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Driving forces on phase boundaries: "The Eshelby principle for an interface"

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Abstract In the quasistatic case, the "Eshelby principle for an interface", namely, that the total, including the external loading, "driving force" (energy-momentum tensor) must vanish on the boundary, so that it can move incrementally with associated generation of eigenstrain, is demonstrated explicitly for a half-space plane boundary under dilatational eigenstrain.

1 Introduction

"Driving forces" on defects, also called Eshelby forces, have been first introduced by Eshelby (1951) as *the force on an elastic singularity*, and a whole new field of defect mechanics was initiated. The mechanics of the defects are governed by the energy-momentum tensor, and the conservation laws that express the invariance of the energy functional under infinitesimal perturbations of the defect (e.g. Eshelby (1975, 1970) also in Markenscoff and Gupta (2006), see also Eshelby's (1977) Collected Works). This influenced dramatically the field of fracture mechanics, e.g. Freund (1990), when the defect is a crack, as well as the mechanics of other defects, such as dislocations, and the modeling of the evolution of the microstructure during plastic deformation in solids etc.

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The application of the energy-momentum tensor as the "driving force" for interfaces, considered as a particular kind of defect, was presented by Eshelby (1970), (also Markenscoff and Gupta (2006)), and Eshelby (1977) with the expression giving the energy-release-rate required to create a new volume of eigenstrain. Gupta and Markenscoff (2008) showed that the implication of Noether's theorem (see, e.g. Gelfand and Fomin 2000) is that the change in the energy functional is equal to the J integral, if and *only if* equilibrium is maintained in the domain during an infinitesimal translation of the defect.

Eshelby, in 1970, not only gave the expression for the "driving force" on an interface with normal n, (derived from the energy-momentum tensor, and coinciding with the derived expression in the phase transformation literature, e.g., Truskinovsky 1982):

$$F = [W] - T \cdot \left[\frac{\partial u}{\partial n}\right] \quad \text{(47) in Eshelby (1970)} \qquad (1)$$

(where brackets denote jumps, W the strain energy density, $\boldsymbol{T}$ the traction vector, and $\boldsymbol{u}$ the displacement vector), but also formulated what we call here the "Eshelby principle for an interface" that defines the equilibrium position of an interface. Quoting from Eshelby (1970) (also in Markenscoff and Gupta (2006)):

> *"equation* (45) *can be used to find the equilibrium position of phase and twin boundaries, in the presence of stresses produced by the transformation itself, or applied externally, or both. Since equation* (45) *must be zero for any small*

δξ, the boundary must take up shape for which eq. (47) *is zero all along it."*

which means that the vanishing of the "driving force", in the presence of external loading, determines the equilibrium position . In dynamics (Markenscoff and Ni 2010a; Markenscoff 2009), it will relate the external loading to the velocity of the inclusion boundary.

The aim of this paper is to demonstrate how this principle applies to a simple case of a plane inclusion boundary in a linear elastic solid, so as to show how the eigenstrain is indeed "generated" by appropriate external loading. It will demonstrate how the externally applied stresses, to the amount that will induce the vanishing of the total energy-momentum tensor, determine the incremental eigenstrain associated with the infinitesimal perturbation of the boundary. It will be demonstrated by explicit calculation in the static case of a plane three-dimensional inclusion boundary perturbed by an incremental displacement $\delta\xi$(Fig. 1) normal to the boundary.

2 Eigenstrain "generation" in perturbed inclusion boundary under applied loading

We consider a three-dimensional half-space inclusion with dilatational eigenstrain. For the inclusion to be constrained, external tractions must be applied at all the boundaries at infinity. These tractions are determined by considering the finite problem, of a spherical inclusion of radius a, and letting the radius tend to infinity, and are shown in the Figure. For a plane two-dimensional boundary, the limit was obtained by Dundurs and Markenscoff (2009), and for a three-dimensional boundary with dilatational eigenstrain by Markenscoff and Ni (2010a).

It must be noted here that, by this limiting process from the finite inclusion well-defined problem, the minimum energy solution is obtained for the half-space constrained inclusion. Any additional superposed self-equilibrated compatible fields will increase the total energy (Dundurs and Markenscoff 2009; Mura 1987). The tractions on the lateral boundaries $x_2 \to \pm\infty$, and $x_3 \to \pm\infty$ experience jumps, so that the necessary compatibility is maintained on the inclusion boundary. These limiting stress fields of the constrained half-space Eshelby inclusion, coincide with the Eshelby (1957) inclusion solution for the inside domain, and the Hill (1961) jump conditions for the outside. This renders the elasticity problem of the half-space phase boundary a unique well-defined one of minimum energy.

To demonstrate how the externally applied stress and the vanishing energy-momentum tensor determine the incremental eigenstrain, we show it by explicit calculation in the static case of a plane three-dimensional boundary perturbed by $\delta\xi$(Fig. 1). We assume that the six applied stresses $\sigma_{11}^{(\mathrm{ext})/(\mathrm{int})}$, $\sigma_{22}^{(\mathrm{ext})/(\mathrm{int})}$,and $\sigma_{33}^{(\mathrm{ext})/(\mathrm{int})}$ at infinity are known, and are those required by the limit of the static spherical inclusion with dilatation eigenstrain ε^* obtained in Gupta and Markenscoff (2008) as the static part of the dynamically expanding spherical inclusion. They can also be obtained from the Eshelby (1957) solution for the interior plus the Hill (1961) jump conditions, or, equivalently, by solving the coupled system of continuity of tractions and compatibility of the deformation of the interface Markenscoff (1998).

In the special case of dilatational eigenstrain, the interior strains for a sphere of radius a under dilatational eigenstrain ε^* are (e.g. Mura 1987, Eq. (11.44)),

$$\varepsilon_{11}^{(i)} = \varepsilon_{22}^{(i)} = \varepsilon_{33}^{(i)} = \frac{1+\nu}{3(1-\nu)}\varepsilon^* \tag{2}$$

while the exterior strains are (e.g. Mura 1987, Eq. (11.45)):

$$\varepsilon_{11}^{(e)} = -\frac{2}{3}\frac{1+\nu}{1-\nu}\frac{a^3}{r^3}\varepsilon^*, \quad \varepsilon_{22}^{(e)} = \varepsilon_{33}^{(e)} = -\frac{1}{3}\frac{1+\nu}{1-\nu}\frac{a^3}{r^3}\varepsilon^* \tag{3}$$

and evaluated at $r = a$:

$$\varepsilon_{11}^{(e)} = -\frac{2}{3}\frac{1+\nu}{1-\nu}\varepsilon^*, \quad \varepsilon_{22}^{(e)} = \varepsilon_{33}^{(e)} = -\frac{1}{3}\frac{1+\nu}{1-\nu}\varepsilon^* \tag{4}$$

so that from (3) and (4) the jump in the strain

$$[\![\varepsilon_{11}]\!] = \left[\!\!\left[\frac{\partial u_1}{\partial x_1}\right]\!\!\right] = -\frac{1+\nu}{1-\nu}\varepsilon^* \equiv -\frac{3\lambda+2\mu}{\lambda+2\mu}\varepsilon^* \tag{5}$$

The stress components are found from the stress–strain relations

$$\sigma_{ij} = 2\mu\left(\varepsilon_{ij} - \varepsilon_{ij}^* + \frac{\lambda}{2\mu}\left(\varepsilon_{kk} - \varepsilon_{kk}^*\right)\delta_{ij}\right), \tag{6}$$

or equivalently,

$$\sigma_{ij} = 2\mu\left(\varepsilon_{ij} - \varepsilon_{ij}^* + \frac{\nu}{1-2\nu}\left(\varepsilon_{kk} - \varepsilon_{kk}^*\right)\delta_{ij}\right). \tag{7}$$

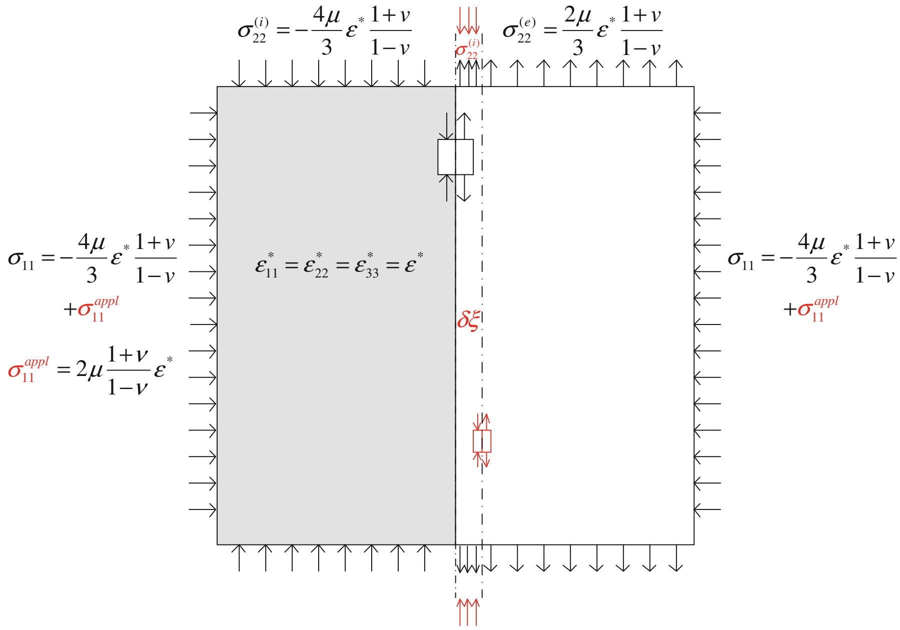

Fig. 1 Expanding half-space inclusion with dilatational eigenstrain

The interior stresses are (see, also Mura Eq. 11.21a by the Eshelby tensor):

$$
\begin{aligned}
\sigma_{11}^{(i)} &= \sigma_{22}^{(i)} = \sigma_{33}^{(i)} \\
&= -\frac{\mu}{1-\nu}\varepsilon^* \left[\frac{16}{15} + 2\frac{5\nu+1}{15} + 2\frac{5\nu+1}{15}\right] \\
&= -\frac{4\mu}{3}\varepsilon^* \frac{1+\nu}{1-\nu} \qquad (8)
\end{aligned}
$$

Without knowing the exterior solution as in Mura (11.45), the exterior stresses are found from the continuity of tractions

$$\sigma_{11}^{(i)} = \sigma_{11}^{(e)}, \qquad (9)$$

and the compatibility of the deformation at the interface

$$\varepsilon_{22}^{(i)} = \varepsilon_{22}^{(e)}, \quad \varepsilon_{33}^{(i)} = \varepsilon_{33}^{(e)}. \qquad (10)$$

From (6) or (7) and (9) we obtain the jump in the strain, as already given in (5) . The jump in the hoop stress is obtained from the constitutive relation (6) or (7) and the jump in the strain (5):

$$[\![\sigma_{22}]\!] = 2\mu\frac{3\lambda+2\mu}{\lambda+2\mu}\varepsilon^* \equiv 2\mu\frac{1+\nu}{1-\nu}\varepsilon^* \qquad (11)$$

so that the exterior stresses, in addition to (9), are :

$$\sigma_{33}^{(e)} = \sigma_{22}^{(e)} = \frac{2\mu}{3}\varepsilon^* \frac{1+\nu}{1-\nu}. \qquad (12)$$

Now, we assume that the constrained half-space Eshelby inclusion with dilatational eigenstrain is kept in place by the above stresses at the boundaries at infinity (see Fig. 1) so that the elastic field is the unique minimum energy one. In the value of these stresses ε^* is considered as a parameter. We want to move the boundary by $\delta\xi$ and create dilatational eigenstrain ε^*.

As the boundary of the inclusion is displaced by an increment $\delta\xi$, the stresses at $x_2, x_3 \to \pm\infty$ switch sign by an external agent, so that on the left of the boundary is compression and the hoop stresses experience jumps at the interface (Fig. 1). We will show that, by applying an additional external stress $\sigma_{11}^{\mathrm{appl}}$, of magnitude such as to have the total energy-momentum tensor ("driving force") equal to zero, the eigenstrains in the incremental region will be indeed dilatation of the same magnitude, i.e.,

$$\varepsilon_{11}^* = \varepsilon_{22}^* = \varepsilon_{33}^* = \varepsilon^* \qquad (13)$$

under the assumption that the eigenstrain remains on the left of the interface.

The proof is by solving the governing system of equations for the *unknown strains and eigenstrsains*, separately on the left and right of the inclusion boundary (assuming the inclusion on the left) (see Fig. 1), given as known the values of the stresses (those of Eqs. (8) and (12)), where ε^* is a considered as a parameter.

The inverted stress–strain relations are:

$$\varepsilon_{ij} - \varepsilon_{ij}^* = \left(\sigma_{ij} - \delta_{ij}\sigma_{kk}\frac{\nu}{1+\nu}\right)/2\mu \qquad (14)$$

and the interface conditions of continuity of tractions and compatibility (for the dilational eigenstrain half-space inclusion) are:

$$\sigma_{11}^{(i)} = \sigma_{11}^{(e)} \quad \varepsilon_{22}^{(i)} = \varepsilon_{22}^{(e)} \quad \varepsilon_{33}^{(i)} = \varepsilon_{33}^{(e)} \tag{15}$$

The solution of the above system of Eqs. (14) and (15) for the (unknown) strains and eigenstrains, with the stresses assumed known, yields for the eigenstrains:

$$\varepsilon_{33}^{*} = \varepsilon_{22}^{*} = \varepsilon^{*}, \tag{16}$$

while leaving ε_{11}^{*} undetermined, due to the first equation in (15). This implies that an additional equation (to the previous Eqs. (14) and (15)) is needed to determine the eigenstrain ε_{11}^{*}, and this is the Eq. (1) (see Eshelby 1970; Markenscoff and Gupta 2006) set equal to zero, so that the total "driving force" on the interface vanishes . For this, the application of an external loading is necessary, since, without it, the "driving force", or, which may be called here the " self-force", is negative and given by Gavazza (1977):

$$-\frac{2\mu(3\lambda+2\mu)\varepsilon^{*2}}{(\lambda+2\mu)} \tag{17}$$

and is independent of the radius of the sphere. The negative sign implies that the interface is stable until the external loading negates the self-force, at which point it becomes unstable. Let us consider that an *additional* uniform $\sigma_{11}^{\text{appl}}$ stress is applied and calculate according to (1) the total "driving force" on the interface with all the interaction terms, and then, let us set this total "driving force" equal to zero . The calculation yields:

$$-\frac{2\mu(3\lambda+2\mu)\varepsilon^{*2}}{(\lambda+2\mu)} + \sigma_{11}^{\text{appl}}\varepsilon^{*} = 0 \tag{18}$$

The *LHS* of equation (18) which was derived from the "driving force" expression (1) by including *all* the interaction energy terms (due to the dilatational eigenstrain and a superposed uniaxial stress $\sigma_{11}^{\text{appl}}$), has *only* two that survive: the first one, which is the self-force (coinciding with the value given by Gavazza (1977), also Markenscoff and Ni (2010a,b)), and the second-one, which, due to its form, may be considered as the counterpart of the Peach-Koehler force for the inclusion.

If the value for the additional applied stress $\sigma_{11}^{\text{appl}}$ is the one that satisfies the above equation, i.e. is equal to

$$\sigma_{11}^{\text{appl}} = \frac{2\mu(3\lambda+2\mu)\varepsilon^{*}}{(\lambda+2\mu)} \equiv 2\mu\frac{1+\nu}{1-\nu}\varepsilon^{*} \tag{18}$$

then, the eigenstrain ε_{11}^{*} is determined from (18) as:

$$\varepsilon_{11}^{*} = \varepsilon^{*} \tag{19}$$

and the desired dialational eigenstrain has been generated. □

3 Conclusions

We thus showed that, if, at infinity, incremental tractions are applied so that the stress is compressive on the inclusion side and tensile on the matrix (see Fig. 1)—ensuring that the compatibility of the interface is maintained—, and, if an additional external stress $\sigma_{11}^{\text{appl}}$, is applied, of magnitude such as to have the total "driving force" (energy-momentum tensor) equal to zero, then, the eigenstrains in the incremental region will be indeed dilatation of the same magnitude, i.e.,

$$\varepsilon_{11}^{*} = \varepsilon_{22}^{*} = \varepsilon_{33}^{*} = \varepsilon^{*} \tag{20}$$

under the assumption that the eigenstrain remains on the left of the interface after perturbation, thus showing how eigenstrain is "generated".

In dynamically expanding inclusions, with inertia effects included, to determine the position of the moving inclusion boundary, we apply the same principle with the dynamic energy-momentum tensor (Markenscoff and Ni 2010a). For the plane boundary (as the limit of the radius of the spherical expanding inclusion tends to infinity), the tractions that need to be applied at infinity for the eigenstrain region to expand were obtained in Markenscoff and Ni (2010a) by the limiting process from the sphere. Again, the total driving force (with inertia effects) equal to zero will uniquely determine the eigenstrain on the left of the boundary. More recently the treatment was expanded (Markenscoff and Ni 2010b) to half-space inclusions and strips of general eigenstrain in general subsonic motion. A recent publication (Yang et al. 2009) obtains the relation of the remote dynamic stress to the motion of a martensitic phase transformation boundary by an analytic-computational model based on *constrained* Eshelby inclusions via the Hill self-consistent method.

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