The multi-moment maps of the nearly Kähler $S^3 \times S^3$

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Abstract

The homogeneous nearly Kähler structure on $S^3 \times S^3$ is the only known complete 6-dimensional strictly nearly Kähler structure which is invariant under the action of a 3-torus. We investigate the multi-moment maps associated to such an action and compare with the usual moment map of a toric manifold.

Keywords Nearly Kähler · Torus actions · Moment maps · Homogeneous spaces

Mathematics Subject Classification 53C25 · 53C30 · 53D20

1 Introduction

An almost Hermitian manifold (M, g, J) is *nearly Kähler* if $\nabla^g J$ is skew-symmetric. We say a nearly Kähler manifold is *strict* if it is not Kähler. The minimum dimension admitting strict nearly Kähler manifolds is 6, where the nearly Kähler condition is equivalent to the existence of an SU_3 structure (ω, Ω) satisfying

$$d\omega = 3 \operatorname{Re} \Omega, \quad d \operatorname{Im} \Omega = -2\omega \wedge \omega,$$

where ω is the Kähler form, and Ω is a complex volume form [9].

There are only a handful of known examples of compact strictly Kähler 6-manifolds. The homogeneous spaces S^6 , $S^3 \times S^3$, $\mathbb{CP}^3$ and SU_3/T^2 admit strict nearly Kähler structures, and there are no other homogeneous strict nearly Kähler 6-manifolds [2]. There are locally homogeneous structures on quotients of $S^3 \times S^3$ [3], and cohomogeneity one structures on S^6 and $S^3 \times S^3$, which are conjectured to be the only cohomogeneity one examples [5].

If one wants to look for higher cohomogeneity examples, one could look for strict nearly Kähler 6-manifolds admitting the action of a 3-torus T^3 by isometries. Of the list of known examples in the previous paragraph, only the homogeneous $S^3 \times S^3$ and some of its locally homogeneous quotients admit such a symmetry group. The purpose of this paper is to explore these examples.

We will make analogies to the case of toric Kähler manifolds. On a compact Kähler manifold M , every Killing vector field K is symplectic. Moreover, if M is simply-connected,

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then K is Hamiltonian. Thus a $2m$ -dimensional compact simply-connected Kähler manifold M^{2m} admitting an effective isometric $\mathbb{T}^m$ action is toric. Such a manifold could be studied with use of the moment map μ , which is a $\mathbb{T}^3$ -equivariant map from the manifold to the dual Lie algebra of the torus, $\mathfrak{t}^*$. Each fiber of μ is a $\mathbb{T}^m$ orbit, and the image of μ is the polytope which is the convex hull of the μ -image of the fixed points of the $\mathbb{T}^m$ action.

The Kähler form ω of a strictly nearly Kähler 6-manifold M is not closed, so there can be no moment map. However, there is a generalization of the moment map called the *multi-moment map* [6], which can be applied to invariant closed forms of any degree. If M is not the round $\mathbb{S}^6$, then every Killing vector field preserves the whole SU_3 structure [7]. Thus if M admits an effective isometric $\mathbb{T}^3$ action, then the exact forms $d\omega$ and $\omega \wedge \omega$ are preserved by the $\mathbb{T}^3$ action. Thus we can associate $d\omega$ and $\omega \wedge \omega$ to multi-moment maps, ν and σ respectively, induced from the $\mathbb{T}^3$ action.

We compute ν and σ for the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. We find in this case that ν has many qualitative similarities to the toric moment map μ . In particular, we find that the multi-moment map image $\Delta := \nu(\mathbb{S}^3 \times \mathbb{S}^3)$ of $\mathbb{S}^3 \times \mathbb{S}^3$ is convex and that its boundary $\partial\Delta$ contains the 1-skeleton of a regular tetrahedron. However, Δ bulges beyond the faces of the tetrahedron, and $\partial\Delta$ is smooth away from the vertices. In fact, Δ is the 3-elliptope, which is the compact convex solid body whose boundary is contained in Cayley's nodal cubic surface. Along $\partial\Delta$, each ν -fiber is a $\mathbb{T}^3$ orbit, but in the interior, each fiber contains two orbits. The other multi-moment map σ distinguishes between these two orbits in the interior fibers, so the map $\nu \oplus \sigma$ has connected fibers, but its image is only the boundary of a convex region $\hat{\Delta}$. However, $\hat{\Delta}$ can be interpreted using the multi-moment maps of the G_2 structure on the cone over M . The following table compares the fiber types for the multi-moment map of $\mathbb{S}^3 \times \mathbb{S}^3$ to the moment map of a toric 3-manifold:

Fiber of a point in ...	μ toric 6-manifold	ν for nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$
A vertex	{point}	$\mathbb{T}^2$
An edge	$\mathbb{S}^1$	$\mathbb{T}^3$
A face	$\mathbb{T}^2$	$\mathbb{T}^3$
The interior	$\mathbb{T}^3$	$\mathbb{T}^3 \sqcup \mathbb{T}^3$

Finally, we find that for the action of a 3-torus $\mathbb{T}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ to descend to a locally homogeneous quotient $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3)$, then the finite group Γ must be a subgroup of $\mathbb{T}$. It follows that the multi-moment maps on $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3)$ and $\mathbb{S}^3 \times \mathbb{S}^3$ have the same image, and the fibers have the same diffeomorphism type.

2 Homogenous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

We begin by reviewing the definition of the homogenous nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$, following the work in [4].

We identify $\mathbb{S}^3$ with the unit sphere in the quaternions $\mathbb{H}$. For any $p \in \mathbb{S}^3$, $T_p \mathbb{S}^3 \subset T_p \mathbb{H}$ is the image of $T_1 \mathbb{S}^3$ by the pushforward of left-multiplication by p . Identifying $T_p \mathbb{S}^3 \subset T_p \mathbb{H}$ with $p^\perp \subset \mathbb{H}$, this pushforward is simply quaternionic multiplication by p . Thus the basis $\{i, j, -k\}$ of $\text{Im } \mathbb{H}$ which is identified with $T_1 \mathbb{S}^3$ gives a frame for $T_{(p,q)} \mathbb{S}^3 \times \mathbb{S}^3 = T_p \mathbb{S}^3 \oplus T_q \mathbb{S}^3$:

$$\begin{aligned} E_1(p, q) &= (pi, 0), & F_1(p, q) &= (0, qi), \\ E_2(p, q) &= (pj, 0), & F_2(p, q) &= (0, qj), \end{aligned}$$

$$E_3(p, q) = (pk, 0), \quad F_3(p, q) = (0, qk),$$

where i, j, k are imaginary quaternions satisfying $ij = k$. Note that this frame differs by the one in [4] by a sign in E_3 and F_3 . Let $\{E^1, E^2, E^3, F^1, F^2, F^3\}$ be the dual frame.

The almost complex structure for the homogenous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is given in this frame by

$$J = \frac{1}{\sqrt{3}} \sum_{n=1}^3 (-E_n \otimes E^n + F_n \otimes F^n + 2F_n \otimes E^n - 2E_n \otimes F^n).$$

The metric g is given by the average of $g_{\mathbb{H}^2}$ and $g_{\mathbb{H}^2}(J \cdot, J \cdot)$, where

$$g_{\mathbb{H}^2} = \sum_{n=1}^3 \left((E^n)^2 + (F^n)^2 \right)$$

is the flat metric from $\mathbb{H}^2$ restricted to $\mathbb{S}^3 \times \mathbb{S}^3$. This gives

$$g = \frac{4}{3} \sum_{n=1}^3 \left((E^n)^2 - E^n F^n + (F^n)^2 \right),$$

$$\omega = \frac{4}{\sqrt{3}} \sum_{n=1}^3 E^n \wedge F^n.$$

2.1 Torus actions

For unit quaternions $a, b, c \in \mathbb{S}^3$, the map

$$F_{a,b,c} : \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3 \times \mathbb{S}^3 : (p, q) \mapsto (apc^{-1}, bqc^{-1})$$

is a holomorphic isometry [8].

Lemma 2.1 *The map*

$$F : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \text{Aut}(\mathbb{S}^3 \times \mathbb{S}^3, J) \cap \text{Isom}(\mathbb{S}^3 \times \mathbb{S}^3, g) : (a, b, c) \mapsto F_{a,b,c}$$

is a homomorphism with kernel $\{(\sigma, \sigma, \sigma)\}_{\sigma \in \pm 1}$.

Proof It is clear from the definition of F that $F_{a,b,c} \circ F_{a',b',c'} = F_{aa',bb',cc'}$, so that F is a homomorphism. To compute the kernel, let $(a, b, c) \in \ker F$, so that $F_{a,b,c} = Id$. Then

$$(1, 1) = F_{a,b,c}(1, 1) = (ac^{-1}, ab^{-1}),$$

so that $a = b = c$. For any $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$,

$$(p, q) = F_{a,a,a}(p, q) = (apa^{-1}, aqa^{-1}).$$

Since this is true for all $p, q \in \mathbb{S}^3$, we find that a lies in the center $\{\pm 1\}$ of $\mathbb{S}^3$. This gives the result, identifying σ with a . $\square$

Since the projection of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ onto any of its factors is a homomorphism, any abelian subgroup of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ must be a product of abelian subgroups of $\mathbb{S}^3$. But every non-trivial

abelian subgroup of $\mathbb{S}^3$ is of the form $\{e^{At}\}_{t \in \mathbb{R}}$ for some unit imaginary quaternion A . Thus, a maximal torus in $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ is of the form

$$\{(e^{At_1}, e^{Bt_2}, e^{Ct_3})\}_{(t_1, t_2, t_3) \in \mathbb{R}^3},$$

for some $A, B, C \in \mathbb{S}^2$, identifying $\mathbb{S}^2$ with the unit imaginary quaternions. A routine computation shows that the image of such a torus under F is generated by the Killing vector fields

$$\begin{aligned} K_1 &= (Ap, 0), \\ K_2 &= (0, Bq), \\ K_3 &= (-pC, -qC). \end{aligned}$$

3 Multi-moment maps

A multi-moment map associated to a closed r -form β invariant under the action of a torus $\mathbb{T}$ is a smooth map $\tau : M \rightarrow \Lambda^{r-1}\mathfrak{t}^*$ defined by the extending linearly the relation

$$\langle d\tau, X^1 \wedge \cdots \wedge X^{r-1} \rangle = \tau(X_M^1, \dots, X_M^{r-1}, \cdot),$$

where for each j , X_M^j is the vector field on M induced by $X^j \in \mathfrak{t}$. Note that when the multi-moment map exists, it is unique up to a constant. Also note that when $r = 2$, we recover the usual moment map up to a sign convention.

We are interested in computing the multi-moment maps for the $\mathbb{T}^3$ action preserving the exact forms $d\omega$ and $\omega \wedge \omega$. In the following result, which has been translated into our simple case, we see that in the exact case, multi-moment maps always exist, and take a convenient form:

Proposition 3.1 ([6] Proposition 3.1) *Let M be a manifold admitting the action of a torus $\mathbb{T}$ that preserves an r -form α . Then the map $\tau : M \rightarrow \Lambda^r \mathfrak{t}^*$ given by the expression*

$$\langle \tau, X^1 \wedge X^2 \wedge \cdots \wedge X^r \rangle = (-1)^r \alpha(X_M^1, X_M^2, \dots, X_M^r)$$

is a multi-moment map for $d\alpha$ under the action of $\mathbb{T}$.

3.1 The multi-moment map ν for $d\omega$

In this subsection, we describe the multi-moment map ν of $d\omega$. We will describe its image and the structure of its fibers.

For $X \in \mathbb{S}^2$, let us define a map

$$\pi_X : \mathbb{S}^3 \rightarrow \mathbb{S}^2 : p \mapsto \bar{p}Xp.$$

When $X = i$, this is the usual Hopf fibration. For general X , π_X also identifies $\mathbb{S}^3$ as a $\mathbb{S}^1$ bundle over $\mathbb{S}^2$.

Proposition 3.2 *With respect to a suitable basis for $\Lambda^2 \mathfrak{t}^*$, $\nu = \bar{\nu} \circ (\pi_A \times \pi_B)$, where*

$$\bar{\nu} : \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \Lambda^2 \mathfrak{t}^* : (x, y) \mapsto (x \cdot y, x \cdot C, y \cdot C).$$

Proof For any $p \in \mathbb{S}^3$, $Ap = p\bar{p}Ap = p\pi_A(p)$. Thus the Killing vector fields can be written in terms of the frame $(E_1, E_2, E_3, F_1, F_2, F_3)$ as

$$\begin{aligned} K_1 &= (\pi_A(p) \cdot i)E_1 + (\pi_A(p) \cdot j)E_2 + (\pi_A(p) \cdot k)E_3, \\ K_2 &= (\pi_B(q) \cdot i)F_1 + (\pi_B(q) \cdot j)F_2 + (\pi_B(q) \cdot k)F_3, \\ -K_3 &= (C \cdot i)(E_1 + F_1) + (C \cdot j)(E_2 + F_2) + (C \cdot k)(E_3 + F_3), \end{aligned}$$

where $\cdot$ is the dot product on $\mathbb{H}$. This allows us to compute

$$\begin{aligned} \frac{\sqrt{3}}{4}\omega(K_1, K_2) &= \pi_A(p) \cdot \pi_B(q), \\ \frac{\sqrt{3}}{4}\omega(K_1, K_3) &= -\pi_A(p) \cdot C, \\ \frac{\sqrt{3}}{4}\omega(K_2, K_3) &= \pi_B(q) \cdot C. \end{aligned}$$

Choosing the basis $\left\{ \frac{\sqrt{3}}{4}(K_1 \wedge K_2)^*, \frac{\sqrt{3}}{4}(K_3 \wedge K_1)^*, \frac{\sqrt{3}}{4}(K_2 \wedge K_3)^* \right\}$ for $\Lambda^2 \mathfrak{t}^*$, Proposition 3.1 allows us to write the multi-moment map in the claimed form. $\square$

Note that $v^{-1}(0, 0, 0)$ is the union of the Lagrangian torus orbits. The example of a Lagrangian torus in [4] can be found with the values $A = B = i$ and $C = j$.

Let $\Delta = v(\mathbb{S}^3 \times \mathbb{S}^3) = \bar{v}(\mathbb{S}^2 \times \mathbb{S}^2)$ with interior $\dot{\Delta}$. Clearly $\Delta \subset [-1, 1]^3$. The following lemma gives more information about $\bar{v}$ and Δ :

Lemma 3.3 *Let $(X, Y, Z) \in [-1, 1]^3$. Then*

$$\bar{v}^{-1}(X, Y, Z) \cong \begin{cases} \emptyset & \text{if } F(X, Y, Z) < 0 \\ \mathbb{S}^1 \sqcup \mathbb{S}^1 & \text{if } F(X, Y, Z) > 0 \\ \{point\} & \text{if } (X, Y, Z) \in V \\ \mathbb{S}^1 & \text{if } (X, Y, Z) \in \ker F \setminus V \end{cases},$$

where

$$\begin{aligned} F(X, Y, Z) &:= 1 + 2XYZ - X^2 - Y^2 - Z^2, \\ V &:= \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}. \end{aligned}$$

Proof Let $(X, Y, Z) = (\cos(\theta), \cos(\phi), \cos(\psi)) \in [-1, 1]^3$. Without loss of generality, by changing coordinates we may assume that $C = i$.

Let $(x, y) \in \bar{v}^{-1}(X, Y, Z)$. The relations

$$\cos(\phi) = Y = x \cdot i, \quad \cos(\psi) = Z = y \cdot i$$

allow us to write

$$x = \cos(\phi)i + \sin(\phi)e^{i\xi}j, \quad y = \cos(\psi)i + \sin(\psi)e^{i\eta}j,$$

for real numbers ξ and η . Now

$$\cos(\theta) = X = x \cdot y = \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\cos(\alpha), \quad (1)$$

where $\alpha = \eta - \xi$. So

$$\begin{aligned} F(X, Y, Z) &= (1 - Y^2)(1 - Z^2) - (X - YZ)^2 \\ &= \sin^2(\phi) \sin^2(\psi) - (\sin(\phi) \sin(\psi) \cos(\alpha))^2 = \sin^2(\phi) \sin^2(\psi) \sin^2(\alpha). \end{aligned}$$

Note that $F(X, Y, Z) \geq 0$, giving the first case of our claim.

If $F(X, Y, Z) > 0$, then none of $\sin(\phi)$, $\sin(\psi)$ or $\sin(\alpha)$ vanish. We observe that (X, Y, Z) determines $\cos(\alpha)$, thus α up to a sign. Since $\sin(\alpha) \neq 0$, there are two choices for α . Since there is a $\mathbb{S}^1$ of choices for $\xi + \eta$, we find that $\bar{v}^{-1}(X, Y, Z) \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$.

If $F(X, Y, Z) = 0$, then $0 = \sin(\phi) \sin(\psi) \sin(\alpha)$. If $\sin(\phi) \neq 0 \neq \sin(\psi)$, then $\sin(\alpha) = 0$. $\cos(\alpha)$ is then determined by (1), giving a single choice for α . Since there is a $\mathbb{S}^1$ of choices for $\xi + \eta$, we find that $\bar{v}^{-1}(X, Y, Z) \cong \mathbb{S}^1$. If $\sin(\phi) = 0 \neq \sin(\psi)$, then $x = i$ and there are no restrictions on η . Thus $\bar{v}^{-1}(X, Y, Z) \cong \mathbb{S}^1$. Similarly, if $\sin(\phi) \neq 0 = \sin(\psi)$, then $\bar{v}^{-1}(X, Y, Z) \cong \mathbb{S}^1$. If $\sin(\phi) = \sin(\psi) = 0$, then we find that $\bar{v}^{-1}(X, Y, Z)$ is a single point. Note that this is the case when $Y^2 = Z^2 = 1$. The condition $F(X, Y, Z) = 0$ then forces $(X, Y, Z) \in V$. $\square$

Lemma 3.4

$$\begin{aligned} \Delta &= \{(X, Y, Z) \in [-1, 1]^3 : F(X, Y, Z) \geq 0\}, \\ \partial\Delta &= \{(X, Y, Z) \in [-1, 1]^3 : F(X, Y, Z) = 0\}. \end{aligned}$$

Proof The first statement follows directly from the previous lemma. To prove the second statement, we compute

$$F(X, Y, Z)|_{Z^2=1} = -(X - YZ)^2 \leq 0.$$

Similar computations show that $F|_{\partial[-1, 1]^3} \leq 0$. Thus

$$\partial\Delta = \left(\partial[-1, 1]^3 \cap F^{-1}([0, \infty)) \right) \cup ([-1, 1]^3 \cap \ker F) = [-1, 1]^3 \cap \ker F.$$

$\square$

Note that $\ker F$ describes Cayley's nodal cubic surface, pictured in Fig. 1. $\partial\Delta$ is then the intersection of this surface with $[-1, 1]^3$, which is the closure of the compact component of the set of regular points of the cubic.

The previous lemma gives a description of Δ which can be recognized as being a definition of a convex body called a 3-elliptope. In the following lemma, we review this correspondence:

Lemma 3.5 Δ can be identified with the 3-elliptope

$$\mathcal{E}_3 := \left\{ (X, Y, Z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & X & Y \\ X & 1 & Z \\ Y & Z & 1 \end{pmatrix} \text{ is positive semi-definite} \right\}.$$

Proof By Sylvester's criterion, a Hermitian matrix is positive semi-definite if and only if all of the principal minors are non-negative. Noting that $F(X, Y, Z)$ is the determinant of the matrix in the definition of $\mathcal{E}_3$, we find that

$$\mathcal{E}_3 = \{(X, Y, Z) \in \mathbb{R}^3 : (1 - X^2), (1 - Y^2), (1 - Z^2), F(X, Y, Z) \geq 0\} = \Delta,$$

where the last equality follows from the previous lemma. $\square$

Since $\mathcal{E}_3$ is obviously convex, we immediately obtain:

Fig. 1 The real points of Cayley's nodal cubic surface [1]

Corollary 3.6 Δ is convex.

We finish this section by describing vertices and edges of Δ , which are well known properties of the Cayley cubic.

Proposition 3.7 The set of singular points of $\partial\Delta$ is

$$V := \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}.$$

Proof By Lemma 3.4 The singular points of $\partial\Delta$ are the points where F and ∇F both vanish. The set of points where ∇F vanishes are $\{(0, 0, 0)\} \cup V$. The result follows since $F(0, 0, 0) = 1 \neq 0$, while F vanishes on V . $\square$

Proposition 3.8 The line segment between any two points in V lies in $\partial\Delta$.

Proof We will show that the line segment between $(1, 1, 1)$ and $(1, -1, -1)$ lies in $\partial\Delta$, with the other line segments following similarly. This line segment is parametrized by

$$\gamma : [-1, 1] \rightarrow \mathbb{R} : t \mapsto (1, t, t).$$

The claim follows easily from Lemma 3.4 and $F|_{X=1} = -(Y - Z)^2$. $\square$

By the previous two propositions, we find that $\partial\Delta$ contains the 1-skeleton of the regular tetrahedron with vertices V , but that the edges of this tetrahedron are not singular in ∂V . In Fig. 2, we see that Δ is a regular tetrahedron with convexly bulging sides.

3.2 The multi-moment map σ for $\omega \wedge \omega$

In this section, we want to compute the multimoment map σ for $\omega \wedge \omega = -\frac{1}{2}d \operatorname{Im} \Omega$. In order to apply Proposition 3.1 again, we will solve directly for a potential 3-form for $\omega \wedge \omega$.

Fig. 2 The multi-moment map image $\Delta = \nu(\mathbb{S}^3 \times \mathbb{S}^3)$ with the tetrahedral 1-skeleton indicated

We continue to work in the frame $(E_1, E_2, E_3, F_1, F_2, F_3)$, although for convenience we will make the identifications $E_n = E_{n+3}$ and $F_n = F_{n+3}$ for every $n \in \mathbb{Z}$.

Lemma 3.9

$$dE^n = 2E^{n-1} \wedge E^{n+1}, \quad dF^n = 2F^{n-1} \wedge F^{n+1}.$$

Proof The Lie bracket in this frame was computed in [4]. Noting that we treat E^3 and F^3 as the negatives of their convention, the Lie bracket is given by

$$[E_{n-1}, E_{n+1}] = -2E_n, \quad [E_n, F_m] = 0, \quad [F_{n-1}, F_{n+1}] = -2F_n.$$

For vector fields X and Y , we have

$$dE^n(X, Y) = XE^n(Y) - YE^n(X) - E^n([X, Y]).$$

Choosing $X, Y \in \{E_1, E_2, E_3, F_1, F_2, F_3\}$, we find that

$$dE^n = -E^n([E_{n-1}, E_{n+1}])E^{n-1} \wedge E^{n+1} = 2E^{n-1} \wedge E^{n+1}.$$

The computation of dF^n is similar. □

Lemma 3.10

$$\omega \wedge \omega = d \left(-\frac{8}{9} \sum_{n=1}^3 (E^n \wedge F^{n-1} \wedge F^{n+1} + F^n \wedge E^{n-1} \wedge E^{n+1}) \right).$$

Proof

$$\begin{aligned} \omega \wedge \omega &= \left(\frac{4}{\sqrt{3}} \sum_{n=1}^3 E^n \wedge F^n \right)^2 = -\frac{16}{9} \sum_{n,m=1}^3 E^n \wedge E^m \wedge F^n \wedge F^m \\ &= -\frac{32}{9} \sum_{n=1}^3 E^{n-1} \wedge E^{n+1} \wedge F^{n-1} \wedge F^{n+1} \\ &= -\frac{8}{9} d \sum_{n=1}^3 (E^n \wedge F^{n-1} \wedge F^{n+1} + F^n \wedge E^{n-1} \wedge E^{n+1}). \end{aligned}$$

□

Lemma 3.11 *The multi-moment map σ associated to the form $\omega \wedge \omega$ is given by $\det(\pi_A(p), \pi_B(q), C)$, with respect to some convenient basis.*

Proof Choose the basis $\frac{3}{16}(K_1 \wedge K_2 \wedge K_3)^*$ for $\Lambda^3 \mathfrak{t}^*$. With respect to this basis, the previous lemma and Proposition 3.1 tells us that σ is given by

$$\begin{aligned} & -\frac{3}{16} \left(-\frac{8}{9} \sum_{n=1}^3 (E^n \wedge F^{n-1} \wedge F^{n+1} + F^n \wedge E^{n-1} \wedge E^{n+1}) \right) (K_1, K_2, K_3) \\ & = \frac{1}{6} \sum_{n=1}^3 (E^n \wedge F^{n-1} \wedge F^{n+1} + F^n \wedge E^{n-1} \wedge E^{n+1}) (K_1, K_2, K_3). \end{aligned}$$

We will compute the first term first. Since $F^n(K_1) = 0$, we get

$$\begin{aligned} \frac{1}{6} \sum_{n=1}^3 (E^n \wedge F^{n-1} \wedge F^{n+1}) (K_1, K_2, K_3) & = \frac{1}{6} \sum_{n=1}^3 E^n(K_1)(F^{n-1} \wedge F^{n+1})(K_2, K_3) \\ & = -\frac{1}{2} (i^* \wedge j^* \wedge k^*)(\pi_A(p), \pi_B(q), -C) \\ & = \frac{1}{2} \det(\pi_A(p), \pi_B(q), C). \end{aligned}$$

Similarly

$$\frac{1}{6} \sum_{n=1}^3 (F^n \wedge E^{n-1} \wedge E^{n+1}) (K_1, K_2, K_3) = \frac{\det(\pi_A(p), \pi_B(q), C)}{2}.$$

Combining these, we get the result. $\square$

Lemma 3.12 $|\sigma| = \sqrt{F} \circ \nu$.

Proof We do the computation in the case that $C = i$. Let $(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that

$$\bar{\nu}(x, y) = (X, Y, Z) = (\cos(\theta), \cos(\phi), \cos(\psi)).$$

Then as in Lemma 3.3, we have

$$x = \cos(\phi)i + \sin(\phi)e^{i\xi}j, \quad y = \cos(\psi)i + \sin(\psi)e^{i\eta}j,$$

for real numbers ξ and η satisfying

$$\cos(\theta) = \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\cos(\alpha), \quad (2)$$

where $\alpha = \eta - \xi$. Then

$$\det(x, y, i) = \sin(\theta)\sin(\phi)\sin(\alpha).$$

But in Lemma 3.3, we saw that

$$F(X, Y, Z) = \sin^2(\theta)\sin^2(\phi)\sin^2(\alpha) = (\det(x, y, i))^2.$$

This combined with the previous lemma gives the desired result. $\square$

Note that the fibers of $\nu \oplus \sigma$ are connected, since when a fiber of ν has two connected components, the sign of σ distinguishes between them.

Lemma 3.13 $\sqrt{F} : \Delta \rightarrow \mathbb{R}$ is concave, so that the image of $\nu \oplus \sigma$ is the boundary of a convex region $\hat{\Delta} := \{(p, t) \in \Delta \times \mathbb{R} : t^2 \leq F(p)\}$.

Proof By the previous lemma, The image of $\nu \oplus \sigma$ is equal to $\{(p, t) \in \Delta \times \mathbb{R} : t^2 = F(p)\} = \partial \hat{\Delta}$. To show that $\hat{\Delta}$ is convex, we will show that $\text{Hess}(\sqrt{F})$ is negative definite.

We first compute

$$\nabla F = 2 \begin{pmatrix} -X + YZ \\ -Y + XZ \\ -Z + XY \end{pmatrix}, \quad \text{Hess}(F) = 2 \begin{pmatrix} -1 & Z & Y \\ Z & -1 & X \\ Y & X & -1 \end{pmatrix}.$$

Now $(0, 0, 0) \in \Delta$ with $F(0, 0, 0) = 1$. Thus $\text{Hess}(\sqrt{F})(0, 0, 0) = -\text{Id}$ is negative definite. Since Δ is connected, to show that $\text{Hess}(\sqrt{F})$ is negative definite as required, it suffices to show that $\det(\sqrt{F^3} \text{Hess}(\sqrt{F}))$ is non-vanishing on the interior of Δ .

$$\begin{aligned} \sqrt{F^3} \text{Hess}(\sqrt{F}) &= \frac{1}{2} F \text{Hess} F - \frac{1}{4} \nabla F \otimes \nabla F \\ &= \begin{pmatrix} -(1 - Y^2)(1 - Z^2) & (Z - XY)(1 - Z^2) & (Y - XZ)(1 - Y^2) \\ (Z - XY)(1 - Z^2) & -(1 - X^2)(1 - Z^2) & (X - YZ)(1 - X^2) \\ (Y - XZ)(1 - Y^2) & (X - YZ)(1 - X^2) & -(1 - X^2)(1 - Y^2) \end{pmatrix}, \\ \det(\sqrt{F^3} \text{Hess}(\sqrt{F})) &= -(1 - X^2)(1 - Y^2)(1 - Z^2)F^2. \end{aligned}$$

Since the functions $\{1 - X^2, 1 - Y^2, 1 - Z^2, F\}$ are all positive on the interior of Δ , we see that $\det(\sqrt{F^3} \text{Hess}(\sqrt{F}))$ does not vanish there, completing the proof. $\square$

In the remainder of the section we will interpret $\hat{\Delta}$ in terms of G_2 geometry. To any nearly Kähler manifold (M, g, ω) , The metric cone

$$(M \times (0, \infty), g_{\text{cone}} = dr^2 + r^2 g)$$

(with r parameterizing $(0, \infty)$) admits a parallel G_2 structure $\varphi = \frac{1}{3}d(r^3\omega)$. The condition that this structure is parallel means that φ and its hodge dual $\psi := *\varphi = -\frac{1}{4}d(r^4 \text{Im } \Omega)$ are both closed. In the case when M is nearly toric, then action of the torus $\mathbb{T}$ lifts to $M \times (0, \infty)$, and this lifted action preserves φ and ψ . Applying Proposition 3.1, we find that $\hat{\nu} := r^3\nu$ and $\hat{\sigma} := r^4\sigma$ are multi-moment maps for the $\mathbb{T}^3$ action preserving the closed forms φ and ψ respectively (choosing a convenient basis for $\Lambda^2\mathfrak{t}^*$ and $\Lambda^3\mathfrak{t}^*$).

Since $\nu \oplus \sigma$ has connected fibers, so does $\hat{\nu} \oplus \hat{\sigma}$. Moreover, the image $(\hat{\nu} \oplus \hat{\sigma})(M \times (0, 1))$ is the interior of $\hat{\Delta} \setminus \{0\}$. Thus $\hat{\Delta}$ is the closure of $(\hat{\nu} \oplus \hat{\sigma})(M \times (0, 1))$.

4 Torus actions on locally homogeneous quotients of $\mathbb{S}^3 \times \mathbb{S}^3$

Recall that $\mathbb{S}^3 \times \mathbb{S}^3$ is a homogeneous space realized as the quotient

$$\mathbb{S}^3 \times \mathbb{S}^3 \cong (\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\mathbb{S}^3,$$

where $\mathbb{S}^3$ is a Lie group under its identification as the unit quaternions and $\mathbb{S}^3$ is realized as a subgroup of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ via the diagonal embedding. From [3], a locally homogeneous strict nearly Kähler 6-manifold is of the form

$$\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3) := \Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\mathbb{S}^3$$

for some finite subgroup Γ of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ freely acting on $\mathbb{S}^3 \times \mathbb{S}^3$. We will show that

Theorem 4.1 *The action of $\mathbb{T}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ descends to $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3)$ if and only if Γ is a subgroup of $\mathbb{T}$.*

Before we prove this theorem, we begin by studying a simpler situation on $\mathbb{S}^3$.

Lemma 4.2 *Let $F \subset \mathbb{S}^3$ be a finite set. Let $C \leq \mathbb{S}^3$ be a circle subgroup. If F is preserved under the action by conjugation by C , then $F \subset C$.*

Proof Using the identification of $\mathbb{S}^3$ with the unit quaternions, C can be written as $\{e^{\theta x}\}_{\theta \in [0, 2\pi)}$ for some unit imaginary quaternion x . $e^{\theta x}$ acts by conjugation on $\mathbb{S}^3$ by rotating the imaginary quaternions by an angle of 2θ about the axis generated by x . Since the only finite orbits of this action are fixed points, F must be contained in the set of fixed points of this action. This is the set $\text{Span}\{1, x\} \cap \mathbb{S}^3 = C$. $\square$

Proof of Theorem 4.1 First assume that the action of $\mathbb{T}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ descends to $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3)$. Since $\mathbb{T}$ and Γ are both subgroups of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ acting on the left of $\mathbb{S}^3 \times \mathbb{S}^3 \cong (\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\mathbb{S}^3$, this implies that the action of $\mathbb{T}$ on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ descends to $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)$. Thus we have

$$t\Gamma g = \Gamma tg, \quad \forall t \in \mathbb{T}, g \in \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3.$$

Using $g = 1$, this relation tells us that Γ is fixed by the conjugation action of $\mathbb{T}$. For each $i \in \{1, 2, 3\}$ consider the projection $\pi_i : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3$ onto the i th factor. From our discussion in Sect. 2.1, $\pi_i(\mathbb{T})$ is a circle subgroup of $\mathbb{S}^3$. Thus we can apply the previous lemma to find $\pi_i(\Gamma) \subset \pi_i(\mathbb{T})$. Combining this result for each i implies that $\Gamma \leq \mathbb{T}$.

Conversely, if $\Gamma \leq \mathbb{T}$, then

$$t\Gamma(p, q) = \Gamma t(p, q), \quad \forall t \in \mathbb{T}, (p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$$

follows from $\mathbb{T}$ being abelian. $\square$

In the situation of the previous theorem, we have Γ a finite subgroup of the torus $\mathbb{T}$. Then $\mathbb{T}/\Gamma$ is also a torus. Since the multi-moment map ν on $\mathbb{S}^3 \times \mathbb{S}^3$ is $\mathbb{T}$ -equivariant, it naturally descends to the multi-moment map ν_Γ on $\Gamma \backslash (\mathbb{S}^3 \times \mathbb{S}^3)$, so that both multi-moment maps have the same image Δ . For any $P \in \Delta$, the fibers are related by $\nu_\Gamma^{-1}(P) = \Gamma \backslash \nu^{-1}(P)$. Since the quotient of a torus by a finite subgroup is still a torus, the fibers have the same diffeomorphism types.

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