Polyhedral hyperbolic metrics on surfaces

François Fillastre

Published online: 15 November 2008
© Springer Science+Business Media B.V. 2008

Erratum to: Geom Dedicata

DOI: 10.1007/s10711-008-9252-2

The original version of the article presented some ambiguity due to omitted aspects. The text below serves to clarify.

In the last section of [4] it is proved that the map $\mathcal{I}$ is a finite-sheeted covering map between $\mathcal{P}$ and $\mathcal{M}$. As $\mathcal{M}$ is simply connected it is deduced that $\mathcal{I}$ is a homeomorphism. The fact that $\mathcal{P}$ is connected is missing. Here, we provide a proof which is a simple adaptation of the argument for the case of the sphere [1, 9.1.2]. The Fuchsian case with finite vertices was already done in [3].

For the parabolic case it is convenient to introduce a projective model of the hyperbolic-Sitter space, which is a direct extension of a model of the hyperbolic space known as paraboloid model [5, 2.3.13] or as parabolic model [2, 10.25]. Let ℓ be a light-like vector of $\mathbb{R}_1^4$, the Minkowski space of dimension 4. The paraboloid model can be obtained by a central projection in $\mathbb{R}_1^4$ of $\widetilde{\text{HS}}_\ell^3$ onto a light-like hyperplane parallel to $\ell^\perp$. In this model $\widetilde{\text{HS}}_\ell^3$ is homeomorphic to $\mathbb{R}^3$, geodesics are sent to straight lines, the boundary $\ell^\perp$ of dS_ℓ^3 is sent to infinity and we choose coordinates such that ℓ is sent to $(0, 0, \infty)$. We denote by D the part of the boundary at infinity of $\mathbb{H}^3$ sent to the paraboloid of equation $2z = -x^2 - y^2$. The hyperbolic space is sent below D and dS_ℓ^3 is sent above D . Horospheres centered at ℓ are sent to paraboloids $2z = -x^2 - y^2 + t$, where t is a real number. The orthogonal projection onto the horospheres centered at ℓ along the lines starting from ℓ corresponds to the vertical projection in this model.

The online version of the original article can be found under doi:[10.1007/s10711-008-9252-2](https://doi.org/10.1007/s10711-008-9252-2).

F. Fillastre (✉)

Département de Mathématiques, Université de Cergy-Pontoise/Saint-Martin,
2, avenue Adolphe Chauvin, 95302 CERGY-PONTOISE Cedex, France
e-mail: francois.fillastre@u-cergy.fr
URL: <http://www.u-cergy.fr/ffillastre/>

For the Fuchsian case we consider the Klein projective model, the plane fixed by the Fuchsian groups is the horizontal one and we take for D the upper half part of the unit sphere. The orthogonal projection onto caps corresponds to the vertical projection in this model.

Let $(P, G) \in \mathcal{P}$. We denote by $\mathcal{P}^{(P,G)}$ the subset of $\mathcal{P}$ made with elements (Q, G) such that Q has its vertices on the same vertical lines as the vertices of P . We also denote by $\overline{\mathcal{P}}^{(P,G)}$ the corresponding subset in the space of invariant convex HS-polyhedra (i.e. we remove the condition that edges must meet hyperbolic space).

The *height* of a vertex v is the signed vertical Euclidean distance between D and v . For an ideal vertex the height is zero. It is positive for a strictly hyperideal vertex and negative for a finite vertex. For example in the parabolic case if v has coordinates (a, b, c) , its height is $c + (a^2 + b^2)/2$. Each element of $\overline{\mathcal{P}}^{(P,G)}$ is defined by the $(n + m + p)$ heights of vertices which are inside a fundamental domain for the action of G [the polyhedral surface is then the convex hull of the orbits for G of these $(n + m + p)$ vertices]. In the following we identify $\overline{\mathcal{P}}^{(P,G)}$ with a subset of $\mathbb{R}^{n+m+p}$.

The condition for an element of $\mathbb{R}^{n+m+p}$ to correspond to an element of $\overline{\mathcal{P}}^{(P,G)}$ is that each vertex must lie outside the convex hull of the other vertices. That means that if v is one of the $(n + m + p)$ vertices and v_1, v_2, v_3 are any other vertices such that v is contained inside the vertical prism defined by v_1, v_2, v_3 , then the plane spanned by v_1, v_2, v_3 must lie below v . This gives (an infinite number of) affine conditions on the heights of the $(n + m + p)$ vertices. We then get a convex subset of $\mathbb{R}^{n+m+p}$ and if $(x_1, \dots, x_{n+m+p})$ are the coordinates of the Euclidean space, we have to intersect this convex subset with $x_i < 0$ if the i th coordinate corresponds to a finite vertex, with $x_i > 0$ if it corresponds to a strictly hyperideal vertex and with $x_i = 0$ if it is a finite vertex. Hence, $\overline{\mathcal{P}}^{(P,G)}$ is a convex subset of $\mathbb{R}^{n+m}$.

Let us call (P^i, G) the ideal convex invariant polyhedron whose vertices are the vertical projection of the ones of P onto D [(P^i, G) is the origin of $\mathbb{R}^{n+m}$ for the coordinates we introduced]. It is lying on the boundary of $\overline{\mathcal{P}}^{(P,G)}$ as slightly pushing up and down suitable vertices of (P^i, G) gives an element of $\overline{\mathcal{P}}^{(P,G)}$.

Hence in $\overline{\mathcal{P}}^{(P,G)}$ there is a segment s from (P, G) to 0. Actually s is lying in $\mathcal{P}^{(P,G)}$: along s the heights of the strictly hyperideal vertices decrease, and if we start from a polyhedron with all edges meeting hyperbolic space, this property is preserved all along the deformation.

Let us denote by $\mathcal{P}^i$ the set of ideal convex invariant polyhedra with $(n + m + p)$ vertices in a fundamental domain. We described a continuous path in $\mathcal{P}$ from any element of $\mathcal{P}$ such that the other endpoint of its completion is in $\mathcal{P}^i$. This last space is path-connected as it is homeomorphic to the Teichmüller space of $(n + m + p)$ marked points on the compact surface $\bar{S}$. Moreover, any continuous path in $\mathcal{P}^i$ can be approximated by a path in $\mathcal{P}$. Hence, $\mathcal{P}$ is path connected.

References

1. Alexandrov, A.D.: *Convex Polyhedra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2005)
2. Bridson, M.R., Haefliger, A.: *Metric Spaces of Non-positive Curvature*. Volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin (1999)
3. Fillastre, F.: Polyhedral realisation of hyperbolic metrics with conical singularities on compact surfaces. *Ann. Inst. Fourier (Grenoble)* **57**(1), 163–195 (2007)
4. Fillastre, F.: Polyhedral hyperbolic metrics on surfaces. *Geom. Dedicata* **134**, 177–196 (2008)
5. Thurston, W.P.: *Three-dimensional geometry and topology*. In: Levy, S. (ed.) Volume 35 of Princeton Mathematical Series, vol. 1. Princeton University Press, Princeton, NJ (1997)