Geom Dedicata (2010) 147:149–157
DOI 10.1007/s10711-009-9446-2

ORIGINAL PAPER

Perelman's $\bar{\lambda}$-invariant and collapsing via geometric characteristic splittings

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Received: 15 February 2009 / Accepted: 18 November 2009 / Published online: 1 December 2009

Abstract Any closed, orientable, smooth, nonpositively curved manifold M is known to admit a geometric characteristic splitting, analogous to the JSJ decomposition in three dimensions. We show that when this splitting consists of pieces which are Seifert fibered or pieces each of whose fundamental group has non-trivial centre, then M collapses with bounded curvature and has zero Perelman $\bar{\lambda}$-invariant.

Keywords Collapse · Minimal volume · Nonpositive curvature · Yamabe invariant · Perelman invariant

Mathematics Subject Classification (2000) 53C20 · 53C23 · 53C44

1 Introduction

In this paper we study the relationship between the geometric characteristic splitting of any compact connected nonpositively curved manifold M as described by Leeb and Scott in [18] and the possibility that M collapses with bounded curvature in the sense of Cheeger and Gromov in [8,10,22]. We show how this information can be used to prove vanishing results about Perelman's $\bar{\lambda}$ invariant, which was introduced in [23].

A smooth manifold M is said to *collapse with bounded curvature* if there exists a sequence of metrics $\{g_i\}$ for which the sectional curvature is uniformly bounded, but whose volumes tend to zero as i tends to infinity. Cheeger and Gromov have shown that if a manifold M admits a generalised torus action, technically known as a polarised $\mathcal{F}$-structure, then M collapses with bounded curvature [10,8].

Consider a smooth orientable compact and connected n-manifold M with non-positive curvature and convex boundary. It has been shown by Leeb and Scott that M admits a geometric decomposition analogous to the topological Jaco–Shalen–Johanson [13,14] torus

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decomposition in dimension three. Either M has a flat metric, or M can be decomposed along totally geodesic codimension one submanifolds which are flat in the metric induced from M. The resulting pieces of this decomposition are either Seifert fibered or codimension-one atoroidal.

One of the aims of this paper is to understand how far the analogy with the JSJ decomposition holds. We will show that some asymptotic invariants, described below, vanish on codimension-one atoroidal pieces whose fundamental groups have non-trivial centre. This allows us to further distinguish between the codimension-one atoroidal pieces.

As pointed out in [18] it may not be necessary to require that M be non-positively curved for such a decomposition to exist in that Rips and Sela have shown in [24] that an algebraic counterpart holds true, (under some mild but technical hypothesis) any finitely presented group admits a JSJ splitting.

Our main result is

Theorem 1 *Let M be closed orientable and smooth nonpositively curved manifold. Assume that the geometric characteristic splitting of M consists of Seifert fibered pieces and of pieces whose fundamental group have non-trivial centre. Then M admits a polarised $\mathcal{F}$-structure.*

The existence of a polarised $\mathcal{F}$-structure on M implies that the minimal volume of M is zero [8] and therefore also implies that the minimal entropy of M is zero [22]. Here the minimal entropy is the infimum of $\mathrm{h}_{\mathrm{top}}(g)$, the topological entropy of the geodesic flow of (M, g), as we run over all unit volume smooth metrics g on M.

As M is non-positively curved, if it is not flat then its fundamental group contains a free non-abelian subgroup [6] and therefore any smooth metric g on M will have $\mathrm{h}_{\mathrm{top}}(g) > 0$. So the minimal entropy problem cannot be solved for an M as in Theorem 1; the infimum $\mathrm{h}(M)$ will never be attained by any smooth metric on M.

Examples of manifolds for which the minimal entropy problem will be solved are quotients of Euclidean space with a flat metric g, which has $\mathrm{h}_{\mathrm{top}}(g) = 0$, or quotients of Hyperbolic space with a constant negative curvature metric. The latter case was shown by Besson, Courtois and Gallot in a series of papers which found deep connections and consequences in the theory, the interested reader is invited to consult [3,4].

The minimal entropy $\mathrm{h}(M)$ is related to the minimal volume $\mathrm{MinVol}(M)$, volume entropy $\lambda(M)$ and simplicial volume $||M||$ of M in the following string of inequalities, noticed by Gromov, Manning and others [10,20,4,21]

$$\frac{n^{n/2}}{n!}||M|| \leq \lambda(M)^n \leq \mathrm{h}(M)^n \leq (n-1)^n \mathrm{MinVol}(M).$$

If M were flat then M would already admit an $\mathcal{F}$-structure, since by Bieberbach's theorem M is finitely covered by T^n. On the other hand, if the sectional curvatures K_M of M were negative at every point of M, then $||M|| \neq 0$ by a theorem of Thurston [10] which was expanded by Inoue and Yano [12], so M cannot admit $\mathcal{F}$-structures. Therefore the interesting scenario is precisely when $K_M \leq 0$, M is not flat and does not admit a metric with strictly negative curvature.

The existence of an $\mathcal{F}$-structure on M also implies that the Yamabe invariant (or *sigma constant*) of M is non-negative [19,22]. The Yamabe invariant of M is positive if and only if M admits a metric of positive scalar curvature [25]. According to Gromov and Lawson any compact smooth manifold which carries a metric of non-positive sectional curvature cannot carry a metric of positive scalar curvature [11]. Therefore for any compact smooth manifold which admits both a metric of non-positive sectional curvature and an $\mathcal{F}$-structure the Yamabe invariant vanishes. We also obtain information about Perelman's $\bar{\lambda}$ invariant, because in this case they coincide [17,1].

Corollary 2 *Assume M is a compact smooth nonpositively curved manifold whose geometric characteristic splitting consists of pieces which are Seifert fibered or pieces whose fundamental group have non-trivial centre, then*

$$\bar{\lambda}(M) = \mathrm{MinVol}(M) = 0.$$

Therefore the vanishing of $\bar{\lambda}(M)$ detects a certain lack of hyperbolicity in the sense that neither M, nor the pieces of its geometric characteristic splitting, may admit a smooth metric of negative sectional curvature.

As a consequence of the description of the characteristic flat submanifold of M in [18] and Gromov's Cutting Off Theorem [10], we obtain an estimate for the simplicial volume of M.

Proposition 3 *Let M be any closed orientable nonpositively curved manifold. Let N denote the complement in M of the pieces of the geometric characteristic splitting which are Seifert fibered and pieces each of whose fundamental group has non-trivial centre, then $||M|| = ||N||$.*

Perhaps the most intriguing of the invariants mentioned above is presently $\mathrm{h}(M)$, as it is not yet known whether it is homotopy invariant or if it depends on the differentiable structure of M. It should be pointed out that the simplicial volume and the volume entropy are homotopy invariant [5,7], whereas the minimal volume was shown by Bessières to be sensitive to changes in the differentiable structure of M [2]. In fact even the vanishing of the minimal volume is not an invariant of topological type, as shown by Kotschick [16].

2 Definitions

2.1 The geometric characteristic splitting

Being consistent with [18], we recall the following

Definition 4 A manifold N of dimension n is **Seifert fibered** if N is a Seifert bundle over a 2-dimensional orbifold with fiber a flat $(n-2)$-manifold.

It follows from the definition that if N is Seifert fibered then it is foliated by $(n-2)$-dimensional closed flat manifolds, each leaf F has a foliated neighbourhood U which has a finite cover whose induced foliation is a product $F \times D^2$.

Definition 5 A manifold N of dimension n is **codimension-one atoroidal** if any π_1-injective map of a $(n-1)$-torus into M is homotopic into the boundary of N.

2.2 $\mathcal{F}$-structures

An $\mathcal{F}$-structure is a generalisation of an S^1-action. The existence of an $\mathcal{F}$-structure on a manifold implies some of its asymptotic invariants vanish [10,8,22].

Definition 6 An $\mathcal{F}$-structure on a closed manifold M is given by,

1. A finite open cover $\{U_1, \dots, U_N\}$;
2. $\pi_i : \widetilde{U}_i \to U_i$ a finite Galois covering with group of deck transformations Γ_i, $1 \le i \le N$;
3. A smooth torus action with finite kernel of the k_i-dimensional torus, $\phi_i : T^{k_i} \to \mathrm{Diff}(\widetilde{U}_i)$, $1 \le i \le N$;
4. A homomorphism $\Psi_i : \Gamma_i \to \mathrm{Aut}(T^{k_i})$ such that

$$\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)$$

   for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$ and $x \in \widetilde{U}_i$;
5. For any finite sub-collection $\{U_{i_1}, \dots, U_{i_l}\}$ such that $U_{i_1 \dots i_l} := U_{i_1} \cap \dots \cap U_{i_l} \neq \emptyset$ the following compatibility condition holds: let $\widetilde{U}_{i_1 \dots i_l}$ be the set of points $(x_{i_1}, \dots, x_{i_l}) \in \widetilde{U}_{i_1} \times \dots \times \widetilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \dots = \pi_{i_l}(x_{i_l})$. The set $\widetilde{U}_{i_1 \dots i_l}$ covers $\pi_{i_j}^{-1}(U_{i_1 \dots i_l}) \subset \widetilde{U}_{i_j}$ for all $1 \le j \le l$, then we require that ϕ_{i_j} leaves $\pi_{i_j}^{-1}(U_{i_1 \dots i_l})$ invariant and it lifts to an action on $\widetilde{U}_{i_1 \dots i_l}$ such that all lifted actions commute.

An $\mathcal{F}$-structure is said to be *pure* if for each $x \in M$ all the orbits of all actions at x have the same dimension.

We will say an $\mathcal{F}$-structure is *polarised* if the smooth torus action ϕ_i above are fixed point free for every U_i.

2.3 $\mathcal{F}$-structures on flat manifolds

The isometry group of $\mathbb{E}^n$ is the semidirect product of $\mathbb{R}^n$ and $\mathrm{O}(n)$. Let $\rho : \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ be the map $\rho(B)(x) = Bx$. Let $\Gamma \subset \mathrm{Iso}(\mathbb{E}^n)$ be a cocompact lattice and $M := \mathbb{E}^n/\Gamma$ a compact flat manifold. Let $p : \Gamma \to \mathrm{O}(n)$ be the homomorphism $p(t, \alpha) = \alpha$, where $(t, \alpha) \in \mathbb{R}^n \times \mathrm{O}(n)$. The Bieberbach theorem ensures that Γ meets the translations in a lattice (the kernel of p is isomorphic to $\mathbb{Z}^n$) and $p(\Gamma)$ is a finite group G. Then M is finitely covered by the torus $\mathbb{R}^n/\ker(p)$ and the deck transformation group of this finite cover is G.

Notice that for any $\alpha \in G$, $\rho(\alpha)$ maps $\ker(p)$ to itself because

$$(u, \alpha) \circ (s, I) \circ (u, \alpha)^{-1} = (\rho(\alpha)s, I)$$

and thus if $(s, I) \in \Gamma$, then $(\rho(\alpha)s, I) \in \Gamma$.

It follows that the map $\rho : \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ induces a map

$$\psi : G \to \mathrm{Aut}(T^n = \mathbb{R}^n/\ker(p)).$$

As an action ϕ of T^n on $\mathbb{R}^n/\ker(p)$ we take $x \mapsto x + t$. To see that this defines an $\mathcal{F}$-structure we check the condition $\alpha(\phi(t)(x)) = \phi(\psi(\alpha)(t))(\alpha(x))$ for $\alpha \in G$ which just says $\alpha(x + t) = \alpha(x) + \alpha(t)$.

This $\mathcal{F}$-structure on M extends to an $\mathcal{F}$-structure on the product $M \times I$, where I denotes any interval.

2.4 Asymptotic invariants

The simplicial volume $||M||$ of a closed orientable manifold M is defined as the infimum of $\Sigma_i |r_i|$ where r_i are the coefficients of any *real* cycle representing the fundamental class of M. For the definition and relevant properties in case M has boundary, the reader is invited to consult [10]. To avoid confusion we will keep the same notation in this case.

For a closed connected smooth Riemannian manifold (M, g), let Vol(M, g) denote its volume and let K_g its sectional curvature. We define the following minimal volumes [10]:

$$\mathrm{MinVol(M)} := \inf_{\mathrm{g}}\{\mathrm{Vol(M, g)} \quad : \quad |\mathrm{K_g}| \leq 1\}$$

and

$$\mathrm{Vol_K(M)} := \inf_{\mathrm{g}}\{\mathrm{Vol(M, g)} \quad : \quad \mathrm{K_g} \geq -1\}.$$

The vanishing of $\mathrm{Vol_K(M)}$ implies that the simplicial volume of M is also zero, using Bishop's comparison theorem. If M admits an $\mathcal{F}$-structure then $\mathrm{Vol_K(M)} = 0$ [22].

The minimal entropy $\mathrm{h}(M)$ of a closed smooth manifold M is defined as the infimum of the topological entropy $\mathrm{h_{top}}(g)$ of the geodesic flow of g over the family of C^∞ Riemannian metrics g on M with unit volume.

The geometric meaning of $\mathrm{h_{top}}(g)$ is best expressed by Mañé's formula [20,21]

$$\mathrm{h_{top}}(g) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(p, q)\, dp\, dq.$$

Here $n_T(p, q)$ is defined to be the number of geodesic arcs of length $\leq T$ joining the points p and q of M. So that if $\mathrm{h_{top}}(g) > 0$ then there are on average exponentially many geodesics between any two points of M.

2.5 Perelman and Yamabe invariant(s)

Assume (M^n, g) is a smooth compact manifold of dimension $n \geq 3$. G. Perelman considered in [23, p. 7] the smallest eigenvalue λ_g of the elliptic operator $4\Delta_g + s_g$, here $\Delta = d^*d = -\nabla \cdot \nabla$ is the positive-spectrum Laplace–Beltrami operator associated with g and s_g is the scalar curvature of g. Perelman defined

$$\bar{\lambda}(M) := \sup_g \lambda_g \mathrm{Vol}(M, g)^{2/n}.$$

Where the supremum is taken over all smooth metrics g on M.

Amongst his various remarkable contribuitions to the understanding of the Ricci flow, Perelman observed that $\lambda_g \mathrm{Vol}(M, g)^{2/n}$ is non-decreasing along the Ricci flow whenever it is non-positive. A detailed proof can be found in [15, 8.I.2.3, p. 22].

Now we review the definition of the Yamabe invariant. Consider a fixed conformal class of metrics γ on the smooth closed manifold M, and let the Yamabe constant of (M, γ) be

$$\mathcal{Y}(M, \gamma) = \inf_{g\in\gamma} \frac{\int_M s_g dvol_g}{(\mathrm{Vol}(M, g))^{2/n}}.$$

The *Yamabe invariant* is then defined to be

$$\mathcal{Y}(M) = \sup_\gamma \mathcal{Y}(M, \gamma),$$

where the supremum is taken over all conformal classes of metrics on M.

A result of Paternain and Petean [22, Theorem 7.2] states that if a smooth compact manifold M admits an $\mathcal{F}$-structure then $\mathcal{Y}(M) \geq 0$.

It was noted by Kotschick in [17] that the Perelman and Yamabe invariants essentially coincide. The precise relationship between them was shown by Akutagawa, Ishida and LeBrun in [1] to be

$$\bar{\lambda}(M) = \begin{cases} \mathcal{Y}(M) & \text{if } \mathcal{Y}(M) \leq 0 \\ +\infty & \text{if } \mathcal{Y}(M) > 0. \end{cases}$$

On the other hand, a well known fact about the Yamabe invariant is that $\mathcal{Y}(M) > 0$ if and only if M admits a smooth metric of positive sectional curvature, see for example [25]. Therefore when M admits an $\mathcal{F}$-structure and cannot admit a metric of positive scalar curvature, we get $\mathcal{Y}(M) = \bar{\lambda}(M) = 0$.

This is precisely the case for non-positively curved manifolds, as mentioned in the introduction, since Gromov and Lawson have shown in [11] that non-positively curved manifolds can not carry a metric of positive scalar curvature. So when a non-positively curved manifold admits an $\mathcal{F}$-structure, we have $\mathcal{Y}(M) = \bar{\lambda}(M) = 0$.

In particular, applied to the construction in Theorem 1, the above discussion provides a proof of Corollary 2.

Remark 7 The same argument shows $\bar{\lambda}(M) = 0$ when M admits an $\mathcal{F}$-strucure, M is enlargeable in the sense of [11] and its universal covering space is spin because under these conditions M cannot admit a metric of positive scalar curvature.

3 Proofs

We begin by recalling some aspects of the description of the geometric characteristic splitting which we will use later, for consistency we will keep the notation in [18] throughout. The reader may consult [6] for facts about non-positively curved manifolds and [18] for further details about the characteristic geometric splitting.

Let X be a simply connected non-positively curved manifold. For every isometry ϕ of X denote by $\mathrm{MIN}(\phi)$ the set where $d_\phi : x \to d(x, \phi x)$ assumes its infimum. If A is an abelian subgroup of the isometry group of X, define

$$\mathrm{MIN}(A) := \bigcap_{\phi \in A} \mathrm{MIN}(\phi) \cong E \times Y.$$

Here E is a Euclidean space and Y is a simply connected manifold of non-positive curvature with convex boundary.

Let $C(A)$ be the centraliser of A and $N(A)$ its normaliser. That $N(A)$ acts on X preserving the metric splitting $\mathrm{MIN}(A) \cong E \times Y$ is shown in [18]. A flat is called A-invariant if the action of any element of A fixes it as a set. We will call a flat submanifold F of X a Γ-flat if the action of Γ on F has compact quotient.

We will first treat the case when the geometric characteristic splitting of M consists of Seifert fibered pieces only.

Lemma 8 *Let M be a closed non-positively curved smooth n-manifold. Assume M only has Seifert fibered pieces in its geometric characteristic splitting. Then M admits a pure polarised $\mathcal{F}$-structure.*

Proof Let N be one of the Seifert fibered pieces of the geometric characteristic splitting of M. Then N is diffeomorphic to the Seifert fibered manifold S_A, here A is the abelian subgroup of $\Gamma = \pi_1(N)$ which defines the fibration as in [18]. Recall $S_A = H_A/N(A)$, where H_A is the closed convex hull of all A-invariant Γ-flats and $N(A)$ is the normaliser of A.

For a closed convex subset Z of Y we have $H_A = Z \times E \subset Y \times E = \mathrm{MIN}(A)$. Bieberbach's theorem implies that on $E \times Y$ the abelian group A acts by translations on the Euclidean factor E.

The group Γ acts by deck transformations on the universal covering $\tilde{H}_A = \tilde{Z} \times E$ of H_A. Therefore $\mathrm{Iso}(\tilde{H}_A) = \mathrm{Iso}(\tilde{Z}) \times \mathrm{Iso}(E)$. So (by a slight abuse of notation) we can view Γ as a subgroup of $\mathrm{Iso}(\tilde{H}_A)$, the isometry group of $\tilde{H}_A$. Say $\dim(E) = k$ and consider the projection homomorphism

$$\mathrm{Iso}(\tilde{H}_A) = \mathrm{Iso}(\tilde{Z}) \times \mathrm{Iso}(E) \to \mathrm{Iso}(E) \to \mathrm{O}(k).$$

Then we have a homomorphism $\Gamma \to \mathrm{O}(k)$ with image a finite group G. Its kernel is a finite index subgroup $\Gamma_0 \subset \mathrm{Iso}(\tilde{Z}) \times \mathbb{R}^k$. It follows that the manifold

$$N_0 := \tilde{H}_A/\Gamma_0 \cong (\tilde{Z} \times E)/\Gamma_0 \cong (\tilde{Z}/\Gamma_0) \times T^k$$

is a finite cover of N with G as a deck transformation group. In this way we obtain an $\mathcal{F}$-structure on N, with $\Psi : G \to \mathrm{Aut}(T^k)$ as in (2.3), to comply with all the requirements in the $\mathcal{F}$-structure definition. As we are assuming every piece of the geometric characteristic splitting of M is Seifert fibered, this construction furnishes M with an $\mathcal{F}$-structure.

This $\mathcal{F}$-structure is pure because the dimension of the tori which define the local actions is $n-2$ over each Seifert piece and can be taken to be $n-2$ over neighbourhoods of the flat hypersurfaces. It is also polarised; T^{n-2} acts freely on itself. □

We now consider the case when the pieces of the geometric characteristic splitting have fundamental groups with non-trivial centre. The strategy is the same; each piece will be finitely covered by a smooth manifold which splits off a torus. The resulting $\mathcal{F}$-structure, although polarised, will not always be pure because the dimension of these tori may vary from piece to piece.

Lemma 9 *Let M be a closed smooth orientable manifold of non-positive sectional curvature. Assume that the fundamental group of every piece of its geometric characteristic splitting has non-trivial centre. Then M admits a polarised $\mathcal{F}$-structure.*

Proof Let N denote a piece of the geometric characteristic splitting. By the main result of Eberlein in [9] if the centre of $\pi_1(N)$ is non-trivial, then there exists a finite covering N^0 of N such that N^0 is diffeomorphic to $N^* \times T^k$. Therefore $N^0 \to N$ induces a polarised $\mathcal{F}$-structure on N. The details of this construction are found by adapting the construction of the main result in [26]. This is true for every piece of the geometric characteristic splitting, by hypothesis.

Let S denote a flat hypersurface of M which is a component of the geometric characteristic flat manifold V found in the geometric splitting. Choose a small enough ϵ-neighbourhood $N(S)$ of S in M, so that $N(S)$ is diffeomorphic to $S \times (-\epsilon, \epsilon)$ and is disjoint from other such flat characteristic hypersurface components. Then $N(S)$ admits an $\mathcal{F}$-structure induced from the flat $\mathcal{F}$-structure on S which was described in (2.3).

All the above $\mathcal{F}$-structures commute on overlaps and are polarised, therefore they endow M with a polarised $\mathcal{F}$-structure. □

3.1 Proof of Theorem 1

Proof Assume M satisfies the hypothesis. We can separate the pieces of the geometric characteristic splitting of M into two sets: the Seifert fibered ones and the ones whose fundamental groups have non-trivial centre. On these two sets of pieces we have described polarised $\mathcal{F}$-structures in Lemmas (8), (9). Lets denote these by $\mathcal{F}_s$ and $\mathcal{F}_c$, respectively.

It remains to show that if a Seifert fibered piece N_s is contiguous to a piece N_c whose fundamental group has non-trivial centre, then $\mathcal{F}_s$ and $\mathcal{F}_c$ can be completed to a polarised $\mathcal{F}$-structure on N_s and N_c glued along their common boundary. This will imply that M admits a polarised $\mathcal{F}$-structure, since the choice of the contiguous pieces N_s and N_c was arbitrary.

To see this we will use (as in the proof of Lemma (9)) the flat hypersurface S which bounds both N_s and N_c. In a small enough ϵ-neighbourhood $N(S)$ of S in $N_s \cup_S N_c$, we have that $N(S)$ is diffeomorphic to $S \times (-\epsilon, \epsilon)$. Therefore $N(S)$ admits a polarised $\mathcal{F}$-structure induced from the flat $\mathcal{F}$-structure on S, which was explained in (2.3).

Notice that the torus actions from this flat $\mathcal{F}$-structure, from $\mathcal{F}_s$ and from $\mathcal{F}_c$ are all acting on tori (in the usual way). Hence the induced torus actions on overlaps commute when lifted to each covering set. That is to say, the structures all commute and are polarised. Therefore M admits a polarised $\mathcal{F}$-structure. □

Next we will see that the simplicial volume does not detect the pieces of the geometric characteristic splitting which are Seifert fibered or have fundamental groups with non-trivial centre.

3.2 Proof of Proposition 3

Proof Let V denote the characteristic flat submanifold of M which determines its geometric splitting. Recall that the fundamental group of each flat component of V injects into $\pi_1(M)$. Also note that fundamental groups of flat manifolds are amenable, so that we can now cut V off from M, as in [10], and its simplicial volume will remain unaffected. Let N_i denote the components of $M - V$ and N denote the complement in M of the Seifert pieces and of the pieces each of whose fundamental group has non-trivial centre. Then the above discussion implies that

$$||M|| = ||M - V|| = \sum_i ||N_i||.$$

If the piece N_i is Seifert fibered or its fundamental group has non-trivial centre, N_i admits an $\mathcal{F}$-strucutre and therefore $||N_i|| = 0$. So that by Gromov's Cutting Off Theorem (in reverse, pasting V back to N) we have $||M|| = ||N||$. □

Acknowledgments Thanks to D. Jane and D. Kotshick for detailed comments and suggestions on a previous version and to G. Besson, M. Brunnbauer, B. Leeb, G. Paternain and H. Weiß for interesting converstions. The author is grateful to the DFG (Deutsche Forschungsgemeinschaft) for support when this work was written.

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