Underwriting profit margin of P/L insurance in the fuzzy-ICAPM*

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Abstract The ICAPM is used to study the underwriting profit margin of the P/L insurance company, including the insurances of automobile damage, automobile liability and fire, in which the parameters are the symmetric or non-symmetric triangular fuzzy numbers. From the ten-year data of a company in Taiwan we determine the lower and upper limits associated with the various α -level of the fuzzy numbers. Our results show that the best-fitting parameters of the model from our data are the asymmetric triangular fuzzy numbers. The skew factors in each insurance are determined, which could be used to perform the forecasting of the underwriting profit margin. Our results show that the systematic risk in the fuzzy environment (with best-fitting value of skew factor) becomes larger than that in the crisp environment. However, the insurance underwriting leverage and insurance financial leverage in the fuzzy environment are smaller than those in the crisp environment.

Keywords Underwriting profit margin · Insurance capital asset pricing model · Fuzzy set theory

JEL Classification G22 · G32

1. Introduction

Financial pricing models are replacing traditional assessment techniques for underwriting profit margin (UPM) in property-liability (P/L) insurers. Thus, policy prices

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should reflect equilibrium relationships between risk and return or, minimally, avoid the creation of arbitrage opportunities. The first attempts to combine investment income with UPM involved developing a target total rate of return for insurers similar to the target total rate of return for utilities. Although there are many other methods, the Capital Asset Pricing Model (CAPM) developed in the mid-1960s by Sharpe [1964], Lintner [1965], and Mossin [1966] has typically been used to select the Target Total Rate of Return (TRR) D'ARcy and Garven [1990, 1998]. The CAPM expresses expected return on equity as consisting of two components: a risk-free component and a market risk premium, which is essentially a reward for taking on risk. The degree of compensation for risk-taking is measured by the equity beta, which quantifies systematic, as opposed to nonsystematic or diversifiable risk. Diversifiable risk is not compensated by the market, since it can be eliminated through appropriate investment diversification methods D'ARcy and Garven [1998]. In other words, the systematic risk, or market risk, can affect all market investments. A recession or war, for example, might impact all investments in a portfolio. Since we can usually eliminate the nonsystematic risk, we focus primarily on the systematic risk.

The first model of this type was developed by Ferrari [1968], which presents the basic algebraic model of the insurer but does not link the concept of market equilibrium. An important advance in insurance financial pricing was the linkage of algebraic model of insurance firm with the capital asset pricing model. The model has been applied to insurance by Cooper [1974] and Biger and Kahane [1978]. The new version of the model not only assumed a world without taxes, but also that each dollar of premium is invested for exactly one year before being paid out in the form of loss or expense. This latter assumption was relaxed by Kahane [1979]. Other authors included taxes in their versions of the model. Both Hill [1979] and Fairley [1979], for example, include a single average tax rate in their versions of the model. Hill and Modigliani [1981] had also developed CAPM applications to P/L insurance which is named as insurance CAPM (ICAPM). The model considers the impact of taxes and allows for differential tax rates. Urrutia [1986] recognized two tax rates—one for underwriting income and another for investment income. These models provide important insights but are too rigid to be used in realistic situations, especially in the insurance financial research.

This article uses the ICAPM to study the UPM of the P/L insurance in the fuzzy environment. The fuzzy set theory (FST) was introduced by Zadeh [1965]. FST has been used successfully in insurance problems that require much actuarial subjective judgment and those for which measuring the embedded variables is difficult Ostaszewski [1993]. It has been over 20 years since the first article on fuzzy logic has been applied to insurance by DeWit [1982]. Although FST actually does not compete with mathematical probability as a means of evaluating random phenomena, but rather is complementary to probability theory in dealing with real world problems for which the available information is subjective, incomplete, or unreliable Cummins and Derrig [1997].

The article sought to quantify the fuzziness in underwriting. Since then, the universe of discourse has expanded considerably and now includes fuzzy logic applications in non-life insurance (or P/L insurance) involving classification, projected liabilities, underwriting, investments and pricing. Therein, Lemaire [1990] applied fuzzy logic to underwriting and reinsurance decisions whereas Cummins and Derrig [1993] used

fuzzy decision to evaluate several econometric methods of claim cost forecasting. Derrig and Ostaszewski [1994, 1995a] showed that fuzzy clustering methods are suitable for risk classification and Young [1996, 1997] applied fuzzy reasoning to insurance rate decisions. Cummins and Derrig [1997] used fuzzy logic to address the financial pricing of property-liability insurance contracts.

In this paper we use the ICAPM to study the underwriting profit margin of the P/L insurance in which the parameters are the symmetric or non-symmetric triangular fuzzy numbers. We will consider the insurances of automobile damage, automobile liability and fire. In Section 2, after describing the ICAPM we presents a brief review of the fuzzy set theory. We apply the fuzzy theory to formulate the underwriting profit margin of P/L insurance company under the fuzzy environment. In Section 3 we apply the formula to investigate the three lines of insurance from the ten-year data in Taiwan. We determine the lower and upper limits associated with the various α -level of the fuzzy members. Our results show that the best-fitting parameters of the model are the asymmetric triangular fuzzy numbers. We determine the skew factors therein and find that the systematic risk in the fuzzy environment (with best-fitting value of skew factor) becomes larger than that in the crisp environment. However, the insurance underwriting leverage and insurance financial leverage in the fuzzy environment are smaller than those in the crisp environment. The last section is devoted to the discussions.

2. Fuzzy ICAPM

2.1. ICAPM under crisp environment

ICAPM of Hill and Modigliani [1981] is given economic content by assuming that the equilibrium expected return on the P/L insurer's underwriting is determined by:

$$(UPM)_i = \frac{1}{1 - t_u} \left(\frac{p}{s} \right)_i^{-1} \left[\left(1 + \frac{\ell}{s} \right)_i t_a - \left(\frac{\ell}{s} \right)_i \right] (r_f)_i + \beta_u (E(r_m) - (r_f)_i), \quad (2.1)$$

where $(UPM)_i$ is the sample i of underwriting profit margin with different tax rates for underwriting and investment. The variables used in the above formulas include:

$\beta_u = \frac{Cov(r_u, r_m)}{Var(r_m)}$ = amount of systematic risk

r_u = rate of return on underwriting

r_m = rate of return on market

r_f = risk-free rate of return

$E(r_m)$ = expected return on the market portfolio

$E(r_m) - r_f$ = market risk premium

p = premium

ℓ = reserves and other liabilities (excluding equity in unearned premium reserves)

s = equity (or adjusted statutory surplus)

t_u = tax rate on underwriting income

t_a = tax rate on investment income

p/s = insurance underwriting leverage factor

$1 + \ell/s$ = insurance financial leverage factor Cummins and Harrington [1987] and Cummins and Philips [1999]

The insurer must earn the return on underwriting in order to avoid penalizing equity (if the return is too low) or charging policyholders too much (if the return is too high). In practice, the appropriate calculation of effective underwriting profit margin is calculated as the risk premium associated with the systematic risk of the insurance underwriting process, offset by investment income, which is credited at the risk-free rate of return and is generated by both insurance underwriting leverage (p/s) and insurance financial leverage ($1 + \ell/s$). In formula (2.1) the (p/s) ratio, which is sometimes referred to as the insurance exposure, has been advocated on occasion as a rule-of-thumb indicator of insolvency and the insurance financial leverage factor $1 + \ell/s$ dependent on the size of reserves relative to surplus or a partial determinant of the riskiness of the owner's investment in the insurance company.

2.2. Fuzzy set theory and fuzzy numbers

Fuzzy analysis has been a popular method with many successful applications for insurance financial and actuarial science (Cummins and Derrig [1993, 1997], Derrig and Ostaszewski [1994], Shapiro [2004], and Young [1996]). In this section we will consider the ICAPM under fuzzy environment. We will work on the fuzzy ICAPM under the assumption that the parameters $r_f, r_u, r_m, \ell/s$ and p/s are the symmetric or asymmetric triangular fuzzy numbers (TFN).

These parameters constitute a reference set X . The fuzzy subset $\tilde{A}$ is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function and is a mapping $\mu_{\tilde{A}} : x \rightarrow [0, 1]$. Therefore an element x has its image within $[0, 1]$, where 0 indicates nonmembership to the fuzzy subset $\tilde{A}$ and 1 indicates absolute membership. Two other conditions are required for an FN. First, it must be a normal fuzzy set, i.e., it exists at least one $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$. Second, it must be convex (i.e., its α -cuts must be convex sets in the real numbers).

Alternatively, a fuzzy subset $\tilde{A}$ can be represented by its level of α -cuts and the fuzzy subset $\tilde{A}_\alpha$ has mathematical expression: $\tilde{A}_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$, $1 \geq \alpha \geq 0$.

Figure 1 shows the shape of membership function for the triangular fuzzy coefficient $\tilde{A}_i$ which is the widely used (Dubois and Prade [1993] Shapiro [2004], and Zimmermann [1999]). As the $\tilde{A}_i$'s have triangular membership functions each fuzzy number coefficient $\tilde{A}_i$ can be uniquely defined by

$$\tilde{A}_i = \{a_i^C, a_i^L, a_i^U\}, \quad (2.2)$$

where a_i^L is the lower limit, a_i^U is the upper limit, and a_i^C is the point having the property that $\mu_{\tilde{A}}(a_i^C) = 1$. The lower spread s_i^L and upper spread s_i^U are defined as

$$s_i^L = a_i^C - a_i^L, \quad s_i^U = a_i^U - a_i^C. \quad (2.3)$$

where a_i^C is the center. The skew factor k_i is defined as s_i^U/s_i^L and thus $k_i = 1$ is the case of the symmetric triangular fuzzy function. The membership function therefore

has the form

$$\mu_{\tilde{A}_i}(x) = \begin{cases} 1 - \frac{x - a_i^C}{k_i s_i^L}, & a_i^C \leq x \leq a_i^C + k_i s_i^L \\ 1 - \frac{a_i^C - x}{s_i^L}, & a_i^C - s_i^L \leq x \leq a_i^C \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

Inverse mappings from any given degree of membership to its corresponding x values can be defined, one on the lower limit of the fuzzy number, another on the upper limit of this number. Thus, a fuzzy number can always be given by its lower and upper representation of each degree of membership:

$$\tilde{A}_i = (a_i^C, a_i^L, a_i^U) = (a_i^C, a_i^L + (a_i^C - a_i^L) \mu_{\tilde{A}}(a_i), (a_i^U - (a_i^U - a_i^C) \mu_{\tilde{A}}(a_i))). \quad (2.5)$$

Note that $(a_i^L, a_i^U)/2 \neq a_i^C$ in the case with asymmetric triangular fuzzy function. Using the above notations we can express the underwriting profit margin of (2.1) under the fuzzy environment.

2.3. ICAPM under fuzzy environment

Under the asymmetric fuzzy environment the fuzzy parameters become

$$W_i = [W_i^C, W_i^L, W_i^U], \quad (2.6)$$

such that

$$W_{\alpha,i} = [W_{\alpha,i}^L, W_{\alpha,i}^U] = [W_i^L + \alpha \cdot (W_i^C - W_i^L), W_i^U - \alpha \cdot (W_i^U - W_i^C)], \quad (2.7)$$

Fig. 1 Membership function for the triangular fuzzy function $\tilde{A}_i$

in which $W_{\alpha,i}$ are the fuzzy parameters $(r_f)_{\alpha,i}$, $(r_m)_{\alpha,i}$, $(r_u)_{\alpha,i}$, $(p/s)_{\alpha,i}$ and $(\ell/s)_{\alpha,i}$. The underwriting profit margin (UPM) in the fuzzy ICAPM can now be expressed as:

$$(UPM)_{\alpha,i} = \frac{1}{1 - t_u} \left(\frac{p}{s} \right)_{\alpha,i}^{-1} \left[\left(1 + \frac{\ell}{s} \right)_{\alpha,i} t_a - \left(\frac{\ell}{s} \right)_{\alpha,i} \right] (r_f)_{\alpha,i} + (\beta_u)_{\alpha} (E(r_m)_{\alpha} - (r_f)_{\alpha,i}), \quad (2.8)$$

where

$$\begin{aligned} E(r_m)_{\alpha} &= [E(r_m)_{\alpha}^L, E(r_m)_{\alpha}^U] \\ &= \left[\text{Min} \left(\frac{1}{n} \sum_{i=1}^n (r_m)_{\alpha,i}^L, \frac{1}{n} \sum_{i=1}^n (r_m)_{\alpha,i}^U \right), \right. \\ &\quad \left. \text{Max} \left(\frac{1}{n} \sum_{i=1}^n (r_m)_{\alpha,i}^L, \frac{1}{n} \sum_{i=1}^n (r_m)_{\alpha,i}^U \right) \right], \end{aligned} \quad (2.9)$$

in which we need the operators “Minimum” and “Maximum” to determine the numbers $E(r_m)_{\alpha}^L$, $E(r_m)_{\alpha}^U$ as the values of $(r_m)_{\alpha,i}$ may be positive or negative.

The value of $(\beta_u)_{\alpha}$ is calculated from the relation

$$(\beta_u)_{\alpha} \equiv \frac{\sum_{i=1}^n ((r_u)_{\alpha,i} - E(r_u)_{\alpha})(r_m)_{\alpha,i} - E(r_m)_{\alpha}}{\sum_{i=1}^n ((r_m)_{\alpha,i} - E(r_m)_{\alpha})^2}, \quad (2.10)$$

in which the $E(r_u)_{\alpha}$ could be evaluated as that in the Eq. (2.9).

In the next section we will apply the above formula to investigate the insurances of automobile damage, automobile liability and fire from the ten-year data of a company in Taiwan, which are announced by the government. The example is used to see how could we find the best-fitting value of skew factor which be useful to perform the forecasting of the underwriting profit margin. The result depends on the input data and thus could be changed for the other company of the country.

We will calculate the lower and upper limits of the underwriting profit margins, i.e. $(UPM)_{\alpha,i}^L$ and $(UPM)_{\alpha,i}^U$, associated to the various α -level of the fuzzy members for the different various values of skew factor k . Our purpose is to determine the best-fitting value of skew factor k form the data and to discuss the difference between the lines of insurance under the crisp environment and that under the fuzzy environment. Note that the value of $(W_i^U - W_i^C)/(W_i^C - W_i^L)$ in (2.7) is equal to the value of skew factor k .

3. Applications

3.1. Underwriting profit margin

After substituting the ten-year data (1994–2003) of the insurances of automobile damage, automobile liability and fire from a P/L insurance company in Taiwan into

Table 1 Lower and upper limits of the underwriting profit margins of ten years

i	1	2	3	4	5	6	7	8	9	10
Insurance of automobile damage with skew factor $k = 0.94$										
$(UPM)_{\alpha,i}^U$	0.085	0.081	0.075	0.075	0.085	0.075	0.073	0.057	0.036	0.024
$(UPM)_{\alpha,i}^L$	0.048	0.046	0.041	0.037	0.041	0.037	0.033	0.026	0.017	0.010
Insurance of automobile liability with skew factor $k = 0.88$										
$(UPM)_{\alpha,i}^U$	0.151	0.144	0.134	0.122	0.132	0.110	0.096	0.081	0.056	0.037
$(UPM)_{\alpha,i}^L$	0.083	0.080	0.073	0.060	0.063	0.052	0.041	0.034	0.023	0.013
Insurance of fire with skew factor $K = 0.91$										
$(UPM)_{\alpha,i}^U$	0.223	0.149	0.103	0.134	0.160	0.010	0.107	0.103	0.033	0.012
$(UPM)_{\alpha,i}^L$	0.129	0.087	0.058	0.069	0.080	0.050	0.050	0.048	0.016	0.004

equation (2.8) we can obtain the values of $(UPM)_{\alpha,i}^L$ and $(UPM)_{\alpha,i}^U$ (i represents the year). The calculated results are shown in Table 1.

Note that, in the above tables we have chosen the different skew factor k which is the best-fitting parameter in each insurance.

To find the best-fitting parameter we first denote the function $(\Delta UPM(k))$ as the ten-year average of the deviation of UPM , i.e.

$$(\Delta UPM(k)) = \sqrt{\frac{1}{n} \sum_{i=1}^n ((UPM)_{defuzzy,i} - (UPM)_i)^2}, \quad (3.1)$$

in which the $(UPM)_{defuzzy,i}$ is the *defuzzification value* of underwriting profit margins which we will evaluated by the Centroid method of defuzzification (Roubens and Vincke [1988], Yager [1978, 1981], and Yager and Filev [1994]). The defuzzification method will convert the fuzzy number “ x ” into a number from the following relation

$$x_{defuzzy} = \frac{\int_{-\infty}^{\infty} x \mu(x) dx}{\int_{-\infty}^{\infty} \mu(x) dx}. \quad (3.2)$$

Using the triangle fuzzy number $\{x^C, x^L, x^U\}$ the membership function (2.4) becomes

$$\mu(x) = \begin{cases} \frac{x - x^L}{x^C - x^L}, & x^L \leq x \leq x^C \\ \frac{x^U - x}{x^U - x^C}, & x^C \leq x \leq x^U \\ 0, & \text{otherwise,} \end{cases} \quad (3.3)$$

and the defuzzification number is

$$x_{defuzzy} = \frac{1}{3}(x^U + x^C + x^L). \quad (3.4)$$

Table 2 k -dependence of $(\Delta UPM(k))$

		Automobile damage insurance			
Skew factor k	0.5	0.93	0.94	0.95	5
$(\Delta UPM(k))$	0.00661	0.00069	0.00068	0.00070	0.01600
		Automobile liability insurance			
Skeew factor k	0.5	0.87	0.88	0.89	5
$(\Delta UPM(k))$	0.00940	0.00108	0.00107	0.00111	0.02689
		Fire insurance			
Skew factor k	0.5	0.90	0.91	0.92	5
$(\Delta UPM(k))$	0.01138	0.00016	0.00013	0.00033	0.03018

In Table 2 we present the k -dependence of $(\Delta UPM(k))$, which explicitly shows the k -dependence of $(\Delta UPM(k))$ in the various insurance. In Fig. 2 we plot the corresponding figure.

We make two comments about the above table.

1. From the Table 2 we see that the best-fitting parameters of the fuzzy ICAPM from the ten-year data of company in Taiwan are the asymmetric triangular fuzzy numbers. The skew factors are $k = (0.94, 0.88, 0.91)$ in the three lines of insurance. Using the best-fitting values of skew factors we could therefore use the defuzzification method will convert the fuzzy number of UPM into a crisp number, which is the lest deviation of the data. We will in the following subsections use the best-fitting skew factors to calculate the underwriting beta, insurance underwriting leverage and insurance financial leverage, which could help us the see that effects of the fuzzy.
2. The best-fitting value of skew factor k in the three lines of insurance surely depend on the input data of company and is relevant to the method of defuzzification Yager [1978, 1981] and Yager and Filev [1994]. It is hoped that the properties found in this paper will not sensitively depend on the defuzzification method. The problem remains to be investigated.

Fig. 2 k -dependence of $(\Delta UPM(k))$. Solid line represents that of automobile damage insurance . Dashing line represents that of fire insurance and dot line represents that of automobile liability insurance

Table 3 Values of underwriting beta

Insurance	Automobile damage	Automobile liability	Fire
$\beta_u(\text{fuzzy-environment})$	0.01262	−0.01692	0.00064
$\beta_u(\text{crisp-environment})$	0.00971	−0.01341	0.00012

3.2. Underwriting beta

We now calculate the value of underwriting beta, β_u , which is the amount of systematic risk to see the effect of fuzzy.

In Table 3 we present the value of $\beta_u(\text{fuzzy} - \text{environment})$ in the fuzzy environment with best-fitting value of skew factor. The value is a defuzzification number which is calculated by the formula (3.4). The value of $\beta_u(\text{crisp} - \text{environment})$ is that calculated in the crisp environment.

From the Table 3 we have seen two properties:

1. The systematic risk in the fuzzy environment (with best-fitting value of skew factor k) become larger than that in the crisp environment. This is the general property as the insurance in the fuzzy environment will suffer more risk.
2. The systematic risk in the insurance of automobile liability is largest and that in the insurance of fire is smallest. The table also shows a positive relation between r_u and r_m in the automobile damage and fire insurances. However, it becomes a negative relation in automobile liability insurance. This relation may be a special property of our input data of company in Taiwan.

3.3. Insurance underwriting leverage and insurance financial leverage

We will now calculate the insurance underwriting leverage p/s and insurance financial leverage $1 + \ell/s$.

In Table 4 we present the values of $p/s(\text{fuzzy} - \text{environment})$ and $1 + \ell/s(\text{fuzzy} - \text{environment})$ in the fuzzy environment with best-fitting value of skew factor. The defuzzification value is calculated by the formula (3.4). The values of $p/s(\text{crisp} - \text{environment})$ and $1 + \ell/s(\text{crisp} - \text{environment})$ are those calculated in the crisp environment.

From the Table 4 we have seen the two properties:

1. The values of $(1 + \ell/s)$ is independent of the kind of insurance. This is because that the input data of the reserves and other liabilities, ℓ , and equity, s , are same in these insurance data.

Table 4 Values of (p/s) and $(1 + \ell/s)$

Insurance	Automobile damage	Automobile liability	Fire
$(p/s)(\text{fuzzy-environment})$	0.55413	0.35217	0.41390
$(p/s)(\text{crisp-environment})$	0.55656	0.35367	0.41520
$(1 + \ell/s)(\text{fuzzy-environment})$	1.56811	1.56466	1.56641
$(1 + \ell/s)(\text{crisp-environment})$	1.57135	1.57135	1.57135

2. The insurance underwriting leverage and insurance financial leverage in the fuzzy environment (with best-fitting value of skew factor k) become smaller than those in the crisp environment. This fact may be traced to the fact that the best-fitting values of skew factor k found in our case are less than 1. This is one of the main result of this paper. In other case of insurances in which the skew factor k may be larger than 1, then the insurance underwriting leverage and insurance financial leverage in the fuzzy environment could become larger than those in the crisp environment. This may be found in other country.

It shall be noted that to obtain the above numerical results we have used the spread $(W_i^U - W_i^L)/W_i^c = 0.2$. However, we have also performed the same calculated while with other spread values. It is found that the qualitative properties of our conclusion (such as that the best-fitting values of skew factor k are less than 1) does not depend on the value of spread. This means that the main properties found in this paper does not depend on the α -level.

4. Conclusion

In this paper we have used the ICAPM to study the underwriting profit margin of the P/L insurance in which the parameters are the symmetric or non-symmetric triangular fuzzy numbers. We have formulated the underwriting profit margin of P/L insurance company under the fuzzy environment. The formula is used to investigate the insurances of automobile damage, automobile liability and fire from the ten-year data in Taiwan. We determine the lower and upper limits associated with the various α -level of the fuzzy numbers and see that the best-fitting parameters of the model from our data are the asymmetric triangular fuzzy numbers. We have found the best-fitting value of skew factors k in the insurances of automobile damage and fire, respectively. The values of skew factor can therefore be used to performed the forecasting of the underwriting profit margin. Our results show that the systematic risk in the fuzzy environment (with best-fitting value of skew factor) becomes larger than that in the crisp environment. However, the insurance underwriting leverage and insurance financial leverage in the fuzzy environment are smaller than those in the crisp environment.

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