Geometriae Dedicata (2020) 207:255–263
https://doi.org/10.1007/s10711-019-00498-2

ORIGINAL PAPER

The horofunction boundary of the infinite dimensional hyperbolic space

Floris Claassens[1]

Received: 26 June 2018 / Accepted: 16 November 2019 / Published online: 27 November 2019

Abstract

In this paper we give a complete description of the horofunction boundary of the infinite dimensional real hyperbolic space, and characterise its Busemann points.

Keywords Infinite dimensional hyperbolic space · Horofunction boundary · Busemann points

Mathematics Subject Classification Primary 51M10; Secondary 53C23

1 Introduction

The geometry of the infinite dimensional real hyperbolic space $\mathbb{H}^\infty$ has been the object of study since it was first suggested by Gromov in [9]. Already much work has been done on the subject and for a detailed study of $\mathbb{H}^\infty$; see [5,6,19,22].

The main goal of this paper is to determine the horofunction boundary of $\mathbb{H}^\infty$ and its Busemann points. The horofunction boundary is a natural way to embed a, possibly non-proper, metric space into a compact topological space. In general the horofunction boundary is much larger than other commonly used boundaries, such as the visual boundary, also known as the bordification [3,21], and the Satake compatification [25]. By now the horofunction boundary has been explicitly determined in a variety of metric spaces, including special cases of, normed spaces [10–12,26], Hilbert's metric spaces [16,29] and Teichmüller spaces [14,30], see [15] for an overview.

Although the horofunction boundary tends to be large, it has proven to be very useful in a variety of applications, see [15] and the references therein. In particular, it has been used in the study of isometry groups of metric spaces, see [20,28,29], the analysis of Denjoy-Wolff type problems in metric spaces, see [1,7,13,16,17], and establishing multiplicative ergodic theorems [8].

Supported by EPSRC Grant EP/M508068/1.

✉ Floris Claassens
f.c.claassens@kent.ac.uk

[1] School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NX, UK

An important subset of the horofunction boundary consists of the Busemann points, which were introduced by Rieffel in [24]. They are horofunctions that are the limits of almost geodesics. It is of interest to understand for which spaces all horofunctions are Busemann points, see for instance [27,31].

Before stating the main result we briefly recall the definition of the real hyperbolic space. Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space and let $V = \mathbb{R} \oplus H$. Let $Q : V \to \mathbb{R}$ be the quadratic form,

$$Q((\lambda, x)) = \lambda^2 - \langle x, x\rangle$$

for $(\lambda, x) \in V$. The vector space V has a natural cone

$$V_+ = \{(\lambda, x) \in V : \|x\| < \lambda\}.$$

Let $B : V \times V \to \mathbb{R}$ be the bilinear form associated with Q,

$$B((\lambda, x), (\mu, y)) = \lambda\mu - \langle x, y\rangle$$

for $(\lambda, x), (\mu, y) \in V$. It is well known that V_+ can be equipped with a pseudometric $d_h : V_+ \times V_+ \to [0, \infty)$ given by

$$\cosh(d_h(u, v)) = \frac{B(u, v)}{\sqrt{Q(u)Q(v)}}$$

for $u, v \in V_+$, which is a metric between pairs of rays in V_+. If we restrict d_h to a hyperboloid

$$\mathbf{H} = \{u \in V_+ : Q(u) = 1\},$$

we obtain $\mathbb{H}^\infty = (\mathbf{H}, d_h)$ the *hyperboloid model* of the infinite dimensional real hyperbolic space. Please note that we do not make any assumptions on the separability or the cardinality of the orthonormal basis of the Hilbert space H, as it does not play a role here.

In this paper we will work with Klein's model of the infinite dimensional real hyperbolic space $\mathbb{H}^\infty = (\mathbf{D}, d_h)$, which is defined on the disc

$$\mathbf{D} = \{(\lambda, x) \in V : \lambda = 1 \text{ and } \|x\| < 1\}.$$

On $\mathbf{D}$ the metric d_h coincides with Hilbert's cross-ratio metric which is defined as follows. Let u and v be different elements of $\mathbf{D}$ and let $l_{u,v}$ be the line through u and v. Let u' and v' be the intersection of $l_{x,y}$ and the boundary of $\mathbf{D}$ such that u is between u' and v and v is between u and v'. The Hilbert distance between u and v is given by

$$\delta(u, v) = \frac{1}{2}\log\left(\frac{\|u' - v\|\|v' - u\|}{\|u' - u\|\|v' - v\|}\right).$$

In the study of Hilbert's metric the factor $\frac{1}{2}$ is usually ignored as it plays no role, except for fixing the curvature of the space.

We can describe the horofunction boundary and Busemann points of $\mathbb{H}^\infty$ as follows.

Theorem 1 *The horofunctions of* $\mathbb{H}^\infty = (\boldsymbol{D}, d_h)$ *are precisely the functions of the form*

$$\xi(v) = \log\left(\frac{B(\hat{u}, v) + \sqrt{(B(\hat{u}, v))^2 - Q(v)(1 - r^2)}}{(1 + r)\sqrt{Q(v)}}\right) \qquad (v \in \boldsymbol{D})$$

where $0 < r \leq 1$ *and* $\hat{u} \in \boldsymbol{D}$ *such that* $0 \leq 1 - r^2 < Q(\hat{u})$ *or* $r = 1$ *and* $\hat{u} \in \partial \boldsymbol{D}$. *Furthermore,* ξ *is a Busemann point if and only if* $r = 1$ *and* $\hat{u} \in \partial \boldsymbol{D}$*, in which case*

$$\xi(v) = \frac{1}{2}\log\left(\frac{B(\hat{u}, v)^2}{Q(v)}\right) \qquad (v \in \boldsymbol{D}).$$

The proof of Theorem 1 uses the order structure, which is connected to the hyperbolic distance as noted by Birkhoff in [2].

2 Preliminaries

Hilbert's metric Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $V = \mathbb{R} \oplus H$ with cone $V_+ = \{(\lambda, x) \in V : \|x\| < \lambda\}$. Recall that a V_+ is *cone* if V_+ is convex, $rV_+ = V_+$ for all $r > 0$ and $V_+ \cap -V_+ \subset \{0\}$. The closure of the cone defines a partial order structure on V by $u \leq v$ if and only if $v - u \in \overline{V_+}$. The order structure can be used to give an alternative formula for d_h as follows. Define the *gauge function*

$$M(u/v) = \inf\{\beta > 0 : u \leq \beta v\} \qquad (u, v \in V_+).$$

Now Birkhoff's version of d_h on V_+ is given by

$$d_h(u, v) = \frac{1}{2}\log(M(u/v)M(v/u)) \qquad (u, v \in V_+).$$

Note that for $\lambda, \mu > 0$ we have that $\delta(\lambda u, \mu v) = \delta(u, v)$. Birkhoff's version of Hilbert's metric was popularised by Bushell in [4] and is well known to coincide with Hilbert's cross ratio metric [18]. Using Birkhoff's version of Hilbert's metric one can generalise Hilbert's metric to the interior of any cone of a normed vector space. It is well known that on $\mathbf{D}$ the topology defined by Hilbert's metric and the norm topology coincide; see [17, Corollary 2.5.6].

Horofunctions and Busemann points For finite dimensional real hyperbolic spaces it is well known that $\partial\mathbb{H}^n$ coincides with the horofunction boundary. In infinite dimensions this is no longer the case as the space is no longer proper, i.e., closed balls are not compact. Let us briefly recall the construction of the horofunction boundary. Let (X, d) a metric space and let $b \in X$ be a base point. Consider the natural embedding $i : X \to C(X)$ given by

$$i(x)(y) = d(x, y) - d(x, b) \qquad (x, y \in X)$$

where $C(X)$ is equipped with the topology of compact convergence; see [23, §46]. By the triangle inequality we have

$$|i(x)(y) - i(x)(y')| = |d(y, x) - d(y', x)| \leq d(y, y'),$$

so $i(X)$ is equicontinuous. By the same methods we also find for all $x, y \in X$ that $|i(x)(y)| \leq d(y, b)$, hence $i(X)(y) = \{i(x)(y) : x \in X\}$ has compact closure in $\mathbb{R}$. By Ascoli's Theorem; see [23, Theorem 47.1] we find that $i(X)$ has compact closure in $C(X)$. The closure $\overline{i(X)}$ is called the *horofunction compactification* of (X, d). The set $\overline{i(X)}\backslash i(X)$ is called the *horofunction boundary* of (X, d) and its elements are called *horofunctions*. The horofunction compactification is metrizable if (X, d) is proper, which means it is sequentially compact. If X is not proper $\overline{i(X)}$ need not be sequentially compact any more, so for the rest of the article we will use nets. It is known that the topology of compact convergence on $\overline{i(X)}$ coincides with the topology of pointwise convergence, see the proof of Ascoli's Theorem in [23, Page 291].

We like to point out that the bordification, as discussed in Bridson and Heafliger [3], is obtained in a similar way by equipping $C(X)$ with the topology of uniform convergence on bounded sets. It is known that the bordification and the horofunction compactification coincide on proper geodesic metric spaces. There is the following useful example by Bader; see [15], illustrating the difference between the bordification and the horocompactification.

Example 1 Consider the intervals $[0, n]$ for $n \in \mathbb{N}$ which are glued together to a point x_0 at the point 0. We can equip this space with a metric d for which $d(x, y) = |x - y|$ if x and y are in the same interval and $d(x, y) = x + y$ if x and y are in different intervals. This space is known as an $\mathbb{R}$-tree, see [3]. Using x_0 as the base point, one can easily verify that there are no horofunctions in the bordification, but every unbounded sequence converges to $\xi(y) = |y|$ in the horocompactification, which is in fact $i(x_0)$.

Definition 1 A net (x_α) in a metric space (X, d) is *almost geodesic* if, for all $\varepsilon > 0$ there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have

$$d(b, x_{\alpha'}) \geq d(b, x_\alpha) + d(x_\alpha, x_{\alpha'}) - \varepsilon.$$

We call a horofunction $\xi \in \overline{i(X)} \backslash i(X)$ a *Busemann point* if there exists an almost geodesic net (x_α) in X such that $\xi = \lim_\alpha i(x_\alpha)$.

Note that if $\mathbb{H} = (\mathbf{D}, d_h)$ has finite dimension a net (x_α) in $\mathbf{D}$ will only give rise to a horofunction if (x_α) is unbounded with respect to the metric, as $\overline{\mathbf{D}}$ is norm compact. For $\mathbb{H}^\infty$ however, $\overline{\mathbf{D}}$ is not proper and in Theorem 2 we will show there are horofunctions which are limits of bounded nets. Busemann points however, always are limits of unbounded nets as observed in [31]. For convenience of the reader we have included the proof.

Proposition 1 *Let (x_α) be a net in a complete metric space X. If (x_α) is almost geodesic and bounded, then (x_α) converges to some $x \in X$.*

Proof Let b be a base point. The first step is to prove that $d(x_\alpha, b)$ converges to some $r \in \mathbb{R}$. To see this we define for an index A the supremum $r_A = \sup_{\alpha \geq A} d(x_\alpha, b)$ which exists as the net is bounded. Let $\varepsilon > 0$ and let A be an index such that for all $\alpha' \geq \alpha \geq A$ we have

$$d(x_{\alpha'}, b) \geq d(x_\alpha, b) + d(x_{\alpha'}, x_\alpha) - \varepsilon.$$

Let $\alpha_A \geq A$ be such that $0 \leq r_A - d(x_{\alpha_A}, b) < \varepsilon$. Then for all $\alpha' \geq \alpha_A$ we have

$$r_A \geq d(x_{\alpha'}, b) \geq d(x_{\alpha_A}, b) + d(x_{\alpha'}, x_{\alpha_A}) - \varepsilon \geq d(x_{\alpha_A}, b) - \varepsilon =\geq r_A - 2\varepsilon.$$

So for all $\alpha', \alpha \geq \alpha_A$ we find that $|d(x_{\alpha'}, b) - d(x_\alpha)| \leq 2\varepsilon$. Hence $(d(x_\alpha, b))$ is a Cauchy net from which it follows that $\lim_\alpha d(x_\alpha, b) = r$ for some $r \in \mathbb{R}$.

Now let $\varepsilon > 0$ and let A be an index such that for all $\alpha' \geq \alpha \geq A$ we have $|r - d(x_\alpha, b)| < \varepsilon$ and

$$d(x_{\alpha'}, b) \geq d(x_\alpha, b) + d(x_{\alpha'}, x_\alpha) - \varepsilon.$$

It follows that

$$d(x_{\alpha'}, x_\alpha) \leq d(x_{\alpha'}, b) - d(x_\alpha, b) + \varepsilon < 3\varepsilon$$

Hence (x_α) is a Cauchy net. The proposition follows by completeness. □

3 Classification of the horofunction boundary of $\mathbb{H}^\infty$

To prove Theorem 1, we will first calculate the gauge functions.

Proposition 2 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $V = \mathbb{R} \oplus H$. For all $(\mu, x), (\gamma, y) \in V_+$ we have*

$$M((\mu, x)/(\gamma, y)) = \frac{\gamma\mu - \langle x, y\rangle + \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2}.$$

Proof We know that

$$\begin{aligned} M((\mu, x)/(\gamma, y)) &= \inf\{\beta > 0 : (\mu, x) \le \beta(\gamma, y)\} \\ &= \inf\{\beta > 0 : (\gamma\beta - \mu)^2 \ge \|\beta y - x\|^2 \text{ and } \gamma\beta - \mu \ge 0\}. \end{aligned}$$

So we have to solve

$$(\gamma\beta - \mu)^2 - \|\beta y - x\|^2 = (\gamma^2 - \|y\|^2)\beta^2 - 2(\gamma\mu - \langle x, y\rangle)\beta + (\mu^2 - \|x\|^2) = 0,$$

which has solutions

$$\beta_\pm = \frac{\gamma\mu - \langle x, y\rangle \pm \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - x^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2}.$$

Note though, that

$$\begin{aligned} \gamma\beta_- - \mu &= \gamma \frac{\gamma\mu - \langle x, y\rangle - \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2} - \mu \\ &= \gamma \frac{(\gamma\mu - \langle x, y\rangle)^2 - (\gamma\mu - \langle x, y\rangle)^2 + (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}{(\gamma\mu - \langle x, y\rangle + \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)})(\gamma^2 - \|y\|^2)} - \mu \\ &= \frac{\gamma(\mu^2 - \|x\|^2)}{\gamma\mu - \langle x, y\rangle + \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}} - \mu \\ &\le \frac{\gamma(\mu^2 - \|x\|^2)}{\gamma\mu - \|x\|\|y\| + \sqrt{(\gamma\mu - \|x\|\|y\|)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}} - \mu \\ &= \frac{\gamma(\mu^2 - \|x\|^2)}{\gamma\mu - \|x\|\|y\| + |\mu\|y\| - \gamma\|x\||} - \mu \\ &= \frac{\mu\|x\|\|y\| - \gamma\|x\|^2 - |\mu^2\|y\| - \mu\gamma\|x\||}{\gamma\mu - \|x\|\|y\| + |\mu\|y\| - \gamma\|x\||}. \end{aligned}$$

We find that if $\mu\|y\| < \gamma\|x\|$, then clearly $\gamma\beta_- - \mu < 0$. If $\mu\|y\| \ge \gamma\|x\|$, then consider

$$\begin{aligned} \gamma\beta_- - \mu &\le \frac{\mu\|x\|\|y\| - \gamma\|x\|^2 - \mu^2\|y\| + \mu\gamma\|x\|}{\gamma\mu - \|x\|\|y\| + \mu\|y\| - \gamma\|x\|} \\ &= \frac{(\mu\|y\| - \gamma\|x\|)(\|x\| - \mu)}{(\gamma + \|y\|)(\mu - \|x\|)} = -\frac{\mu\|y\| - \gamma\|x\|}{\gamma + \|y\|} \le 0. \end{aligned}$$

Hence we find that $M((\mu, x)/(\gamma, y)) = \beta_+$. □

For all $u, v \in V_+$ we can rewrite this result using the quadratic and bilinear forms as

$$M(u/v) = \frac{B(u, v) + \sqrt{B(u, v)^2 - Q(u)Q(v)}}{Q(v)}.$$

Note that if $Q(u) = Q(v) = 1$, then using Proposition 2 we find

$$d_h(u, v) = \log(B(u, v) + \sqrt{B(u, v)^2 - 1}) \\ = \cosh^{-1}(B(u, v)),$$

which shows that indeed on V_+ the hyperbolic metric d_h coincides with Birkhoff's version of the Hilbert metric. We also need the following basic result from functional analysis.

Lemma 1 *Let (x_α) be a net in a Hilbert space H such that x_α converges in the weak topology to some $x \in H$ and $(\|x_\alpha\|)$ converges to some $r \geq 0$. Then $r \geq \|x\|$. Moreover, if $r = \|x\|$, then (x_α) converges to x in the norm topology.*

Proof Note that

$$\|x\|^2 = \lim_\alpha |\langle x, x_\alpha\rangle| \leq \lim_\alpha \|x\|\|x_\alpha\| = r\|x\|.$$

Now suppose that $r = \|x\|$. Then

$$\lim_\alpha \|x - x_\alpha\|^2 = \lim_\alpha \|x\|^2 + \|x_\alpha\|^2 - 2\langle x, x_\alpha\rangle = 0.$$

□

Using this we can now classify the horofunctions of $\mathbb{H}^\infty$.

Theorem 2 *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space and let $V = \mathbb{R} \oplus H$. The horofunction of the Hilbert geometry are the functions of the following form:*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle\hat{x}, y\rangle + \sqrt{(\gamma - \langle\hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1 + r)\sqrt{\gamma^2 - \|y\|^2}}\right)$$

where either $\|\hat{x}\| < 1$ and $\|\hat{x}\| < r \leq 1$ or $\|\hat{x}\| = r = 1$.

Proof Let $((1, x_\alpha))$ be a net in V_+ such that $\delta(\cdot, (1, x_\alpha)) - \delta((1, 0), (1, x_\alpha))$ converges to a horofunction. By taking a subnet we may assume that (x_α) weakly converges to some $\hat{x} \in H$ as the unit ball is weakly compact and $(\|x_\alpha\|)$ converges to some $r \leq 1$. Note that by Lemma 1, $r \geq \|\hat{x}\|$. Let $(\gamma, y) \in V_+$. Using Proposition 2 we find

$$M((1, x_\alpha)/(1, 0)) = 1 + \|x_\alpha\|$$

$$M((1, 0))/(1, x_\alpha)) = \frac{1 + \|x_\alpha\|}{1 - \|x_\alpha\|^2}$$

$$M((1, x_\alpha)/(\gamma, y)) = \frac{\gamma - \langle x, y\rangle + \sqrt{(\gamma - \langle x, y\rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2}$$

$$M((\gamma, y))/(1, x_\alpha)) = \frac{\gamma - \langle x, y\rangle + \sqrt{(\gamma - \langle x, y\rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{1 - \|x_\alpha\|^2}.$$

Hence

$$i((1, x_\alpha))((\gamma, y)) = \frac{1}{2}\log(M((\gamma, y))/(1, x_\alpha))M((1, x_\alpha)/(\gamma, y))) \\ - \frac{1}{2}\log(M((1, 0))/(1, x_\alpha))M((1, x_\alpha)/(1, 0))) \\ = \log\left(\frac{\gamma - \langle x_\alpha, y\rangle + \sqrt{(\gamma - \langle x_\alpha, y\rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{\sqrt{\gamma^2 - \|y\|^2}\sqrt{1 - \|x_\alpha\|^2}}\right)$$

$$- \log\left(\frac{1+\|x_\alpha\|}{\sqrt{1-\|x_\alpha\|^2}}\right)$$
$$= \log\left(\frac{\gamma - \langle x_\alpha, y\rangle + \sqrt{(\gamma - \langle x_\alpha, y\rangle)^2 - (1-\|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{(1+\|x_\alpha\|)\sqrt{\gamma^2 - \|y\|^2}}\right).$$

Taking the limit gives us

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y\rangle + \sqrt{(\gamma - \langle \hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1-r^2)}}{(1+r)\sqrt{\gamma^2 - \|y\|^2}}\right).$$

Note that if $r = \|\hat{x}\| < 1$, then $\xi = i(1, \hat{x})$. So $r > \|\hat{x}\|$, if $\|\hat{x}\| < 1$.

Now suppose that a function is of the form as described above. Note that all we need to do is find a net $((1, x_\alpha))$ in V_+ such that (x_α) converges weakly to $\hat{x}$ and $(\|x_\alpha\|)$ converges to r. Then it will give rise to the desired horofunction by the above. If $\|\hat{x}\| = 1$, consider the sequence $((1, (1 - \frac{1}{n})\hat{x}))$, clearly this sequence converges strongly to $(1, \hat{x})$ and gives rise to a horofunction by the above. If $\|\hat{x}\| < 1$ then let (e_n) be an orthonormal sequence in H, which exists as $\dim(H) = \infty$, and consider the sequence $((1, \hat{x} + \sqrt{r^2 - \|\hat{x}\|^2} e_n))$. Note that this converges weakly to $\hat{x}$ since (e_n) converges weakly to 0. Also note that

$$\lim_{n\to\infty} \|\hat{x} + \sqrt{r^2 - \|\hat{x}\|^2} e_n\|^2 = \lim_{n\to\infty} r^2 + 2\sqrt{r^2 - \|\hat{x}\|^2}\langle \hat{x}, e_n\rangle = r^2.$$

□

Note that the proof of Theorem 2 also shows that ξ is a Busemann point if $\|\hat{x}\| = 1$. We can show that these are the only horofunctions that are Busemann points.

Theorem 3 *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space and let $V = \mathbb{R} \oplus H$, let $\hat{x} \in H$ and $\|\hat{x}\| \le r \le 1$ and let*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y\rangle + \sqrt{(\gamma - \langle \hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1-r^2)}}{(1+r)\sqrt{\gamma^2 - \|y\|^2}}\right)$$

be a horofunction. Then ξ is a Busemann point if and only if $\|\hat{x}\| = r = 1$.

Proof In Theorem 2 we already proved that if $\|\hat{x}\| = r = 1$, then ξ is a Busemann point. Now suppose that ξ is a Busemann point and let $((1, x_\alpha))$ be an almost geodesic net such that $i((1, x_\alpha))$ converges to ξ. Combining Proposition 1 and Theorem 2 gives us that $\delta((1, 0), (1, x_\alpha))$ is not bounded, so $\lim_\alpha \|x_\alpha\| = r = 1$. Note that we can rewrite the horofunction as

$$\xi((1, y)) = \log\left(\frac{1 - \langle \hat{x}, y\rangle}{\sqrt{1 - \|y\|^2}}\right).$$

Now suppose $\|\hat{x}\| < 1$. Let $\epsilon > 0$ and let A be such that for all $\alpha' \ge \alpha \ge A$ we have

$$\varepsilon + \delta((1, 0), (1, x_{\alpha'})) \ge \delta((1, 0), (1, x_\alpha)) + \delta((1, x_\alpha), (1, x_{\alpha'}))$$

As in the proof of Theorem 2, using Proposition 2 we find

$$\varepsilon \ge \log\left(\frac{1+\|x_\alpha\|}{\sqrt{1-\|x_\alpha\|^2}}\right) - \log\left(\frac{1+\|x_{\alpha'}\|}{\sqrt{1-\|x_{\alpha'}\|^2}}\right)$$

$$+\log\left(\frac{1-\langle x_\alpha, x_{\alpha'}\rangle+\sqrt{(1-\langle x_\alpha, x_{\alpha'}\rangle)^2-(1-\|x_\alpha\|^2)(1-\|x_{\alpha'}\|^2)}}{\sqrt{1-\|x_{\alpha'}\|^2}\sqrt{1-\|x_\alpha\|^2}}\right).$$

Taking the exponential we find

$$e^\varepsilon \geq \frac{1-\langle x_\alpha, x_{\alpha'}\rangle+\sqrt{(1-\langle x_\alpha, x_{\alpha'}\rangle)^2-(1-\|x_\alpha\|^2)(1-\|x_{\alpha'}\|^2)}}{(1-\|x_\alpha\|)(1+\|x_{\alpha'}\|)}.$$

As this holds for all $\alpha' \geq \alpha$, we can take the limit with respect to α' to get

$$\begin{aligned} e^\varepsilon &\geq \lim_{\alpha'} \frac{1-\langle x_\alpha, x_{\alpha'}\rangle+\sqrt{(1-\langle x_\alpha, x_{\alpha'}\rangle)^2-(1-\|x_\alpha\|^2)(1-\|x_{\alpha'}\|^2)}}{(1-\|x_\alpha\|)(1+\|x_{\alpha'}\|)} \\ &= \frac{1-\langle x_\alpha, \hat{x}\rangle}{1-\|x_\alpha\|}. \end{aligned}$$

Finally, as this holds for all $\alpha \geq A$, we can take the limit with respect to α to find

$$e^\varepsilon \geq \lim_\alpha \frac{1-\langle x_\alpha, \hat{x}\rangle}{1-\|x_\alpha\|} = \infty,$$

which is a contradiction. □

Theorem 1 follows from Theorems 2 and 3.

To conclude this paper, we note that recently Gutiérrez has found the following description of the horofunction compactification of infinite dimensional Hilbert spaces; see [11].

Theorem 4 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space the horofunction of the norm geometry are of the form*

$$\xi(y) = (\|y\|^2 - 2c\langle y, z\rangle + c^2)^{\frac{1}{2}} - c \qquad (y \in H) \tag{3.1}$$

where $z \in H$, with $\|z\| < 1$ and $0 < c < \infty$ or

$$\xi(y) = -\langle y, z\rangle \qquad (y \in H) \tag{3.2}$$

where $z \in H$ and $\|z\| \leq 1$.

As with $\mathbb{H}^\infty$, a horofunction of the norm geometry $\xi = \lim_\alpha i(x_\alpha)$ is uniquely determined by the weak limit $z = \lim_\alpha \frac{x_\alpha}{\|x_\alpha\|}$ and the limit of the norms $c = \lim_\alpha \|x_\alpha\|$. In particular if (x_α) is unbounded, which in the case of $\mathbb{H}^\infty$ corresponds with $r = 1$ and in the case of Hilbert spaces corresponds with $c = \infty$, we see that the general form (3.1) becomes a simplified form (3.2). Using similar arguments as in the proof of Theorem 3 one can easily check that the Busemann points are exactly the horofunctions of the form (3.2) where $\|z\| = 1$.

References

1. Beardon, A.F.: The dynamics of contractions. Ergod. Theory Dyn. Syst. **17**(6), 1257–1266 (1997)
2. Birkhoff, G.: Extensions of Jentzsch's theorem. Trans. Am. Math. Soc. **85**, 219–227 (1957)

3. Bridson, M.R., Haefliger, A.: Metric Spaces of Non-positive Curvature. Grundleren der Mathematischen Wissenschaften, vol. 319. Springer, Berlin (1999)
4. Bushell, P.J.: Hilbert's metric and positive contraction mappings in Banach spaces. Arch. Ration. Mech. Anal. **52**, 330–338 (1973)
5. Burger, M., Iozziand, A., Monod, N.: Equivariant embeddings of trees into hyperbolic spaces. Int. Math. Res. Not. **22**, 1331–1369 (2005)
6. Das, T., Stratmann, B.O., Urbański, M.: The Bishop–Jones relation and Hausdorff geometry of convex-cobounded limit sets in infinite-dimensional hyperbolic space. Stoch. Dyn. **16**(5), 1650018 (2016)
7. Gaubert, S., Vigeral, G.: A maximin characterisation of the escape rate of non- expansive mappings in metrically convex spaces. Math. Proc. Camb. Philos. Soc. **152**(2), 341–363 (2012)
8. Gouëzel, S., Karlsson, A.: Subadditive and multiplicative ergodic theorems. J. Eur. Math. Soc. **(to appear)**
9. Gromov, M.: Asymptotic Invariants of Infinite Groups. Geometric Group Theory. Volume 2 of London Mathematical Society. Lecture Notes in Series, vol. 182. Cambridge Univ. Press, Cambridge (1993)
10. Gutiérrez, A.W.: The horofunction boundary of finite-dimensional ℓ_p spaces. Colloq. Math. **155**, 51–65 (2019)
11. Gutiérrez, A.: On the metric compactification of infinite-dimensional ℓ_p spaces. Can. Math. Bull. **62**, 491–507 (2019)
12. Gutiérrez, A.: The Metric Compactification of L_p Represented by Random Measures. arXiv:1903.02502
13. Karlsson, A.: Nonexpanding maps and Busemann functions. Ergod. Theory Dyn. Syst. **21**(5), 1447–1457 (2001)
14. Karlsson, A.: Two extensions of Thurston's spectral theorem for surface diffeomorphisms. Bull. Lond. Math. Soc. **46**(2), 217–226 (2014)
15. Karlsson, A.: Elements of a Metric Spectral Theory (2019). arXiv:1904.01398
16. Lemmens, B., Lins, B., Nussbaum, R., Wortel, M.: Denjoy–Wolff theorems for Hilbert's and Thompson's metric spaces. J. Anal. Math. **134**(2), 671–718 (2018)
17. Lemmens, B., Nussbaum, R.: Nonlinear Perron–Frobenius Theory. Cambridge Tracts in Mathematics, vol. 189. Cambridge Univ. Press, Cambridge (2012)
18. Lemmens, B., Nussbaum, R.: Birkhoff's version of Hilbert's metric and its applications in analysis. In: Papadopoulos, A., Troyanov, M. (eds.) Handbook of Hilbert Geometry. IRMA Lectures in Mathematics and Theoretical Physics, vol. 22, pp. 275–303. European Mathematical Society, Zürich (2014)
19. Lemmens, B., Roelands, M., Wortel, M.: Hilbert and Thompson isometries on cones in JB-algebras. Math. Z. **292**, 1511–1547 (2019)
20. Lemmens, B., Walsh, C.: Isometries of polyhedral Hilbert geometries. J. Topol. Anal. **3**(2), 213–241 (2011)
21. Monod, N.: Superrigidity for irreducible lattices and geometric splitting. J. Am. Math. Soc. **19**(4), 781–814 (2006)
22. Monod, N., Py, P.: An exotic deformation of the hyperbolic space. Am. J. Math. **136**(5), 1249–1299 (2014)
23. Munkres, J.R.: Topology, 2nd edn. Prentice-Hall inc., Upper Saddle River (2000)
24. Rieffel, M.A.: Group C^*-algebras as compact quantum metric spaces. Doc. Math. **7**, 605–651 (2002)
25. Satake, I.: On representations and compactifications of symmetric Riemannian spaces. Ann. Math. (2) **71**(1), 77–110 (1960)
26. Walsh, C.: The horofunction boundary of finite-dimensional normed spaces. Math. Proc. Camb. Philos. Soc. **142**(3), 497–507 (2007)
27. Walsh, C.: The horofunction boundary of the Hilbert geometry. Adv. Geom. **8**(4), 503–529 (2008)
28. Walsh, C.: The horofunction boundary and isometry group of the Hilbert geometry. In: Papadopoulos, A., Troyanov, M. (eds.) Handbook of Hilbert Geometry. IRMA Lectures in Mathematics and Theoretical Physics, vol. 22, pp. 127–146. European Mathematical Society, Zürich (2014)
29. Walsh, C.: Gauge-reversing maps on cones, and Hilbert and Thompson isometries. Geom. Topol. **22**(1), 55–104 (2018)
30. Walsh, C.: The asymptotic geometry of the Teichmüller metric. Geom. Dedicata **200**, 115–152 (2019)
31. Walsh, C.: Hilbert and Thompson geometries isometric to infinite dimensional Banach spaces. Ann. Inst. Fourier (Grenoble) **68**(5), 1831–1877 (2018)