 + 3B^4A^{-1}\dot{A}^3] + SA^{-1}\dot{A}\ddot{A}}, \quad (73)$$

where, K , S and M are functions of r alone given by:

$$\begin{aligned} K &= 2B - 3rB', \quad S = B'^2 - 2BB'' + B^2, \\ M &= -6rB^2B_{rrr} + 2B(9rB' + B)B'' + (2BB' - 9rB'^2 - 3rB^2)B' - 4B^3. \end{aligned}$$

Since the left hand side of (73) depends only on r , this implies that the time derivative of the right hand side must vanish. A careful but otherwise trivial analysis reveals that there are only three possibilities, assuming the metric is non-static, namely

1. $M + KS = 0$, in which case $B(r)$ is determined by the resulting ordinary third-order differential equation. $A(t)$ is in principle arbitrary, and $f(r)$ is fixed by (73) once the solution for $B(r)$ to the equation $M + KS = 0$ is given. We have not been able to find an integral for $B(r)$ in closed form, but in this case one has for $f(r)$:

$$\frac{f'}{f} = \frac{B'}{B^2} - \frac{2}{3rB}.$$

2. $\ddot{A} = 0$, which in turn implies $A = t$ without loss of generality, since the two constants of integration may be absorbed by suitable redefinitions of t and B . In this case, $B(r)$ is free, constrained only by the requirements imposed by the DEC, and once it is chosen, $f(r)$ is determined through (73), which implies as in the

previous case

$$\frac{f'}{f} = \frac{B'}{B^2} - \frac{2}{3rB}.$$

3. In this case, both $A(t)$ and $B(r)$ are determined as the solutions of the following two third-order differential equations:

$$\begin{aligned} -2BB'' + B'^2 + B^2 - kB^4 &= 0, \quad k = \text{constant}, \\ AA_{ttt} \left(\frac{k}{3} + \dot{A}\ddot{A} \right) - 2A\dot{A}\ddot{A}^2 + \dot{A}^3 - q\dot{A}\ddot{A} &= 0, \quad k, q = \text{constant}. \end{aligned}$$

Since $A(t)$ and $B(r)$ are fixed, so is $f(r)$, and (73) implies in this case

$$3rB^2 \frac{f'}{f} = \frac{M + (k - 3q)KB^4}{3qB^4}.$$

As in the first case above, we have not been able to find integrals for $A(t)$ or $B(r)$ in closed form.

Example 2

Let us next investigate in some detail the second case above, that is, $A = t$. Substituting this into the EFEs, we get

$$\begin{aligned} \rho &= \frac{1}{8\pi t^2} \left(-\frac{2B''}{B^3} + \frac{B'^2}{B^4} + \frac{1}{B^2} + 3 \right), \quad p_1 = \frac{1}{8\pi t^2} \left(\frac{B'^2}{B^4} - \frac{1}{B^2} - 1 \right), \\ p_2 &= \frac{1}{8\pi t^2} \left(\frac{B''}{B^3} - \frac{B'^2}{B^4} - 1 \right). \end{aligned} \quad (74)$$

On the other hand, the DEC (59)–(63) are all satisfied if and only if the following three inequalities hold:

$$\begin{aligned} -3BB'' + 2B'^2 + B^2 + 4B^4 &\geq 0, \quad -BB'' + B'^2 + B^4 \geq 0, \\ -BB'' + B^2 + 2B^4 &\geq 0 \end{aligned} \quad (75)$$

which upon setting $B \equiv e^b$ are equivalent to

$$-3b'' - b'^2 + 1 + 4e^{2b} \geq 0, \quad -b'' + e^{2b} \geq 0, \quad -b'' - b'^2 + 1 + 2e^{2b} \geq 0. \quad (76)$$

It is easy to see that these conditions can be satisfied, at least for certain ranges of the radial coordinate $r \in [0, R)$, for suitably chosen functions $b(r)$, such as

$$\begin{aligned}
 b(r) &= \frac{3}{2} \ln \left[\frac{2}{3} + \sinh^2 \left(\frac{r-r_0}{c} \right) \right] - \ln c \Leftrightarrow B(r) \\
 &= \frac{\sqrt{3}}{9c} \ln \left[-1 + 3 \cosh^2 \left(\frac{r-r_0}{c} \right) \right]^{\frac{3}{2}}
 \end{aligned} \quad (77)$$

The form of $f(r)$ can be given explicitly up to a quadrature.

As in previous cases, we do not claim that it has any particular significance, but it provides a relatively simple instance of solution with the desired properties.

Example 3

Another simple example with similar characteristics is provided by the following choice of $B(r)$:

$$b(r) = \frac{3}{2} \ln \left(\frac{2}{3} + (r-r_0)^2 \right) \Leftrightarrow B(r) = \frac{\sqrt{3}}{9} \left(2 + 3(r-r_0)^2 \right)^{\frac{3}{2}}, \quad (78)$$

in which case $f(r)$ can be integrated out yielding:

$$f(r) = \exp \left\{ \frac{-15 - 9(r-r_0)^2 + \frac{3}{2}(2+3(r-r_0)^2)\sqrt{4+6(r-r_0)^2} \tanh^{-1} \left(\frac{2}{\sqrt{4+6(r-r_0)^2}} \right)}{\sqrt{6+9(r-r_0)^2} (2+3(r-r_0)^2)} \right\}$$

The DEC is satisfied for certain ranges of the radial coordinate $r \in [0, R)$ assuming $r_0 \neq 0$. Similar remarks to the ones above regarding its physical significance apply also here.

Example 4

Interestingly enough, one can see that the dominant energy condition holds everywhere, i.e., $r \in [0, \infty)$ for the following choice of the function $b(r)$:

$$b(r) = \ln \left[\frac{16e^{-r-2}}{-16 + e^{-4r-4}} \right] \Leftrightarrow B(r) = \frac{16e^{-r-2}}{-16 + e^{-4r-4}}. \quad (79)$$

In this case, $f(r)$ takes the form

$$f(r) = Ce^{e^{r+2} - \frac{1}{16}e^{-3r-2} - \frac{2}{3}e^2 Ei(1, -r) + \frac{1}{24}Ei(1, 3r)e^{-2}}, \quad (80)$$

where C is a constant and Ei stands for the exponential integral defined by $Ei(a, r) = r^{a-1} \Gamma(1-a, r)$.

As in previous cases, we do not claim that it has any particular significance, but it provides a relatively simple instance of solution with the desired properties.

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